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THE MATRIX EQUATIONS $A = XYZ$ AND $B = ZYX$ AND RELATED ONES

BY

J. L. BRENNER AND M. J. S. LIM

'Dedicated to Olga Taussky-Todd in friendship.'

In [15], O. Taussky-Todd posed the problem of title, namely to find X, Y, Z when A, B are given. Clearly if X, Y, Z exist then A, B are either both invertible or both noninvertible.

In section 1, the problem is reviewed in case A, B are both invertible. The problem is seen to be fundamentally one of group theory rather than matrix theory. Application of results of Shoda, Thompson, Ree to the general group-theoretical results allows specialization to certain matrix groups.

In Section 2, examples and counterexamples are given in case A, B are non-invertible. A general necessary condition for solvability (involving ranks) is obtained. This condition may or may not be sufficient. For dim $A=2$, 3 the problem is settled: there is always a solution in the noninvertible case.

1. **The invertible case.** For the equations $A=XYZ$ and $B=ZYX$, solvability is decided by theorems 1.1 and 1.2. Related equations are treated below. They concern the five permutations of X, Y, Z.

THEOREM 1.01. *Let $A=XYZ$, $B=ZYX$ in any group. Then AB^{-1} is a commutator* [16].

Proof. $AB^{-1}=[XY, ZY]$.

THEOREM 1.02. [1]. *Suppose (in some group) AB^{-1} is a commutator $UVU^{-1}V^{-1}$. Then X, Y, Z exist such that $A=XYZ$, $B=ZYX$ [or $B=XZY$ or $B=YXZ$].*

Proof. $X=UB$, $Y=B^{-1}VU^{-1}$, $Z=V^{-1}B$, [or $X=UVB$, $Y=B^{-1}U^{-1}B$, $Z=B^{-1}V^{-1}B$; or: $X=U$, $Y=V$, $Z=U^{-1}V^{-1}B$].

Theorem 1.01 remains true if we replace ZYX by XZY, YXZ, ZXY, or YZX. Theorem 1.02 remains true if we replace ZYX by XZY or YXZ, but not by ZXY or YZX.

1.03 **Proofs.**

$$(XYZ)(XZY)^{-1} = [XYX^{-1}, XZX^{-1}], \qquad (XYZ)(YXZ)^{-1} = [X, Y],$$
$$(XYZ)(ZXY)^{-1} = [XY, Z], \qquad (XYZ)(YZX)^{-1} = [X, YZ].$$

1.04 COUNTEREXAMPLE. There is a group in which AB^{-1} is a commutator, but neither the set $A=XYZ$, $B=ZXY$ nor the set $A=X_1Y_1Z_1$, $B=Y_1Z_1X_1$ is solvable.

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If X, Y, Z existed, then B would be conjugate to A: $B=ZAZ^{-1}$. The matrix example $A=\text{diag}[2, 2]$, $B=\text{diag}[1, 4]$ shows that AB^{-1} can be a commutator $UVU^{-1}V^{-1}$, with B not conjugate to A:

$$U = \begin{bmatrix} 12, & -10 \\ 9, & 6 \end{bmatrix}, \qquad V = \begin{bmatrix} 3, & -2 \\ 0, & 6 \end{bmatrix}.$$

Corollaries of the above results are the following.

Let $SL(n, K)$ denote the multiplicative group of all invertible $n \times n$ matrices over K with determinant unity, $GF(p^n)$ denote the finite field with p^n elements. $SL(n, GF(m))$ will be abbreviated $SL(n, m)$.

THEOREM 1.1. *In the following groups, the equations $A=XYZ$, $B=ZYX$ ($B=XZY$, $B=YXZ$) are always solvable*

(a) [12] $SL(n, 2)$ ($n > 2$); this is false for $n = 2$.
(b) [12] $SL(n, 3)$ ($n > 2$); this is false for $n = 2$.
(c) [11] $SL(n, m)$ ($n = 2$, $m > 3$; also $n > 2$)
(d) [11] $SL(n, m)$ $m > 3$
(e) [9] $SL(n, \mathbb{C})$
(f) [10] $SL(n, K)$, K algebraically closed.

