TRANSFORMATION ALGEBRAS

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Introduction. The purpose of this paper is to show that most results concerning polyadic algebras can be generalized to transformation algebras. The results of this paper will clearly indicate that a great deal can be done in polyadic algebras without ever mentioning the quantifier structure (for instance, terms and operations can be characterized without the help of the quantifier structure, at least in the case where an equality is present). In § 1, we develop the elementary theory; in § 2, we study the different ways of extending a (locally finite) transformation algebra (of infinite degree) to a polyadic algebra; in § 3, we study equality transformation algebras; finally, in § 4, we show how terms and operations can be defined in equality transformation algebras.

0. Preliminaries. The purpose of this section is to establish the notation and the basic terminology to be used throughout this paper. For easier reference, we also state, without proof, one known lemma. The notation for Boolean algebras and polyadic algebras is the one used by Halmos in his four papers **(1–4)**. However, there are two notational changes which we describe now. If I is a set and i, j are elements of I, then the symbol (j/i) denotes the transformation on I which sends i onto j and every other element onto themselves. Previously, in **(3)**, this transformation was denoted by (i/j). The advantage of this new notation is that the "cancellation law" holds: $S(j/k)S(k/i)p = S(j/i)p$ whenever p belongs to some polyadic algebra and i, j, and k are variables so that p is independent of k. Accordingly, if t is a term and K a set of variables, then the symbol (K/t) as defined in **(3)** shall be replaced by (t/K). The second change of notation concerns functional polyadic algebras. Suppose X and I are sets, J a subset of I, and x, y belong to X^I. We define $x \equiv y \bmod J$ (to read: x is equal to y modulo J) to mean $x_i = y_i$ whenever i is not in J. The letter O shall denote the two-element Boolean algebra. A homomorphism from a Boolean algebra B onto O shall be called a *valuation* of B. A *sum* of B is a subset Σ of B so that Σ has a supremum in B which is denoted by $\vee\ \Sigma$. If C is a Boolean algebra and f is a mapping from B into C, then f is said to *preserve* a sum Σ of B whenever $f(\Sigma)$ is a sum of C and $f(\vee\ \Sigma) = \vee f(\Sigma)$. If f is a homomorphism from B into C and if f preserves every sum of B, then f is called a *complete homomorphism.* Suppose C is a subalgebra of B and Σ is a sum of C. Then B is said to *preserve* the sum Σ if the inclusion mapping of C into B preserves the sum Σ.

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Suppose B is a Boolean algebra and V is a set of valuations of B. The smallest topology on B which makes every valuation in V continuous (the topology on O is the discrete one) will be referred to as the V-*topology* or the *topology induced by* V. With the V-topology B becomes a bounded topological Boolean algebra, that is, B viewed as a group under $+$ is a bounded topological group; moreover, the Boolean operations are uniformly continuous. If the V-topology is Hausdorff, the topological completion $\bar{B}$ of B is a compact Hausdorff topological Boolean algebra, and $\bar{B}$ is isomorphic and homeomorphic to the Boolean algebra of all subsets of V, the topology on $\bar{B}$ being induced by the set of all complete valuations of $\bar{B}$. A set V of valuations of B is said to be *separating* (*on* B) if for every p in B distinct from 0, there is a valuation v in V so that $v(p) = 1$. The V-topology is Hausdorff if and only if V is separating. All the results just stated are easy to establish, and, for that reason, we omit the proofs. For more details concerning bounded topological Boolean algebras, see **(6)**. We conclude this section by stating a known result.

(0.1) LEMMA. **(9)** *If B is a Boolean algebra, then there exists a unique complete Boolean algebra A so that B is a subalgebra of A, A preserves every sum of B, and for every element p in A, there exists a subset Σ of B, so that $p = \bigvee \Sigma$. Moreover, if C is a complete Boolean algebra and f is a complete homomorphism from B into C, then f has a unique extension to a complete homomorphism from A into C.*

1. Elementary theory. We begin with the definition of a transformation algebra; it was first formulated by Halmos. A *transformation algebra* is a triple (B, I, S) where B is a Boolean algebra, I a set, and S a mapping from transformations τ on I to endomorphisms $S(\tau)$ of B so that

(S_1) $S(\delta)$ is the identity endomorphism if δ is the identity on I;

