

## ALGEBRAIC INTEGERS AS SPECIAL VALUES OF MODULAR UNITS

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*Abstract* Let  $\varphi(\tau) = \eta(\frac{1}{2}(\tau + 1))^2 / \sqrt{2\pi} \exp\{\frac{1}{4}\pi i\} \eta(\tau + 1)$ , where  $\eta(\tau)$  is the Dedekind eta function. We show that if  $\tau_0$  is an imaginary quadratic argument and  $m$  is an odd integer, then  $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$  is an algebraic integer dividing  $\sqrt{m}$ . This is a generalization of a result of Berndt, Chan and Zhang. On the other hand, when  $K$  is an imaginary quadratic field and  $\theta_K$  is an element of  $K$  with  $\text{Im}(\theta_K) > 0$  which generates the ring of integers of  $K$  over  $\mathbb{Z}$ , we find a sufficient condition on  $m$  which ensures that  $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$  is a unit.

*Keywords:* Dedekind eta function; modular functions; automorphic functions

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### 1. Introduction

The *Dedekind eta function*  $\eta(\tau)$  is defined to be the infinite product

$$\eta(\tau) = \sqrt{2\pi} e^{\pi i/4} q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \tau \in \mathfrak{H}, \quad (1.1)$$

where  $q = e^{2\pi i\tau}$  with  $i = \sqrt{-1}$  and  $\mathfrak{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ . Define a function

$$\varphi(\tau) = \frac{1}{\sqrt{2\pi} e^{\pi i/4}} \frac{\eta((\tau + 1)/2)^2}{\eta(\tau + 1)} = \prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 (1 - q^n), \quad \tau \in \mathfrak{H}, \quad (1.2)$$

which is identical to Jacobi's  $\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2}$ , by Jacobi's triple product identity [1, p. 36]. Motivated by Ramanujan's evaluation of  $\varphi(mi)/\varphi(i)$  for some positive integers  $m$  [10] which are algebraic numbers, Berndt *et al.* proved the following theorem.

**Theorem 1.1 (Berndt *et al.* [2, Theorem 4.4]).** *Let  $m$  and  $n$  be positive integers. If  $m$  is odd, then  $\sqrt{2m}\varphi(mni)/\varphi(ni)$  is an algebraic integer dividing  $2\sqrt{m}$ , while if  $m$  is even, then  $2\sqrt{m}\varphi(mni)/\varphi(ni)$  is an algebraic integer dividing  $4\sqrt{m}$ .*

In this paper we shall first revisit the theorem and improve it when  $m$  is odd, as follows.

**Theorem 1.2.** *Let  $m$  be a positive integer and let  $\tau_0 \in \mathfrak{H}$  be imaginary quadratic. Then  $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$  is an algebraic integer dividing  $4\sqrt{m}$ . In particular, if  $m$  is odd, then  $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$  is an algebraic integer dividing  $\sqrt{m}$ .*

For  $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ , the Siegel function  $g_{(r_1, r_2)}(\tau)$  is defined by

$$g_{(r_1, r_2)}(\tau) = -q^{\mathbf{B}_2(r_1)/2} e^{\pi i r_2(r_1-1)} (1 - q_z) \prod_{n=1}^{\infty} (1 - q^n q_z)(1 - q^n q_z^{-1}), \quad \tau \in \mathfrak{H}, \quad (1.3)$$

where  $\mathbf{B}_2(x) = x^2 - x + 1/6$  is the second Bernoulli polynomial and  $q_z = e^{2\pi i z}$  with  $z = r_1\tau + r_2$ . We shall express the function  $\varphi(m\tau)/\varphi(\tau)$  as a product of certain eta-quotient and Siegel functions (Lemma 2.6 (i)). Also, we shall prove Theorem 1.2 in §3 by using integrality of Siegel functions over  $\mathbb{Z}[j(\tau)]$  [6, §3], where

$$j(\tau) = \left( \frac{\eta(\tau)^{24} + 2^8 \eta(2\tau)^{24}}{\eta(\tau)^{16} \eta(2\tau)^8} \right)^3 = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \dots$$

is the well-known modular  $j$ -function [3, Theorem 12.17].

