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NORMED RIGHT ALTERNATIVE ALGEBRAS OVER THE REALS

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1. One of the most interesting results on real normed division algebras says that every real normed *associative* division algebra is finite dimensional [**6**, Theorem 1.7.6], and hence by a classical theorem of Frobenius either isomorphic to the real field, the complex field, or the algebra of quaternions. Thus the dimension of the algebra can only be either 1, 2 or 4. In this note we show in Theorem 2 that if in the previous result we replace the associativity of the algebra by the weaker assumption that it is *right alternative*, that is, the relation

$$(xy)y = xy^2 \tag{1}$$

holds for all elements x and y in the algebra, then the algebra is still finite dimensional, but in this case to the three previous possibilities a fourth one has to be added, namely, that the algebra be isomorphic to the algebra of Cayley numbers, which is of dimension 8.

We give two proofs of Theorem 2, one of them based on Theorem 1, which asserts that every element of a real normed right alternative algebra with a unit has a non-empty spectrum. Thus Theorem 1 extends a basic result of the theory of normed associative algebras to a wide class of non-associative algebras.

We leave here open the question whether every real normed division algebra is finite dimensional. In other words we do not know whether the finite dimensionality of the algebra can be established without assuming the relation (1). Since, by a theorem of Milnor [**5**], every finite dimensional real division algebra, right alternative or not, can only be of dimension 1, 2, 4 or 8, an affirmative answer to our question would prove that the dimension of every real normed division algebra can only be these powers of 2. However the algebra would not have to be one of the four listed in Theorem 2.

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2. Let $\mathfrak{A}$ be a right alternative algebra over the real field $\mathbf{R}$. The *right multiplication* operator R_a associated to $a \in \mathfrak{A}$ is defined by $R_a x = xa$. For a, b, c in $\mathfrak{A}$ their *associator* is $[a, b, c] = (ab)c - a(bc)$. With this terminology the right alternative law (1) is expressed either by

$$R_a^2 = R_{a^2} \tag{2}$$

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or by

(3) $$[a, b, b] = 0,$$

which in turn implies

$$0 = [a, b + c, b + c] = [a, b, b] + [a, b, c] + [a, c, b] + [a, c, c] = [a, b, c] + [a, c, b],$$

that is,

(4) $$[a, c, b] = -[a, b, c].$$

The following two formulas have been derived by Skorniakov [7] and also by Kleinfeld [4] for right alternative rings of characteristic not 2 (i.e., $2x = 0$ implies $x = 0$); hence they hold in $\mathfrak{A}$.

(5) $$a[(bc)b] = [(ab)c]b$$

(6) $$[a, b, c](cb) = ([a, b, c]b)c.$$

LEMMA ON INVERSES. *Let $\mathfrak{A}$ be a real right alternative algebra with a unit* 1 *and let I be the identity operator on $\mathfrak{A}$. Then*

(i) *$ab = 1 = b'a$ implies $ba = 1$,*

(ii) *$ab = 1 = ba$ if and only if $R_bR_a = I = R_aR_b$,*

(iii) *$ab = 1 = ba$ and $ab' = 1 = b'a$ imply $b' = b$.*

Proof. (i) is an immediate consequence of (5) since $ba = [(b'a)b]a = b'[(ab)a] = b'a = 1$. In (ii), $R_bR_a = I = R_bR_a$ means that $(xa)b = x = (xb)a$ for all $x \in \mathfrak{A}$, and this implies, taking $x = 1$, that $ab = 1 = ba$. Conversely, let us assume $ab = 1 = ba$, and let x be an arbitrary element of $\mathfrak{A}$. Then it follows from (6) and (5) that

$$\begin{aligned}(xa)b - x = [x, a, b] &= ([x, a, b]a)b = ([(xa)b - x]a)b \\ &= \{[(xa)b]a\}b - (xa)b = (xa)b - (xa)b = 0.\end{aligned}$$

Switching now the roles of a and b one obtains similarly that $(xb)a - x = 0$. Thus $(xa)b = x = (xb)a$ for all $x \in \mathfrak{A}$. For the proof of (iii) we note first that according to (ii), $R_bR_a = I = R_aR_{b'}$, and since the multiplication of operators is associative, we have $R_{b'} = (R_bR_a)R_{b'} = R_b(R_aR_{b'}) = R_b$. Hence $xb' = xb$ holds for all $x \in \mathfrak{A}$, and in particular for $x = 1$. Thus $b' = b$.

In the Lemma, (i) expresses that if a has a right inverse b and a left inverse b' then b is an inverse of a. By (ii), b is an inverse of a (in $\mathfrak{A}$) if and only if R_b is an inverse of R_a (in the associative algebra of linear operators on $\mathfrak{A}$). (iii) is nothing else but the expression of the uniqueness of the inverse.