(S_2) $S(\tau\sigma) = S(\tau)S(\sigma)$ whenever τ and σ are transformations on I. As in the polyadic case, we shall often commit the solecism of identifying (B, I, S) with B and we shall also say that B is an I-algebra. It is clear that, if $(B, I, S, \exists)$ is a polyadic algebra, then (B, I, S) is a transformation algebra. For that reason, we shall simply say that a polyadic algebra is a transformation algebra. However, in general, the converse is not true. The elementary algebraic theory for transformation algebras is straightforward and very similar to that of polyadic algebras. In the sequel we shall use expressions like transformation subalgebras, transformation homomorphisms, transformation ideals, and the like without explicit reference. Their meaning is clear. The *degree* of (B, I, S) is the cardinality of I. The concept of independence is formulated as follows: if $p \in B$ and J is a subset of I, then p is said to be *independent* of J, if, for all transformations τ on I with $\tau i = i$ for i in $I - J$, $S(\tau)p = p$. A subset K of I is said to be a *support* for p if p is independent of $I - K$. It is true, as in the polyadic case, that if $p \in B$, K is a support for p and τ is a transformation on I, then $\tau(K)$ supports $S(\tau)p$. It is easy to see that if B happens to be a polyadic algebra, then the concept of support as defined above coincides with the concept of support as defined in **(2)**. If every element in B has finite support, then B is said to be *locally finite.* If C is a Boolean algebra and X a set, then a C-valued functional transformation algebra over X is defined as in the polyadic case, except that no mention is made of quantification. We shall also make use of this concept without explicit reference.

(1.1) THEOREM. *Every transformation algebra is isomorphic to a functional transformation algebra.*

Proof. Let A be the transformation algebra of all functions from I^I into B and define a mapping f from B into A by $(fp)(\tau) = S(\tau)p$ (remember that a transformation on I is an element of I^I and conversely). It is easy to check that f is actually a transformation isomorphism of B into A.

(1.2) COROLLARY. *Any transformation algebra can be embedded in a polyadic algebra.*

Proof. Because of (1.1), it suffices to prove the corollary for a functional transformation algebra. If B is a C-valued functional transformation algebra over a set X, let $\bar{C}$ be a complete Boolean algebra containing C and let A be the polyadic algebra of all functions from X^I into $\bar{C}$. Then A is clearly a polyadic extension of B.

Suppose B is a transformation I-algebra. If i is a variable, p an element of B, then the set of all elements of B of the form $S(j/i)p$ for some j in I shall be called an *existential subset* of B, and it shall be denoted by $\Sigma(i, p)$; if $\Sigma(i, p)$ has a supremum in B, it shall be called an *existential sum* of B. Note that what is called an existential subset here was called a totally existential subset in **(5)**. A valuation v of B will be called an *existential valuation* if it preserves every existential sum of B. The relevance of the concept of existential subset is shown by the following results.

(1.3) LEMMA. *Suppose* $(A, I, S, \exists)$ *is a locally finite polyadic algebra of infinite degree. If* I_0 *is an infinite subset of* I, i *a variable in* I, *and* p *an element of* A, *then*

$$\exists(i)p = \vee\{S(j/i)p : j \in I_0\}.$$

Proof. If i is in I_0, fix the variables in $I - I_0$, and apply **(2**, 10.5**)**. If i is not in I_0, choose k in I_0 so that p is independent of k. Since $S(j/k)S(k/i)p = S(j/i)p$ for all j in I_0, it follows that $\vee\{S(j/i)p\ ; j \in I_0\} = \exists(k)S(k/i)p = \exists(i)p$. This completes the proof of the Lemma.

It follows from (1.3) that if $\Sigma(i, p)$ is an existential subset of A, then $\Sigma(i, p)$ is a sum and $\exists(i)p$ is the supremum of $\Sigma(i, p)$. It follows also that a transformation isomorphism from A onto another polyadic algebra is necessarily a polyadic isomorphism. In particular, this implies that if a transformation algebra can be made into a polyadic algebra, then this can be done in only one way.