On the other hand, let  $K$  be an imaginary quadratic field with discriminant  $d_K$ , and define

$$\theta_K = \begin{cases} \frac{\sqrt{d_K}}{2} & \text{for } d_K \equiv 0 \pmod{4}, \\ \frac{-1 + \sqrt{d_K}}{2} & \text{for } d_K \equiv 1 \pmod{4}, \end{cases} \quad (1.4)$$

which generates the ring of integers of  $K$  over  $\mathbb{Z}$ . Ramachandra showed in [9, §6] that if  $N$  ( $N \geq 2$ ) is an integer with more than one prime ideal factor in  $K$ , then  $g_{(0, 1/N)}(\theta_K)^{12N}$  is a unit. This fact, together with Shimura’s Reciprocity Law (Proposition 4.6), enables us to prove the following theorem in §4.

**Theorem 1.3.** *If  $m$  ( $m \geq 3$ ) is an odd integer whose prime factors split in  $K$ , then  $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$  is a unit.*

## 2. Arithmetic properties of Siegel functions

In this section we shall examine some arithmetic properties of Siegel functions. For the classical theory of modular functions, we refer the reader to [8, 11].

For each positive integer  $N$ , let  $\zeta_N = e^{2\pi i/N}$  and let  $\mathcal{F}_N$  be the field of meromorphic modular functions of level  $N$  whose Fourier coefficients belong to the  $N$ th cyclotomic field  $\mathbb{Q}(\zeta_N)$ .

**Proposition 2.1.** *For each positive integer  $N$ ,  $\mathcal{F}_N$  is a Galois extension of  $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$  whose Galois group is isomorphic to*

$$\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} = G_N \cdot \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\},$$

where

$$G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$

Here, the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in G_N$$

acts on  $\sum_{n=-\infty}^{\infty} c_n q^{n/N} \in \mathcal{F}_N$  by

$$\sum_{n=-\infty}^{\infty} c_n q^{n/N} \mapsto \sum_{n=-\infty}^{\infty} c_n^{\sigma_d} q^{n/N},$$

where  $\sigma_d$  is the automorphism of  $\mathbb{Q}(\zeta_N)$  induced by  $\zeta_N \mapsto \zeta_N^d$ . Also, for an element  $\gamma \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$  let  $\gamma' \in \text{SL}_2(\mathbb{Z})$  be a preimage of  $\gamma$  via the natural surjection  $\text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ . Then  $\gamma$  acts on  $h \in \mathcal{F}_N$  by composition

$$h \mapsto h \circ \gamma'$$

as a fractional linear transformation.

**Proof.** See [8, Chapter 6, Theorem 3]. □

**Proposition 2.2.** Let  $(r_1, r_2) \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$  for some integer  $N \geq 2$ .

- (i)  $g_{(r_1, r_2)}(\tau)$  is integral over  $\mathbb{Z}[j(\tau)]$ . Namely,  $g_{(r_1, r_2)}(\tau)$  is a zero of a monic polynomial whose coefficients are in  $\mathbb{Z}[j(\tau)]$ .
- (ii) Suppose that  $(r_1, r_2)$  has the primitive denominator  $N$  (that is,  $N$  is the smallest positive integer such that  $(Nr_1, Nr_2) \in \mathbb{Z}^2$ ). If  $N$  is composite (that is,  $N$  has at least two prime factors), then  $g_{(r_1, r_2)}(\tau)^{-1}$  is also integral over  $\mathbb{Z}[j(\tau)]$ .
- (iii)  $g_{(r_1, r_2)}(\tau)$  is holomorphic and has no zeros and poles on  $\mathfrak{H}$ . Furthermore,  $g_{(r_1, r_2)}(\tau)$  (respectively,  $g_{(r_1, r_2)}(\tau)^{12N/\text{gcd}(6, N)}$ ) belongs to  $\mathcal{F}_{12N^2}$  (respectively,  $\mathcal{F}_N$ ).

**Proof.**

- (i) See [6, § 3].
- (ii) See [7, Chapter 2, Theorems 2.2 (i)].
- (iii) See [7, Chapter 2, Theorem 1.2, and Chapter 3, Theorem 5.2]. □

**Remark 2.3.** Let  $g(\tau)$  be an element of  $\mathcal{F}_N$  for some positive integer  $N$ . If both  $g(\tau)$  and  $g(\tau)^{-1}$  are integral over  $\mathbb{Q}[j(\tau)]$  (respectively,  $\mathbb{Z}[j(\tau)]$ ), then  $g(\tau)$  is called a *modular unit* (respectively, *modular unit over  $\mathbb{Z}$* ) of level  $N$ . As is well known,  $g(\tau)$  is a modular unit if and only if it has no zeros or poles on  $\mathfrak{H}$  (see [7, Chapter 2, § 2] or [6, § 2]). Hence,  $g_{(r_1, r_2)}(\tau)$  is a modular unit for any  $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ , by (iii). Moreover, if  $(r_1, r_2)$  has a composite primitive denominator, then  $g_{(r_1, r_2)}(\tau)$  is a modular unit over  $\mathbb{Z}$ , by (ii).

