THE MAXIMUM ORDER OF THE GROUP OF A TOURNAMENT

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1. Introduction. To each tournament T_n with n nodes there corresponds the automorphism group $G(T_n)$ consisting of all dominance preserving permutations of the set of nodes. In a recent paper [3], Myron Goldberg and J. W. Moon consider the maximum order $g(n)$ which the group of a tournament with n nodes may have. Among other results they prove that

(1) $\lim g(n)^{1/n}$ exists as $n \to \infty$ and is ≤ 2.5 ;

(2) $g(n) \geq \sqrt{3}^{n-1}$ for $n = 3^k$ $(k = 0, 1, \ldots)$.

Moreover, they conjecture that

(3) $\lim g(n)^{1/n} = \sqrt{3}$.

The object of the present paper is to prove

THEOREM 1. For each positive integer n, $g(n) \leq \sqrt{3}^{n-1}$. Taken together with (2) this implies the truth of the conjecture (3).

In contrast to the graph theoretic approach of [3], our approach is via group theory. It takes as its starting point the Addendum to [3] where it is shown that $g(n)$ can be interpreted as being the largest order of a permutation group of odd order and degree n . By the celebrated theorem of Feit and Thompson [2], any group of odd order is solvable. Thus Theorem 1 is equivalent to

THEOREM 1'. Every solvable permutation group G of odd order and degree n has order $|G| \leq \sqrt{3}^{n-1}$.

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We shall prove the result in this latter form. It would be interesting to know if the result is as deep as the use of the Feit-Thompson theorem suggests.

2. Proof of Theorem 1'. The main step in the proof of Theorem 1' is already contained in a previous paper of the author (see [1]). It is shown there that we can use induction on n to reduce the problem to the case where G is a primitive permutation group. In the latter case it is shown that $G = AG_1$ with $A \cap G_1 = 1$, where A is a normal elementary abelian p-subgroup of order $p^k = n$ and G_1 is the stability subgroup of G fixing one symbol. Moreover, A equals its centralizer in G, and so G_1 is isomorphic to a subgroup of the group Aut A of all automorphisms of A. Finally, since the order of Aut A for an elementary abelian p-group is known, we obtain

(4) $$|G| = |A|\,|G_1| \text{ divides } p^k(p^k-1)(p^k-p)\ldots(p^k-p^{k-1}).$$

(See [1] Section 2 for details.)

The remaining step is to prove that (4) together with the hypothesis that $|G|$ is odd implies $|G| \leq \sqrt{3}^{\,n-1}$ with $n = p^k$. Direct calculation shows

$$|G| \leq p^k(p^k-1)\ldots(p^k-p^{k-1}) < p^{k+k^2} \leq \sqrt{3}^{\,p^k-1}$$

unless $p^k = 3, 3^2, 5$ or 7. However, since $|G|$ is odd, we have in the exceptional cases

$$|G| \leq 3 = \sqrt{3}^{\,3-1} \quad \text{if } n = 3,$$

$$|G| \leq 3^3 < \sqrt{3}^{\,9-1} \quad \text{if } n = 3^2,$$

$$|G| \leq 5 < \sqrt{3}^{\,5-1} \quad \text{if } n = 5,$$

and

$$|G| \leq 21 < \sqrt{3}^{\,7-1} \quad \text{if } n = 7.$$

Thus the inequality holds in all cases, and the theorem is proved.

3. Remarks. The inequality (2) can be proved in a direct group-theoretic manner by constructing imprimitive permutation

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groups of suitable order (cf. the end of Section 2 in [1]). Theorem 1' shows that we actually have equality in (2) and a simple check of the inequalities in the proof above shows that we cannot have equality except when $n = 3^k$. Again a straightforward case-by-case analysis of the proof above gives an easy way of calculating the values of $g(n)$ for moderate values of n.

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