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(1.4) LEMMA. *If τ is a transformation on I so that $\tau(I)$ is infinite, then $S(\tau)$ preserves the existential sums of A; if $\tau(I) = I$, and if Σ is an existential sum of A, then $S(\tau)(\Sigma)$ is an existential sum of A.*

Proof. Let $\tau(I) = I_0$ and let $\Sigma(i, p)$ be an existential sum of A. Suppose σ is a finite transformation on I so that σ agree with τ on a finite support J of p, and let k be a variable so that $\sigma^{-1}k = \{k\}$ and k is not in $\tau(J)$. Then $S(\tau) \exists(i)p = S(\sigma) \exists(i)p = \exists(k)S(\sigma)S(k/i)p = \vee\{S(j/k)S(\sigma)S(k/i)p: j \in I_0\} = \vee\{S(\tau)S(j/i)p : j \in I\}$. Since $\exists(i)p$ is the supremum of $\Sigma(i, p)$, it follows that $S(\tau)$ preserves $\Sigma(i, p)$. This proves the first part of the Lemma. The second part is immediate.

The following theorem is more or less the converse of (1.4).

(1.5) THEOREM. *Suppose (B, I, S) is a locally finite transformation algebra of infinite degree. If every existential subset of B is a sum and if $S(\tau)$ preserves the existential sums whenever $\tau(I) = I$, then there is a unique mapping $\exists$ from subsets of I to quantifiers of B so that $(B, I, S, \exists)$ is a polyadic algebra.*

Proof. Let A be the transformation algebra of all finite dimensional functions from I^I into B. Define a mapping f from B into A by $(fp)(\tau) = S(\tau)p$ for every transformation τ on I and every p in B. Let $\tilde{B} = f(B)$. Clearly f is a transformation isomorphism. Since uniqueness of the quantifier mapping $\exists$ is an immediate consequence of (1.3), to complete the proof it suffices to show that $\tilde{B}$ is a functional polyadic algebra. For that purpose, let p in B, i in I, $q = \vee\, \Sigma(i, p)$, and σ and τ transformations on I so that $\tau(I) = I$ and $\sigma \equiv \tau \bmod (I - J)$ where J is a finite support for both p and q, and $i \in J$. Then $(fq)(\sigma) = S(\sigma)q = S(\tau)q = \vee\{S(\tau)S(j/i)p: j \in I\} = \vee\{S(\pi)p: \pi \equiv \tau \mathit{ mod } i\} = \vee\, \{(fp)(\pi): \pi \equiv \sigma \bmod i\} = (\exists(i)fp)(\sigma)$ which shows that $\exists(i)$ is defined on $\tilde{B}$ and therefore, since $\tilde{B}$ is locally finite $\exists(J)$ is defined for every subset J of I. It follows that $\tilde{B}$ is a functional, polyadic algebra. This completes the proof of the theorem.

We close this section by observing that if B has an equality (see § 3), then the hypothesis, in the preceding theorem, that $S(\tau)$ preserves the existential sums whenever $\tau(I) = I$ can be removed. The results of § 3 will clearly indicate that this hypothesis is necessarily satisfied in the presence of an equality.

2. Polyadic completions of a transformation algebra. The purpose of this section is to study the different ways of extending a transformation algebra to a polyadic algebra. We begin with two Lemmas.

(2.1) LEMMA. *Suppose A is a locally finite polyadic I-algebra of infinite degree. If A has cardinality less than or equal to that of I, and if V is the set of all existential valuations of A, then V is separating on A.*

Proof. Because of semi-simplicity, it suffices to prove the lemma in the case where A is simple. In view of the cardinality assumption and of the representation theorem for polyadic algebras, we may in turn assume that A is an O-valued algebra over X where X has cardinality less than or equal to that of I. Choose q in A so that $q \neq 0$. Choose x in X^I so that $x(I) = X$ and so that $q(x) = 1$. Define a valuation v of A by $vp = p(x)$ for all p in A; clearly $vq = 1$. It is a straightforward matter to check that v is an existential valuation of A. This shows that V is separating on A.

We make the observation that (2.1) is false if the cardinality assumption is removed. Indeed, let A be the algebra of all finite dimensional functions from X^I into O where I is countably infinite and X has cardinality greater than that of the continuum. Suppose v is an existential valuation of A. Let $\tilde{A}$ be the algebra of all finite dimensional functions from I^I into O and define a mapping f from A into $\tilde{A}$ by $fp(\tau) = vS(\tau)p$. Then f is an isomorphism into $\tilde{A}$ which is a contradiction since A and $f(A)$ have different cardinalities. This shows that A has no existential valuation.