We recall the necessary transformation formulae of Siegel functions.

**Proposition 2.4.** *Let  $r = (r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ .*

(i) *We have*

$$g_{-r}(\tau) = g_{(-r_1, -r_2)}(\tau) = -g_r(\tau).$$

(ii) *For*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

*we get*

$$\begin{aligned} g_r(\tau) \circ S &= \zeta_{12}^9 g_{rS}(\tau) = \zeta_{12}^9 g_{(r_2, -r_1)}(\tau), \\ g_r(\tau) \circ T &= \zeta_{12} g_{rT}(\tau) = \zeta_{12} g_{(r_1, r_1+r_2)}(\tau). \end{aligned}$$

*Hence, we obtain that, for any  $\gamma \in \text{SL}_2(\mathbb{Z})$ ,*

$$g_r(\tau) \circ \gamma = \varepsilon g_{r\gamma}(\tau)$$

*with  $\varepsilon$  a 12th root of unity (depending on  $\gamma$ ).*

(iii) *For  $s = (s_1, s_2) \in \mathbb{Z}^2$  we have*

$$g_{r+s}(\tau) = g_{(r_1+s_1, r_2+s_2)}(\tau) = (-1)^{s_1 s_2 + s_1 + s_2} e^{-\pi i (s_1 r_2 - s_2 r_1)} g_r(\tau).$$

(iv) *Let  $r \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$  for some integer  $N \geq 2$ . Each element*

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{in} \quad \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$$

*acts on  $g_r(\tau)^{12N/\text{gcd}(6,N)}$  by*

$$(g_r(\tau)^{12N/\text{gcd}(6,N)})^\alpha = g_{r\alpha}(\tau)^{12N/\text{gcd}(6,N)} = g_{(r_1 a + r_2 c, r_1 b + r_2 d)}(\tau)^{12N/\text{gcd}(6,N)}.$$

**Proof.** (i)–(iii) See [6, Proposition 2.4].

(iv) See [7, Chapter 2, Proposition 1.3]. □

**Remark 2.5.** The expression  $r\alpha$  in (iv) is well defined by (i) and (iii).

**Lemma 2.6.**

(i) *We can express  $\varphi(\tau)$  as*

$$\varphi(\tau) = -\frac{1}{\sqrt{2\pi}} \eta(\tau) g_{(1/2, 1/2)}(\tau).$$

(ii) *We get*

$$g_{(0, 1/2)}(\tau) g_{(1/2, 0)}(\tau) g_{(1/2, 1/2)}(\tau) = 2e^{\pi i/4}.$$

(iii) If  $m$  ( $m \geq 3$ ) is an odd integer, then we have the relation

$$\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} = (-1)^{(m-1)/2} \prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau).$$

**Proof.** (i) By the definition (1.3) we have

$$\begin{aligned} g_{(1/2,1/2)}(\tau) &= -q^{\mathbf{B}_2(1/2)/2} e^{-\pi i/4} (1 + q^{1/2}) \prod_{n=1}^{\infty} (1 + q^{n+1/2})(1 + q^{n-1/2}) \\ &= -e^{-\pi i/4} q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{n-1/2})^2. \end{aligned}$$

One can then obtain the assertion by the definition (1.1) of  $\eta(\tau)$  and the infinite product expansion (1.2) of  $\varphi(\tau)$ .

(ii) It follows from the definition (1.3) that

$$\begin{aligned} g_{(0,1/2)}(\tau)g_{(1/2,0)}(\tau)g_{(1/2,1/2)}(\tau) &= -2e^{-3\pi i/4} \prod_{n=1}^{\infty} (1 + q^n)^2(1 - q^{n-1/2})^2(1 + q^{n-1/2})^2 \\ &= 2e^{\pi i/4} \prod_{n=1}^{\infty} (1 + q^n)^2(1 - q^{2n-1})^2 \\ &= 2e^{\pi i/4} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^n)^2} \cdot \frac{(1 - q^n)^2}{(1 - q^{2n})^2} \\ &= 2e^{\pi i/4}. \end{aligned}$$