The *spectrum* of an element x of a right alternative algebra $\mathfrak{A}$ with a unit over $\mathbf{R}$ is defined, as in the associative case, to be the set of all $(\alpha, \beta) \in \mathbf{R}^2$ for which $x^2 - 2\alpha x + (\alpha^2 + \beta^2) \cdot 1$ is not invertible. We have

THEOREM 1. *Let $\mathfrak{A}$ be a real normed right alternative algebra with a unit. Then the spectrum of each element $x \in \mathfrak{A}$ is not empty.*

Proof. Let $\mathscr{L}(\mathfrak{A})$ denote the normed algebra of all linear bounded operators on $\mathfrak{A}$. It is clear that for $a \in \mathfrak{A}$, $R_a \in \mathscr{L}(\mathfrak{A})$ with $||R_a|| = ||a||$. The algebra $\mathscr{L}(\mathfrak{A})$ is associative; $I = R_1$ is its unit and, as is well-known, each element of $\mathscr{L}(\mathfrak{A})$ has a non-empty spectrum. Using these facts it becomes very easy to prove, by contradiction, that there exist real numbers α_0, β_0 for which $x^2 - 2\alpha_0 x + (\alpha_0^2 + \beta_0^2) \cdot 1$ is not invertible. Indeed, if for all $(\alpha, \beta) \in \mathbf{R}^2$, $w = x^2 - 2\alpha x + (\alpha^2 + \beta^2) \cdot 1$ were invertible, then by (2) and (ii), $R_w = R_x^2 - 2\alpha R_x + (\alpha^2 + \beta^2)I$ would be invertible in $\mathscr{L}(\mathfrak{A})$ for all $(\alpha, \beta) \in \mathbf{R}^2$, which contradicts the non-emptyness of the spectrum of R_x in $\mathscr{L}(\mathfrak{A})$.

THEOREM 2. *Let $\mathfrak{A}$ be a normed right alternative division algebra over the reals. Then $\mathfrak{A}$ is finite dimensional and isomorphic to either the reals, the complex numbers, the quaternions or the Cayley numbers.*

First proof. That $\mathfrak{A}$ is a division algebra means that for $a \neq 0$ and b arbitrary the equations

$$ax = b, \qquad ya = b$$

have unique solutions x, y in $\mathfrak{A}$. By a theorem of Skorniakov [**7**] the assumption that $\mathfrak{A}$ is a right alternative division algebra over the reals implies that $\mathfrak{A}$ is alternative, has a unit, and each non-zero element is invertible. Let now x be an arbitrary element of $\mathfrak{A}$. By Theorem 1 there exist real numbers α, β for which $x^2 - 2\alpha x + (\alpha^2 + \beta^2) \cdot 1$ is not invertible. Consequently $x^2 - 2\alpha x + (\alpha^2 + \beta^2) \cdot 1 = 0$. Thus x satisfies a quadratic equation with real coefficients, where the coefficient of 1 is zero only for $x = 0$, so that $\mathfrak{A}$ is an alternative quadratic algebra over the reals. But this implies, according to a theorem of Albert [**1**, Theorem 1], that $\mathfrak{A}$ is finite dimensional and isomorphic either to the reals, the complex numbers, the quaternions or the Cayley numbers.

Second proof. We know already from the first proof that $\mathfrak{A}$ is an alternative division algebra. Hence by the Bruck-Kleinfeld-Skorniakov Theorem [**8**; **2**; **3**] $\mathfrak{A}$ is either associative or a Cayley-Dickson algebra of dimension 8 over its centre C, which is a field (C is by definition the set of all elements of $\mathfrak{A}$ which commute and associate with all the elements of $\mathfrak{A}$).

In the associative case, we have, as already pointed out in § 1, that $\mathfrak{A}$ is either isomorphic to the reals, the complex numbers or the quaternions. In the non-associative case the centre is a normed field and hence the dimension of C over $\mathbf{R}$ is 1 or 2 [**6**, Theorem 1.7.5]. On the other hand since the dimension of $\mathfrak{A}$ over $\mathbf{R}$ is equal to the dimension of $\mathfrak{A}$ over C times the dimension of C over $\mathbf{R}$, we conclude that the dimension of $\mathfrak{A}$ over $\mathbf{R}$ is finite and is either 8 or 16. But it is quite well-known that an alternative division algebra over the reals cannot have dimension 16. This dimension argument not only shows that the dimension of C over $\mathbf{R}$ is one but also that $\mathfrak{A}$ is an alternative division algebra of dimension 8 over the reals, and hence isomorphic to the Cayley numbers. This completes the second proof of the theorem.

Remark. Theorems 1 and 2 remain valid if instead of assuming $\mathfrak{A}$ right alternative we assume it to be *left alternative*, i.e., $x(xy) = x^2y$ for all x and y in $\mathfrak{A}$. Of course in the Lemma the right multiplication operators should be replaced by left multiplication operators and (i) changed to: $ab = 1 = b'a$ implies $ab' = 1$.

Added in proof. The question left open in § 1 has been raised in 1969 as a conjecture by I. Kaplansky in his Hawaii lectures on *Algebraic and analytic aspects of operator algebras* (Regional Conference Series in Mathematics, Monograph 2, published by the American Mathematical Society).

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