(2.2) LEMMA. *Suppose A is a locally finite polyadic algebra of infinite degree and B is a transformation subalgebra which generates A polyadically. If V is a set of existential valuations of A, and if A is topologized with the V-topology, then B is dense in A.*

Proof. Clearly, we may assume that V contains all existential valuations of A. In view of (1.4), it follows that $S(\tau)$ is continuous whenever τ is a transformation on I so that $\tau(I) = I$. Denote by $\bar{B}$ the closure of B; then $\bar{B}$ is a transformation subalgebra. To see that $\bar{B} = A$, it suffices to prove that $\bar{B}$ is a polyadic subalgebra. For that purpose, let p in $\bar{B}$ and i in I. For each finite subset J, let $p_J = \vee \{S(j/i)p : j \in J\}$. Because every valuation in V preserves the existential sums, it follows that the net $\{p_J : J \in D\}$ converges to $\exists(i)p$ where D is the directed set of all finite subsets of I, the ordering on D being set-theoretical inclusion. Hence $\exists(i)p$ belongs to $\bar{B}$ which shows that $\bar{B} = A$.

If B is a transformation algebra, then a *polyadic completion* of B is a pair (A, h) where A is a polyadic algebra and h a transformation homomorphism from B into A such that $h(B)$ generates A; (A, h) shall be called *faithful* if h is a transformation monomorphism. When no confusion is liable to arise, A itself shall be called a polyadic completion of B. We shall give now, under suitable hypotheses, a complete classification of all possible polyadic completions of a given locally finite transformation algebra of infinite degree.

For the remainder of this section, (B, I, S) shall be a fixed locally finite transformation algebra of infinite degree.

Suppose (A_1, h_1) and (A_2, h_2) are polyadic completions of B. Then a *homomorphism* from (A_1, h_1) onto (A_2, h_2) is a polyadic homomorphism f from A_1 onto A_2 such that $fh_1 = h_2$; if f is an isomorphism, then (A_1, h_1) and (A_2, h_2)

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are said to be *isomorphic*. We are now ready to prove the main theorem of this section concerning the existence of polyadic completions.

(2.3) THEOREM. *Suppose V is a non-empty set of valuations of B. Then there exists a unique (up to isomorphism) polyadic completion (A, h) of B having the following two properties:*

(2.4) *for every v in V, there is a (necessarily unique) existential valuation $\bar{v}$ of A so that $\bar{v}h = v$;*

(2.5) *for every p in A distinct from 0, there is a valuation v in V and a transformation τ on I so that $\bar{v}S(\tau)p = 1$.*

(A, h) shall be called the *polyadic completion (of B) with respect to V* or else the *V-polyadic completion (of B)*.

Proof. Let $\tilde{V}$ be the set of all valuations of B of the form $vS(\tau)$ where v is in V and τ is a transformation on I so that $\tau(I) = I$. Assume first that $\tilde{V}$ is separating on B. Topologize B with the $\tilde{V}$-topology and let $\bar{B}$ be the topological completion of B; then $\bar{B}$ is actually isomorphic and homeomorphic with the Boolean algebra of all subsets of $\tilde{V}$, the topology on $\bar{B}$ being induced by the set of all complete valuations. If τ is a transformation on I so that $\tau(I) = I$, then $S(\tau)$ is uniformly continuous on B and therefore $S(\tau)$ has a unique extension to a complete endomorphism $\bar{S}(\tau)$ of $\bar{B}$. Let A^* be the set of all p in $\bar{B}$ for which there is a finite subset J of I so that $S(\tau)p = S(\sigma)p$ whenever $\tau(I) = \sigma(I) = I$ and $\tau \equiv \sigma \bmod (I - J)$; then A^* is a Boolean algebra. It is easy to see that there is a unique way of extending $\bar{S}$ to all transformations on I so that $(A^*, I, \bar{S})$ becomes a locally finite transformation algebra. Moreover, if p is in A^*, i in I, and $q = \vee \{\bar{S}(j/i)p : j \in I\}$ where the supremum is taken in $\bar{B}$, then q is in A^*. This follows from the fact that $\bar{S}(\tau)$ is a complete endomorphism of $\bar{B}$ whenever $\tau(I) = I$. It follows from (1.5) that there is a unique quantifier mapping $\exists$ so that $(A^*, I, \bar{S}, \exists)$ is a polyadic algebra. Let A be the polyadic subalgebra generated by B; if h is the identity on B, then (A, h) is a polyadic completion of B having the properties (2.4) and (2.5). In the general case, that is, when $\tilde{V}$ is not necessarily separating, let M be the set-theoretical intersection of all ideals of the form $v^{-1}(0)$ where $v \in \tilde{V}$. Let $\tilde{B} = B/M$ and let h be the natural projection from B onto $\tilde{B}$. Since M is a transformation ideal, h is a transformation homomorphism. Let W be the set of all valuations of $\tilde{B}$ so that $v \in W$ if and only if there is $\tilde{v}$ in $\tilde{V}$ such that $\tilde{v}p = vhp$ for every p in B. If τ is a transformation on I so that $\tau(I) = \mathrm{I}$, and if v is in W, then $vS(\tau)$ is in W. Clearly W is separating on $\tilde{B}$. We are then reduced to the first case which concludes the proof as far as existence is concerned. Uniqueness follows from (2.2) together with the fact that if (A, h) is a polyadic completion satisfying (2.4) and (2.5), then $h^{-1}(0) = M$.