(iii) By the definition (1.3) we obtain

$$\begin{aligned} \frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} &= \frac{-q^{m\mathbf{B}_2(1/2)/2} e^{-\pi i/4} (1 + q^{m/2}) \prod_{n=1}^{\infty} (1 + q^{mn+m/2})(1 + q^{mn-m/2})}{-q^{\mathbf{B}_2(1/2)/2} e^{-\pi i/4} (1 + q^{1/2}) \prod_{n=1}^{\infty} (1 + q^{n+1/2})(1 + q^{n-1/2})} \\ &= q^{(1-m)/24} \prod_{n=1}^{\infty} \left( \frac{1 + q^{m(n-1/2)}}{1 + q^{n-1/2}} \right)^2 \end{aligned}$$

and

$$\begin{aligned} &\prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau) \\ &= \prod_{k=1}^{m-1} \left( -q^{\mathbf{B}_2(1/2)/2} \exp \left\{ \pi i \left( \frac{1}{2} + \frac{k}{m} \right) \left( -\frac{1}{2} \right) \right\} \right. \\ &\quad \left. \times (1 + q^{1/2} \zeta_m^k) \prod_{n=1}^{\infty} (1 + q^{n+1/2} \zeta_m^k)(1 + q^{n-1/2} \zeta_m^{-k}) \right) \end{aligned}$$

$$\begin{aligned} &= (-1)^{m-1} e^{\pi i(1-m)/2} q^{(1-m)/24} \prod_{k=1}^{m-1} \prod_{n=1}^{\infty} (1 + q^{n-1/2} \zeta_m^k)(1 + q^{n-1/2} \zeta_m^{-k}) \\ &= (-1)^{(1-m)/2} q^{(1-m)/24} \prod_{n=1}^{\infty} \prod_{k=1}^{m-1} (1 + q^{n-1/2} \zeta_m^k)^2 \quad \text{because } m \text{ is odd} \\ &= (-1)^{(1-m)/2} q^{(1-m)/24} \prod_{n=1}^{\infty} \left( \frac{1 + q^{m(n-1/2)}}{1 + q^{n-1/2}} \right)^2 \end{aligned}$$

by the identity

$$\frac{1 + X^m}{1 + X} = \frac{1 - (-X)^m}{1 - (-X)} = \prod_{k=1}^{m-1} (1 - (-X) \zeta_m^k).$$

This proves (iii). □

**3. Proof of Theorem 1.2**

Let

$$\Delta(\tau) = \eta(\tau)^{24} = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad \tau \in \mathfrak{H},$$

be the modular discriminant function.

**Proposition 3.1.** *Let  $\tau_0 \in \mathfrak{H}$  be imaginary quadratic.*

- (i)  $j(\tau_0)$  is an algebraic integer.
- (ii) Let  $a, b$  and  $d$  be integers with  $ad > 0$  and  $\gcd(a, b, d) = 1$ . Then

$$\frac{a^{12} \Delta((a\tau_0 + b)/d)}{\Delta(\tau_0)}$$

is an algebraic integer dividing  $(ad)^{12}$ .

**Proof.**

- (i) See [8, Chapter 5, Theorem 4].
- (ii) See [8, Chapter 12, Theorem 4] or [4]. □

**Remark 3.2.** Case (ii) is the most important special case of the prime factorizations of  $\Delta(\alpha\tau_0)/\Delta(\tau_0)$  ( $\alpha \in M_2^+(\mathbb{Z})$ ); this was proved by Hasse for all factorizations.

**Proposition 3.3.** *Let  $m$  be a positive integer and  $\tau_0 \in \mathfrak{H}$  be imaginary quadratic.*

- (i)  $\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$  is an algebraic integer dividing  $\sqrt{m}$ .
- (ii)  $2g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$  is an algebraic integer dividing 4. In particular, if  $m$  is odd, then  $g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$  is a unit.

**Proof.** (i) Applying Proposition 3.1 (ii) with  $(a, b, d) = (m, 0, 1)$ , we see that

$$m^{12} \frac{\Delta(m\tau_0)}{\Delta(\tau_0)} = \left( \sqrt{m} \frac{\eta(m\tau_0)}{\eta(\tau_0)} \right)^{24}$$

is an algebraic integer dividing  $m^{12}$ . We then get the assertion by taking the 24th root.