Let $GL(n, K)$ denote the multiplicative group of all invertible $n \times n$ matrices over K. Again, $GL(n, m)$ will denote $GL(n, GF(m))$.

THEOREM 1.2. *If A, B have the same determinant (but not otherwise), i.e. if A, B belong to the same coset of $GL(n, K)$ over $SL(n, K)$, the equations $A=XYZ$, $B=ZYX$ (XZY, YXZ) are solvable (in GL) in the following cases*

(a) [13] $GL(n, m)$, $n = 2$, $m = 3$,
(b) [9], [14], $GL(n, \mathbb{C})$.

THEOREM 1.3. [8] *In a connected semi-simple algebraic group defined over an algebraically closed field, every element is a commutator. Therefore in such a group $A=XYZ$, $B=ZYX$ are simultaneously solvable.*

THEOREM 1.4. *Let A, B be quaternions of the same norm. Then quaternions X, Y, Z exist such that $A=XYZ$, $B=ZYX$* [3].

THEOREM 1.5. *Given $A=XYZW$, $B=WZYX$. Then*

$$AB^{-1} = [XYZ, WYZ][WYW^{-1}, WZW^{-1}]$$

THEOREM 1.6. *Given $A=XYZWM$, $B=MWZYX$. Then*

$$AB^{-1} = [XYZW, MYZW][MYZM^{-1}, MWZM^{-1}]$$

See [6a].

2. **The singular case.** If XYZ is not invertible, neither is ZYX. In this section we first give a condition on the ranks of A, B that is necessary for the existence of a

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simultaneous solution of $A=XYZ$, $B=ZYX$. Then an example of two 4×4 singular matrices A, B is given for which no solution exists. Next it is shown that the general problem reduces to the case $A=\text{diag}(I, 0)$.

For 2×2 and 3×3 matrices the problem is resolved; there is always a solution if A, B are both singular. Also in the important case that A, B are diagonal, singular, and of the same rank, there is always a solution. The products of the non-zero diagonal elements can be different.

In every case we have examined, there is a solution provided the necessary condition holds.

2.01 Theorem. *If A, B are $n\times n$ matrices with (real or) complex entries, and if $n\times n$ X, Y, Z exist such that $A=XYZ$, $B=ZYX$, then* $|\text{rank } A-\text{rank } B|\le 2n/3$.

Proof. The general assertion will follow from the application of theorem 2.05 to the particular assertion of theorem 2.01 in the case $A=0$. In this case, it must be shown that rank $B\le 2n/3$.

Assume *per contra* that rank $B\,[2n/3]+k$, k a positive integer. Thus rank $X\ge [2n/3]+k$. From the relation $A=0=XYZ$, it now follows that $r_1=\text{rank } YZ\le n-[2n/3]-k$. Write $r_1=n-[2n/3]-k-l$, $l\ge 0$.

Let $r_2=\text{rank } ZY$. From [3], it follows that $|r_2-r_1|\le[(n-r_1)/2]$. Thus

$$r_2 \le r_1+[(n-r_1)/2] = n-[2n/3]-k-l+[[n/3]+(k+l)/2] \le [2n/3].$$

This relation contradicts the assumption rank $B>[2n/3]$, and the theorem follows.

2.02 Remark. The bound is sharp, as the example $X=\text{diag}[0, 1, 1]$,

$$Y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Z = \text{diag}[1, 1, 0]$$

shows. By forming direct sums, an example with difference $[2n/3]$ can be constructed for any value of n.

2.03 Remark. An example can be constructed such that

$$\prod_1^k X_i = 0, \qquad \coprod_1^k X_i \text{ has rank } [(k-1)n/k].$$

2.04 Corollary. *If A is the 4×4 0-matrix and B has rank 3, the equations $A=XYZ$, $B=ZYX$ have no simultaneous solution.*

2.05 Theorem. *Let P, Q be invertible $n\times n$ matrices. Set $A_1=PAQ$, $B_1=PBQ$. The equations $A=XYZ$, $B=ZYX$ have a simultaneous solution if and only if the equations $A_1=X_1Y_1Z_1$, $B_1=Z_1Y_1X_1$ have a simultaneous solution.*

Proof. $X_1=PXQ$, $Y_1=Q^{-1}YP^{-1}$, $Z_1=PZQ$.