(2.6) THEOREM. *If B has cardinality less than or equal to that of I, then any*

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polyadic completion of B is the polyadic completion of B with respect to some set of valuations.

Proof. Suppose (A, h) is a polyadic completion of B. Let $\bar{V}$ be the set of all existential valuations of A. Let V be the set of all valuations of B of the form vh where $v \in \bar{V}$. Since B has cardinality less than or equal to that of I, A has also the same property. By (2.1), (A, h) satisfies (2.4) and (2.5). By uniqueness, (A, h) is the polyadic completion of B with respect to V.

The following theorem describes the main properties of the polyadic completions of B.

(2.7) THEOREM. *If V_1 and V_2 are sets of valuations of B so that V_2 is a subset of V_1, then the V_2-polyadic completion is a quotient of the V_1-polyadic completion; if (A, h) is the V-polyadic completion, then (A, h) is faithful if and only if $\tilde{V}$ is separating where $\tilde{V}$ is the set of all valuations of the form $vS(\tau)$ with v in V and τ a transformation on I so that $\tau(I) = I$; if B has cardinality less than or equal to that of I, then a polyadic completion of B is simple if and only if it is the polyadic completion with respect to a single valuation.*

Proof. The proof of the first part follows easily from a continuity argument. The proof of the second part consists in a straightforward verification. To prove the third part, suppose (A, h) is the polyadic completion of B with respect to a single valuation. It follows from (2.5) that A cannot contain closed elements distinct from 0 and 1. Therefore, A is simple. Conversely, suppose A is simple. Since A has also a cardinality less than or equal to that of I, by (2.1), we can assert that there is at least one existential valuation $\bar{v}$ of A. Let v be a valuation of B defined by $v = \bar{v}h$. Then (A, h) satisfies (2.4) and (2.5) with respect to v. By uniqueness, it follows that (A, h) is the polyadic completion with respect to v. This completes the proof of the theorem.

3. Equality transformation algebras. If B is a locally finite transformation algebra of infinite degree, then a *predicate* of B is defined as in **(3)**. An *equality* for B is a reflexive and substitutive binary predicate of B. An equality E for B is necessarily symmetric and transitive. Moreover, $p \wedge E(i, j) = S(j/i)p \wedge E(i, j) = S(i, j)p \wedge E(i, j)$ whenever i and j are variables and $p \in B$. The proof of these elementary facts goes exactly as in the polyadic case (treated in **(4)**, § 2)) without any change.

For the remainder of this section, (B, I, S) shall be a fixed locally finite transformation algebra of infinite degree with an equality E.

Suppose (A, h) is a polyadic completion of B and suppose moreover that Σ is a sum of B; (A, h) is said to *preserve* Σ if h preserves Σ. The main purpose of this section is to show that there exists one and only one faithful polyadic completion of B which preserves every sum of B. From this result follows an interesting refinement of (1.1) (see 3.4). We first prove two Lemmas.

(3.1) LEMMA. *If* (A, h) *is a polyadic completion of* B, *then* hE *is an equality for* A.

Proof. What is meant here is that, if $F(i, j) = hE(i, j)$ for all variables i and j, then F is an equality for A. Clearly, we may assume that A is faithful and that h is the identity endomorphism of B onto B. Let A_0 be the set of all p in A so that $p \wedge E(i, j) = S(j/i)p \wedge E(i, j)$ whenever i and j are variables. Since A_0 is a polyadic subalgebra of A and since A_0 contains B, it follows that $A_0 = A$. This completes the proof of the Lemma.