(ii) We obtain by Lemma 2.6 (ii) that

$$\begin{aligned} 2 \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)} &= e^{-\pi i/4} g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0) g_{(1/2,1/2)}(\tau_0) \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)} \\ &= e^{-\pi i/4} (g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)) g_{(1/2,1/2)}(m\tau_0). \end{aligned}$$

By Propositions 2.2 (i) and 3.1 (i) we know that the values  $g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)$ ,  $g_{(1/2,1/2)}(\tau_0)$ ,  $g_{(0,1/2)}(m\tau_0) g_{(1/2,0)}(m\tau_0)$  and  $g_{(1/2,1/2)}(m\tau_0)$  are algebraic integers. Moreover, since

$$\begin{aligned} (g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)) g_{(1/2,1/2)}(\tau_0) &= (g_{(0,1/2)}(m\tau_0) g_{(1/2,0)}(m\tau_0)) g_{(1/2,1/2)}(m\tau_0) \\ &= 2e^{\pi i/4} \end{aligned}$$

by Lemma 2.6 (ii), both  $g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)$  and  $g_{(1/2,1/2)}(m\tau_0)$  are algebraic integers dividing 2. Hence, the value  $2g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$  is an algebraic integer dividing  $2 \cdot 2 = 4$ .

Next, suppose that  $m$  ( $m \geq 3$ ) is odd. Recall the relation

$$\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} = (-1)^{(m-1)/2} \prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau)$$

given in Lemma 2.6 (iii). Since each vector

$$\left( \frac{1}{2}, \frac{1}{2} + \frac{k}{m} \right)$$

has a composite primitive denominator,  $g_{(1/2,1/2+k/m)}(\tau)$  is a modular unit over  $\mathbb{Z}$  by Proposition 2.2 (ii); hence, so is  $g_{(1/2,1/2)}(m\tau)/g_{(1/2,1/2)}(\tau)$ . Therefore, the value  $g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$  is a unit by Proposition 3.1 (i).  $\square$

Now we are ready to prove Theorem 1.2. Let  $m$  be a positive integer and let  $\tau_0 \in \mathfrak{H}$  be imaginary quadratic. By Lemma 2.6 (i) we have

$$2\sqrt{m} \frac{\varphi(m\tau_0)}{\varphi(\tau_0)} = \sqrt{m} \frac{\eta(m\tau_0)}{\eta(\tau_0)} \cdot 2 \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)}.$$

Thus, it follows from Proposition 3.3 (i) and (ii) that  $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$  is an algebraic integer dividing  $4\sqrt{m}$ . Likewise, if  $m$  is odd, then

$$\sqrt{m} \frac{\varphi(m\tau_0)}{\varphi(\tau_0)} = \sqrt{m} \frac{\eta(m\tau_0)}{\eta(\tau_0)} \cdot \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)} \tag{3.1}$$

is an algebraic integer dividing  $\sqrt{m}$ . This completes the proof of Theorem 1.2.

On the other hand, when  $\tau_0 = ni$  we are able to improve Theorem 1.1 as a corollary.

**Corollary 3.4.** *Let  $m$  and  $n$  be positive integers. If  $m$  is odd, then  $\sqrt{m}\varphi(mni)/\varphi(ni)$  is an algebraic integer dividing  $\sqrt{m}$ , while if  $m$  is even, then  $2\sqrt{m}\varphi(mni)/\varphi(ni)$  is an algebraic integer dividing  $4\sqrt{m}$ .*

**Remark 3.5.** Berndt *et al.* [2] used only the argument of Proposition 3.1 (ii) in order to achieve Theorem 1.1.

**4. Proof of Theorem 1.3**

**Lemma 4.1.** *Let  $m$  ( $m \geq 2$ ) be an integer. Then we have the following identities.*

(i)

$$\prod_{\substack{a,b \in \mathbb{Z}, \\ 0 \leq a,b < m, (a,b) \neq (0,0)}} g_{(a/m,b/m)}(\tau)^{12m} = m^{12m}.$$

(ii)

$$\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau) = i^{m-1} \left( \sqrt{m} \frac{\eta(m\tau)}{\eta(\tau)} \right)^2.$$

**Proof.** (i) See [7, Example, p. 45].

(ii) We deduce from the definition (1.3) that

$$\begin{aligned} & \prod_{k=1}^{m-1} g_{(0,k/m)}(\tau) \\ &= \prod_{k=1}^{m-1} (-q^{\mathbf{B}_2(0)/2} \zeta_{2m}^{-k} (1 - \zeta_m^k) \prod_{n=1}^{\infty} (1 - q^n \zeta_m^k) (1 - q^n \zeta_m^{-k})) \\ &= i^{m-1} m q^{(m-1)/12} \prod_{n=1}^{\infty} \left( \frac{1 - q^{mn}}{1 - q^n} \right)^2 \\ & \quad \text{by the identity } \frac{1 - X^m}{1 - X} = 1 + X + \dots + X^{m-1} = \prod_{k=1}^{m-1} (1 - X \zeta_m^k) \\ &= i^{m-1} \left( \sqrt{m} \frac{\eta(m\tau)}{\eta(\tau)} \right)^2 \quad \text{by the definition (1.1).} \end{aligned}$$

□

**Remark 4.2.** Let  $\tau_0 \in \mathfrak{H}$  be imaginary quadratic. By Propositions 2.2 (i), 3.1 (i) and Lemma 4.1 (i) we see that  $\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau_0)$  is an algebraic integer dividing  $m$ . It then follows from Lemma 4.1 (ii) that  $\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$  is an algebraic integer dividing  $\sqrt{m}$ . This gives another proof of Proposition 3.3 (i) without using the usual argument of Hasse (namely, Proposition 3.1 (ii)).