2.06 Theorem. *The equations $A=XYZ$, $B=ZYX$ have a simultaneous solution if and only if the equations $A'=X_2Y_2Z_2$, $B'=Z_2Y_2X_2$ have a simultaneous solution.*

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Proof. $X_2=Z'$, $Y_2=Y'$, $Z_2=X'$.

2.07 THEOREM. *Let A, B be diagonal, singular, and of equal rank. Then $A=XYZ$, $B=ZYX$ have a simultaneous solution.*

Proof. By theorem 2.05, it is sufficient to consider the case $A=\text{diag}[I_k, O_{n-k}]$,

$$B = \text{diag}[O_r, b_{r+1}, \ldots, b_k, b_{k+1}, \ldots, b_{k+r}, O_{n-k-r}],$$

where O_j is the $j\times j$ zero matrix, and $b_i\neq 0$, $i=r+1, \ldots, r+k$. If Z is the circulant matrix corresponding to the permutation $(1, 2, \ldots, n)$, i.e. with first row $(0, 1, 0, \ldots, 0)$, then $A=XYZ$ requires $XY=AZ^{-1}$; $B=ZYX$ requires $Z^{-1}B=YX$. Direct computation of AZ^{-1}, $Z^{-1}B$ shows that, although singular, they are equivalent. Thus X, Y can be found. In fact, a little more is proved: the ranks of A, B can even differ by 1, provided both are singular [4].

2.08 THEOREM. *Let $A=PBYP^{-1}Y^{-1}[A=PYP^{-1}BY^{-1}]$. Then $A=XYZ$, $B=ZYX$ have a simultaneous solution.*

Proof. $X=PY^{-1}$, $Z=BYP^{-1}Y^{-1}[X=P, Z=BY^{-1}]$.

Note the contrast with theorem 1.01. In this theorem, A, B need not be invertible.

Note. The referee has called our attention to the paper [2] concerning solutions of the equation $T=PAQ-QAP$. This equation is amenable to attack by the methods of this article whenever T is of the form $S(I\oplus 0)U$. Write $T=S(I\oplus 0)U$, $P=SP_1U$, $Q=SQ_1U$, $A=U^{-1}A_1S^{-1}$.

LEMMA. *The equation $I\oplus 0=P_1A_1Q_1-Q_1A_1P_1$ is solvable in finite-dimensional nonsingular matrices of dimension >1. (There is clearly no solution for dimension 1, except when $I\oplus 0\equiv 0$.)*

Proof. The first step is to solve $I=PAQ-QAP$. Since $0=PAQ-QAP$ is trivially solvable, the lemma will follow. But $I=PAQ-QAP$ is solvable provided $I=V-W$ for some V, W such that VW^{-1} is a commutator. This is easy: set $V=(1+a)I$, $W=aI$, where $(1+a)^n=a^n$; $VW^{-1}=\varepsilon I$.

It can even be shown that $I=V-W$ can be solved subject to the further restrictions (i) det $V=$det $W=\Delta$, (ii) V, W real if Δ is real. This extends the result of [2] from the unitary space to the real space.

It should now be clear (direct sums) that there are many commutators $C=XYX^{-1}Y^{-1}$ in $\mathscr{H}$ such that $C-I$ is invertible, $C-I=Z^{-1}$. Thus $V=I+Z$, $W=Z$ satisfy $VW^{-1}=C$, so that finally every operator T of the form $S(I\oplus 0)U$, S, U invertible in (real or complex) Hilbert space can be written in the form $PAQ-QAP$, with P, A, Q invertible.

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Department of Mathematics,
University of Victoria,
Victoria, B.C., Canada

J. L. Brenner
10 Phillips Rd.
Palo Alto, California 94303

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