(3.2) LEMMA. *If* τ *is a finite transformation on* I, *then* $S(\tau)$ *is a complete endomorphism of* B.

Proof. If τ is a transposition, then $S(\tau)$ is an automorphism of B and hence $S(\tau)$ preserves the sums of B. Therefore, since every finite transformation is a finite product of transpositions and replacements, it suffices to prove the Lemma for the case, where $\tau = (j/i)$ and $j \neq i$. Let Σ be a sum of B and let $p_0 = \vee \Sigma$. Suppose q is an element in B so that $S(j/i)p \leqslant q$ for all p in Σ. We are to show that $S(j/i)p_0 \leqslant q$. For that purpose, let k be a variable distinct from i and j and so that p and q are independent of k. Since $S(j/i)p$ is independent of i for all p, we have $S(j/i)p \leqslant S(k/i)q$ for all p in Σ. But $p \wedge E(i, j) \leqslant S(j/i)p$ for all p in Σ which implies that $p \wedge E(i, j) \leqslant S(k/i)q$ for all p in Σ. It follows that $p_0 \wedge E(i, j) \leqslant S(k/i)q$. Applying $S(j/i)$ to both sides of the last inequality, we get $S(j/i)p_0 \leqslant S(k/i)q$. It follows that $S(i/k)S(j/i)p_0 \leqslant S(i/k)S(k/i)q = q$. But since $S(i/k)S(j/i)p_0 = S(j/i)p_0$, it follows finally that $S(j/i)p_0 \leqslant q$. This shows that $S(j/i)$ preserves the sum Σ.

We make the observation that the preceding lemma is false for polyadic algebras with no equality and *a fortiori* for transformation algebras with no equality. Counterexamples can be furnished. We are now ready to prove the main result of this section.

(3.3) THEOREM. *There exists a unique* (*up to isomorphism*) *faithful polyadic completion of* B *which preserves every sum of* B.

Proof. Let $\bar{B}$ be the McNeille completion of B (see 0.1), and let $\tilde{A}$ be the polyadic algebra of all finite dimensional functions from I^I into $\bar{B}$. Define a mapping h from B into $\tilde{A}$ by $(hp)(\tau) = S(\tau)p$ for all p in B and all τ in I^I. The mapping h is a transformation isomorphism from B into $\tilde{A}$. Let A be the polyadic subalgebra of $\tilde{A}$ generated by $h(B)$. Clearly, (A, h) is a faithful polyadic completion of B. Using (3.2), and using the fact that an element of $\tilde{A}$ is completely determined by its values on finite transformations, it is easy to see that (A, h) preserves the sums of B. This proves existence. Uniqueness follows from (0.1) and (3.2).

The polyadic completion (A, h) given by the preceding theorem shall be referred to as the *McNeille polyadic completion of* B.

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We conclude this section by giving an application of (3.3). First, we need two definitions. An element p in B is said to be *undecidable* if $p \neq 0$ and if there exists an element q in B so that $q \neq 1$ and $S(\tau)p \leqslant q$ for all transformations τ on I. It follows from **(2**, 10.3**)** that, if B is a polyadic algebra, then p is undecidable if and only if $\exists(I)p \neq 0$ and $\exists(I)p \neq 1$. Suppose A is a locally finite functional transformation algebra of functions from X^I into a Boolean algebra C. The functional transformation algebra A is said to be an *existential functional* (*transformation*) *algebra* if, whenever an existential sum $\Sigma(i, p)$ of A has a supremum q in A, then $q(x)$ is the supremum (in C) of $\{p(y) : x \equiv y \bmod i\}$ for all x in X^I. Clearly, if A is a functional polyadic algebra, then A is necessarily an existential functional algebra. The following theorem is a direct generalization of the representation theorem for locally finite polyadic algebras of infinite degree.

(3.4) THEOREM. *The transformation algebra B is isomorphic to an existential functional transformation algebra; B is isomorphic to an O-valued existential functional transformation algebra if and only if B has no undecidable elements.*

Proof. The first part of the theorem follows immediately from **(2**, 10.9**)** and (3.3). To prove the second part, in view of **(2**, 17.3**)**, it suffices to prove that the McNeille polyadic completion A of B is simple if and only if B has no undecidable element. This follows easily from **(2**, 10.3**)** and (0.1).