From now on, we let  $K$  be an imaginary quadratic field and  $\theta_K$  be as in (1.4). We denote by  $H_K$  and  $K_{(N)}$  the Hilbert class field and the ray class field modulo  $N$  ( $N \geq 1$ ) of  $K$ , respectively.

**Proposition 4.3 (main theorem of complex multiplication).**

$$K_{(N)} = K\mathcal{F}_N(\theta_K) = K(h(\theta_K) : h \in \mathcal{F}_N \text{ is defined and finite at } \theta_K).$$

**Proof.** See [8, Chapter 10, Corollary to Theorem 2] or [11, Chapter 6]. □

**Proposition 4.4.**

- (i) If  $N$  ( $N \geq 2$ ) is an integer with more than one prime ideal factor in  $K$ , then  $g_{(0,1/N)}(\theta_K)^{12N}$  is a unit in  $K_{(N)}$ .
- (ii) If  $m$  ( $m \geq 3$ ) is an odd integer, then  $(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$  is an algebraic integer in  $K_{(48m^2)}$ .

**Proof.** (i) See [9, §6].

(ii) We see that

$$\begin{aligned} \left(\frac{\sqrt{m}\varphi(m\tau)}{\varphi(\tau)}\right)^2 &= \left(\frac{\sqrt{m}\eta(m\tau)}{\eta(\tau)}\right)^2 \left(\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)}\right)^2 \quad \text{by Lemma 2.6 (i)} \\ &= (-1)^{(1-m)/2} \prod_{k=1}^{m-1} g_{(0,k/m)}(\tau)g_{(1/2,1/2+k/m)}(\tau)^2 \\ &\quad \text{by Lemmas 4.1 (ii) and 2.6 (iii).} \end{aligned} \tag{4.1}$$

And  $(\sqrt{m}\varphi(m\tau)/\varphi(\tau))^2$  belongs to  $\mathcal{F}_{48m^2}$  by Proposition 2.2 (iii). Therefore,

$$(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$$

lies in  $K_{(48m^2)}$  by Proposition 4.3, which is an algebraic integer by Theorem 1.2. □

**Remark 4.5.** In [5] Jung *et al.* showed that if  $K$  is an imaginary quadratic field other than  $\mathbb{Q}(\sqrt{-1})$  and  $\mathbb{Q}(\sqrt{-3})$ , then the singular value  $g_{(0,1/N)}(\theta_K)^{12N}$  in Proposition 4.4 (i) is in fact a primitive generator of  $K_{(N)}$  over  $K$ , which is called a *Siegel–Ramachandra invariant* (see [7, Chapter 11, §1] or [9]).

On the other hand, we have the following explicit description of Shimura’s reciprocity law, due to Stevenhagen [12], which connects the class field theory with the theory of modular functions.

**Proposition 4.6 (Shimura’s Reciprocity Law).** *Let  $\min(\theta_K, \mathbb{Q}) = X^2 + BX + C \in \mathbb{Z}[X]$ . For every positive integer  $N$  the matrix group*

$$W_{K,N} = \left\{ \begin{pmatrix} t - Bs & -Cs \\ s & t \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : t, s \in \mathbb{Z}/N\mathbb{Z} \right\}$$

gives rise to the surjection

$$\begin{aligned} W_{K,N} &\rightarrow \text{Gal}(K_{(N)}/H_K), \\ \alpha &\mapsto (h(\theta_K) \mapsto h^\alpha(\theta_K)), \end{aligned}$$

where  $h \in \mathcal{F}_N$  is defined and finite at  $\theta_K$ . Its kernel is given by

$$\text{Ker}_{K,N} = \begin{cases} \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} & \text{otherwise.} \end{cases}$$

**Proof.** See [12, §3]. □

**Proposition 4.7.** *If  $m$  ( $m \geq 2$ ) is an integer whose prime factors split in  $K$ , then  $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$  is a unit.*