4. Terms and operations in transformation algebras. With the results so far obtained, it is now possible to develop the theory of terms and operations in transformation algebras, at least in the case where an equality is present. The purpose of this section is to show how most definitions and results of **(3)** can be generalized to locally finite equality transformation algebras of infinite degree.

Throughout this section (B, I, S) shall be a fixed locally finite transformation algebra of infinite degree with an equality E.

We begin by stating the definition of a J-constant of B where J is a finite subset of I. A *J-constant* of B is a mapping c that associates with each subset K of $I - J$ an endomorphism $S(c/K)$ of B so that:

(T_1) if ϕ is the empty subset of I, then $S(c/\phi)$ is the identity on B;

(T_2) $S(c/H \cup K) = S(c/H)S(c/K)$ whenever H and K are subsets of $I - J$;

(T_3) $J \cup (L - K)$ supports $S(c/K)p$ whenever K is a subset of $I - J$ and L supports p;

(T_4) $S(c/K)p = S(c/K \cap L)p$ whenever K is a subset of $I - J$ and L is a support for p;

(T_5) $S(c/K)S(\tau) = S(\tau)S(c/\tau^{-1}K)$ whenever K is a subset of $I - J$ and τ is a transformation that lives on $I - J$.

A *constant* of B is a ϕ-constant where ϕ is the empty set. If B is a polyadic algebra, then a J-constant of B as defined in **(3)** is a J-constant in the preceding sense; this follows from **(3**, (5.1) and (5.2)**)**. The results of this section will imply that the converse is also true (see (4.2)). Now that we have the definition of a J-constant, the definition of a J-term goes exactly as in the polyadic case. Namely, a *J-term* of B is a mapping t that associates with every subset K of I a Boolean endomorphism $S(t/K)$ of B, so that the restriction of t to subsets of $I - J$ is a J-constant of B, and so that $S(t/K)p = S(t/\sigma K)S(\sigma)p$ whenever p is an element of B with finite support L, K is a finite subset of I and σ is a transformation of type $(K, L \cup J)$.

If c is a J-constant and i a variable, we define $E(i, c)$ by $E(i, c) = S(c/k)E(i, k)$ where $k \neq i$ and k is in $I - J$. Using T_3 and T_5, it is easy to see that $E(i, c)$ is unambiguously defined, that is, independent of the choice of k. We define $E(c, i)$ in a similar manner and clearly $E(i, c) = E(c, i)$. Note that $S(c/i)E(i, c) = 1$ and $S(j/i)E(i, c) = E(j, c)$ whenever i and j are variables not in J. We are now ready to prove the main result concerning J-constants.

(4.1) THEOREM. *Suppose c is a J-constant and suppose i is a variable not in J. If $q = E(i, c)$, then, for all p in B, $\Sigma(i, p \wedge q)$ is an existential sum and*

$$S(c/i)p = \vee \Sigma(i, p \wedge q).$$

Proof. Let p be a fixed element of B. We first show that if j is any variable, then $S(j/i)(p \wedge q) \leqslant S(c/i)p$. Assume first j distinct from i and not in J. The required inequality follows then by applying $S(c/i)$ to both sides of the inequality $S(j/i)p \wedge E(i, j) \leqslant p$. In the general case, choose a variable k distinct from i, not in J, and so that p, q, and $S(c/i)p$ are independent of k. The required inequality follows then by applying $S(j/k)$ to both sides of the inequality $S(k/i)(p \wedge q) \leqslant S(c/i)p$. To complete the proof, it suffices to show that if p_0 is an element of B so that $S(j/i)(p \wedge q) \leqslant p_0$ for all variables j, then $S(c/i)p \leqslant p_0$. To prove that, select a variable j not in J and so that p and p_0 are independent of j. If we apply $S(c/j)$ to both sides of the inequality $S(j/i)p \wedge E(j, c) \leqslant p_0$, we get $S(c/j)S(j/i)p \leqslant S(c/j)p_0 = p_0$. Since $S(c/j)S(j/i)p = S(c/i)p$, we have finally $S(c/i)p \leqslant p_0$. This completes the proof of the theorem.

(4.2) COROLLARY. *If B is a polyadic algebra, then a J-constant as defined above is a J-constant as defined in* **(3)**.

Proof. This follows from the preceding theorem together with (1.3) and from the fact that if c is a J-constant (as defined in **(3)**), then $S(c/i)p = \exists(i)(p \wedge E(i, c))$ for all p in B and all variables i not in J.

Our next task is to establish the exact relationship between J-constants of B and J-constants of the polyadic completions of B. If (A, h) is a polyadic completion of B and c is a J-constant of B, then A will be said to *preserve c* if there exists a (necessarily unique) J-constant $\tilde{c}$ of A so that $hS(c/K)p = S(\tilde{c}/K)hp$ whenever K is a subset of $I - J$ and p belongs to B.

(4.3) THEOREM. *Suppose c is a J-constant of B, i a variable not in J, $q = E(i, c)$, and (A, h) a polyadic completion of B. Then A preserves c if and only if A preserves the existential sum $\Sigma(i, q)$.*

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Proof. First note that, by (4.1), the supremum (in B) of $\Sigma(i, q)$ is 1. Assume first that A preserves c. Let $\tilde{c}$ be the unique J-constant of A so that $hS(c/K)p = S(\tilde{c}/K)hp$ for all p in B and all subsets K of $I - J$. By (3.1), hE is an equality for A; let $hE = \tilde{E}$ and let $\tilde{q} = \tilde{E}(i, c)$. By (4.1), the supremum (in A) of $\Sigma(i, \tilde{q})$ is 1. Since $hS(j/i)q = S(j/i)\tilde{q}$ for all j in I, we conclude that h (and therefore A) preserves $\Sigma(i, q)$. Conversely, suppose A preserves the sum $\Sigma(i, q)$. Let $\tilde{q} = hE(i, c)$. It is easy to verify that $\exists!(i)\tilde{q} = 1$. By **(4**, (9.1)**)**, there exists a unique J-constant $\tilde{c}$ of A so that $S(\tilde{c}/i)\tilde{q} = 1$. Since $S(\tilde{c}/i)hp = \exists(i)(hp \wedge \tilde{q})$ for all p in B, it follows from (1.3) and (4.1) that $S(\tilde{c}/i)hp \leqslant hS(c/i)p$ for all p in B which implies that $S(\tilde{c}/i)hp = hS(c/i)p$ for all p in B. Therefore A preserves c. This completes the proof of the theorem.

If (A, h) is the polyadic completion with respect to a set V of valuations of B, it can be shown that A preserves c if and only if every valuation in $\tilde{V}$ preserves the sum $\Sigma(i, q)$ where $\tilde{V}$ is the set of all valuations of the form $vS(\tau)$ with $v \in V$ and with $\tau(I) = I$.

With these theorems at our disposal the theory of J-constants in B can now be reduced to the theory of J-constants in polyadic algebras. Indeed, let A be the McNeille polyadic completion of B; we may assume that B is a subalgebra of A. Since A preserves the sums of B, it follows by (4.3) that a J-constant of B has a unique extension to a J-constant of A. Therefore, if c is a J-constant of B, by **(3**, (6.7)**)**, there exists a unique J-term t of A so that $S(c/K)p = S(t/K)p$ whenever $p \in B$ and K is a subset of $I - J$. By **(3**, (4.1)**)**, $S(t/K)$ sends B into B for all subsets K of I; it follows that a J-constant of B has a unique extension to a J-term of B. If t is a term of B and τ is a transformation on I, then the transform τt of t by τ is a term of A (here, we identify t with its unique extension to a term of A), and by **(3**, (11.6)**)** τt is also a term of B (or more precisely, the restriction of τt to B is a term of B). Similarly, if s and t are terms of B and K is a subset of I, then the result $(s/K)t$ of replacing the variables of K in t by s is a term of B (see **(3**, (13.2)**)**). An *operation* of B is then defined as it is defined by **(3**, (12.3)**)**. We conclude by making the remark that all the definitions and results in **(3)** concerning terms and operations in polyadic algebras are valid in (locally finite) equality transformation algebras (of infinite degree) as long as no essential use of quantification is made.

Errata. The results of this paper were announced in **(6)** and **(7)**; the results announced in **(5)** combined with those of the present paper lead to some applications given in **(8)**. The purpose of this section is to correct an error that appeared in two different forms in **(6)** and **(7)**. The first correction concerns Theorem VI of **(6)**: add to the statement of that theorem the hypothesis that A has cardinality less than or equal to that of I. The second correction concerns Theorem II of **(7)**: add to the statement of that theorem the hypothesis that B has cardinality less than or equal to that of I.

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