**Proof.** We get from Lemma 4.1 (ii) that

$$\left( \frac{\sqrt{m}\eta(m\theta_K)}{\eta(\theta_K)} \right)^{24m} = \prod_{k=1}^{m-1} g_{(0,k/m)}(\theta_K)^{12m}. \tag{4.2}$$

For each  $1 \leq k \leq m - 1$ , let us write

$$\frac{k}{m} = \frac{a}{b} \quad \text{with relatively prime positive integers } a \text{ and } b.$$

Note that  $b$  has more than one prime ideal factor in  $K$ , by the assumption on  $m$ . Thus,  $g_{(0,1/b)}(\theta_K)^{12b}$  is a unit in  $K_{(b)}$ , by Proposition 4.4 (i). On the other hand, since

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in W_{K,b}/\text{Ker}_{K,b} \simeq \text{Gal}(K_{(b)}/H_K),$$

we deduce that

$$\begin{aligned} (g_{(0,1/b)}(\theta_K)^{12b})^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}} &= (g_{(0,1/b)}(\tau)^{12b})^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}(\theta_K) \quad \text{by Proposition 4.6} \\ &= \left( g_{(0,1/b)}\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}(\tau)^{12b} \right) \right)(\theta_K) \quad \text{by Proposition 2.4 (iv)} \\ &= g_{(0,a/b)}(\theta_K)^{12b}, \end{aligned}$$

which is also a unit. Therefore,  $\sqrt{m}\eta(m\theta_K)\eta(\theta_K)$  becomes a unit, by the relation (4.2). □

Now, we can prove Theorem 1.3. Let  $m$  ( $m \geq 3$ ) be an odd integer whose prime factors split in  $K$ . Since both  $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$  and  $g_{(1/2,1/2)}(m\theta_K)/g_{(1/2,1/2)}(\theta_K)$  are units by Propositions 4.7 and 3.3 (ii), the conclusion follows from (3.1) with  $\tau_0 = \theta_K$ .

**Corollary 4.8.** *Let  $m$  ( $m \geq 3$ ) be an odd integer whose prime factors  $p$  satisfy  $p \equiv 1 \pmod{4}$ . Then  $\sqrt{m}\varphi(mi)/\varphi(i)$  is a unit.*

**Proof.** If we take  $K = \mathbb{Q}(\sqrt{-1})$ , then  $\theta_K = i$ . For each prime factor  $p$  of  $m$ , the fact that  $p \equiv 1 \pmod{4}$  implies that  $p$  splits in  $K$  [3, Corollary 5.17]. Therefore, we get the assertion by applying Theorem 1.3.  $\square$

We close this section by evaluating  $\sqrt{m}\varphi(mi)/\varphi(i)$  when  $m = 3$  and 5, explicitly.

**Example 4.9.** We shall first estimate  $\sqrt{3}\varphi(3i)/\varphi(i)$ . If  $K = \mathbb{Q}(\sqrt{-1})$ , then  $\theta_K = i$  and  $H_K = K$  [3, Theorem 12.34]. By Proposition 4.6 we have

$$\begin{aligned} \text{Gal}(K_{(6)}/K) &\simeq W_{K,6}/\text{Ker}_{K,6} \\ &= \left\{ \alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \right\}. \end{aligned}$$

Since

$$\begin{aligned} x &= \left( \sqrt{3} \frac{\varphi(3i)}{\varphi(i)} \right)^{24} = g_{(0,1/3)}(i)^{12} g_{(0,2/3)}(i)^{12} g_{(1/2,5/6)}(i)^{24} g_{(1/2,7/6)}(i)^{24} \quad \text{by (4.1)} \\ &= g_{(0,1/3)}(i)^{24} g_{(1/2,1/6)}(i)^{48} \quad \text{by Proposition 2.4 (i) and (iii)} \\ &\approx 72954, \end{aligned}$$

$x$  lies in  $K_{(6)}$  by Propositions 2.2 (iii) and 4.3. Hence, its conjugates  $x_k = x^{\alpha_k}$  ( $1 \leq k \leq 4$ ) over  $K$  are

$$\begin{aligned} x_1 &= g_{(0,1/3)}(i)^{24} g_{(1/2,1/6)}(i)^{48}, \\ x_2 &= g_{(2/3,1/3)}(i)^{24} g_{(5/6,1/6)}(i)^{48}, \\ x_3 &= g_{(1/3,1/3)}(i)^{24} g_{(1/6,1/6)}(i)^{48}, \\ x_4 &= g_{(2/3,0)}(i)^{24} g_{(5/6,1/2)}(i)^{48} \end{aligned}$$

with some multiplicity by Propositions 4.6 and 2.4 (iv). We claim that the minimal polynomial of  $x$  over  $K$  has integer coefficients. Indeed, since  $x$  is a real algebraic integer by definition (1.2) and Theorem 1.2, we have

$$[\mathbb{Q}(x) : \mathbb{Q}] = \frac{[K(x) : K] \cdot [K : \mathbb{Q}]}{[K(x) : \mathbb{Q}(x)]} = \frac{[K(x) : K] \cdot 2}{2} = [K(x) : K],$$

from which the claim follows. Thus,  $x$  is a zero of the polynomial

$$(X - x_1)(X - x_2)(X - x_3)(X - x_4) = (X^2 - 72954X + 729)^2$$

whose coefficients can be determined by numerical approximation with the aid of a computer. Therefore, we obtain

$$\sqrt{3} \frac{\varphi(3i)}{\varphi(i)} = \sqrt[24]{x} = \sqrt[24]{36\,477 + 21\,060\sqrt{3}} = \sqrt[4]{3 + 2\sqrt{3}}.$$

**Example 4.10.** Next, we consider  $\sqrt{5}\varphi(\sqrt{5}i)/\varphi(i)$ . Let  $K = \mathbb{Q}(\sqrt{-1})$ . By Proposition 4.6 we have

$$\begin{aligned} \text{Gal}(K_{(10)}/K) &\simeq W_{K,10}/\text{Ker}_{K,10} \\ &= \left\{ \alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & -6 \\ 6 & 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \right. \\ &\quad \left. \alpha_5 = \begin{pmatrix} 2 & -5 \\ 5 & 2 \end{pmatrix}, \alpha_6 = \begin{pmatrix} 2 & -7 \\ 7 & 2 \end{pmatrix}, \alpha_7 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \alpha_8 = \begin{pmatrix} 4 & -5 \\ 5 & 4 \end{pmatrix} \right\}. \end{aligned}$$

Since

$$\begin{aligned} x &= \left( \sqrt{5} \frac{\varphi(5i)}{\varphi(i)} \right)^{120} = g_{(0,1/5)}(i)^{120} g_{(0,2/5)}(i)^{120} g_{(1/2,1/10)}(i)^{240} g_{(1/2,3/10)}(i)^{240} \\ &\quad \text{by Proposition 2.4 (i) and (iii)} \\ &\approx 41\,473\,935\,220\,454\,921\,602\,871\,195\,774\,259\,272\,002, \end{aligned}$$

$x$  lies in  $K_{(10)}$  by Propositions 2.2 (iii) and 4.3. Its conjugates  $x_k = x^{\alpha_k}$  ( $1 \leq k \leq 8$ ) over  $K$  are

$$\begin{aligned} x_1 &= x_5 = x_7 = x_8 = g_{(0,1/5)}(i)^{120} g_{(0,2/5)}(i)^{120} g_{(1/2,1/10)}(i)^{240} g_{(1/2,3/10)}(i)^{240}, \\ x_2 &= g_{(4/5,1/5)}(i)^{120} g_{(3/5,2/5)}(i)^{120} g_{(9/10,1/10)}(i)^{240} g_{(7/10,3/10)}(i)^{240}, \\ x_3 &= x_6 = g_{(1/5,1/5)}(i)^{120} g_{(2/5,2/5)}(i)^{120} g_{(1/10,1/10)}(i)^{240} g_{(3/10,3/10)}(i)^{240}, \\ x_4 &= g_{(3/5,2/5)}(i)^{120} g_{(1/5,4/5)}(i)^{120} g_{(3/10,7/10)}(i)^{240} g_{(9/10,1/10)}(i)^{240} \end{aligned}$$

with some multiplicity by Propositions 4.6 and 2.4 (iv). So  $x$  is a zero of the polynomial

$$(X^2 - 41\,473\,935\,220\,454\,921\,602\,871\,195\,774\,259\,272\,002X + 1)^4,$$

which shows that  $x$  is a unit. Therefore, we get

$$\begin{aligned} \sqrt{5} \frac{\varphi(5i)}{\varphi(i)} &= \sqrt[120]{x} \\ &= \sqrt[120]{20\,736\,967\,610\,227\,460\,801\,435\,597\,887\,129\,636\,001 + 9\,273\,853\,844\,735\,993\,106\,095\,069\,260\,699\,853\,880\sqrt{5}} \\ &= \sqrt[10]{682 + 305\sqrt{5}}. \end{aligned}$$

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