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ARTICLE

Degree sequences of sufficiently dense random uniform hypergraphs

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Abstract

We find an asymptotic enumeration formula for the number of simple r-uniform hypergraphs with a given degree sequence, when the number of edges is sufficiently large. The formula is given in terms of the solution of a system of equations. We give sufficient conditions on the degree sequence which guarantee existence of a solution to this system. Furthermore, we solve the system and give an explicit asymptotic formula when the degree sequence is close to regular. This allows us to establish several properties of the degree sequence of a random r-uniform hypergraph with a given number of edges. More specifically, we compare the degree sequence of a random r-uniform hypergraph with a given number edges to certain models involving sequences of binomial or hypergeometric random variables conditioned on their sum.

Keywords: random hypergraph; degree sequence; asymptotic enumeration

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1. Introduction

Hypergraphs are useful for modelling relationships between objects in a complex discrete system, and can offer improvements over graph models in areas such as ecology [10], quantum computing [24] and computer vision [26]. A hypergraph $H = (V, E)$ consists of a finite set V of vertices and a finite set E of edges, where each edge is a subset of the vertex set. Here edges do not contain repeated vertices, and there are no repeated edges. A hypergraph is *r-uniform* if every edge contains r vertices. We present an asymptotic enumeration formula for the number of r-uniform hypergraphs with a specified degree sequence, where the degree of a vertex is the number of edges containing it. Our formula holds for $3 \leq r \leq \frac{1}{2}n$ and $nr^4 \log n \ll d \leq \frac{1}{2}\binom{n-1}{r-1}$, where d is the average degree, under very weak restrictions on how much the degrees can vary. By symmetry, the ranges obtained by complementing the edge set and/or complementing each edge are also covered. Using this formula, we establish some results on the degree sequence of a random r-uniform hypergraph with a given number of edges, verifying a conjecture of Kamčev, Liebenau and Wormald [15] for our parameter range.

To be more precise, we must introduce some notation. Let $[a]$ denote the set $\{1, 2, \ldots, a\}$ for any positive integer a. For infinitely many natural numbers n, let $r(n)$ satisfy $3 \leq r(n) \leq n - 3$ and

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let $\boldsymbol{d}(n) = \big(d_1(n), \ldots, d_n(n)\big)$ be a sequence of positive integers. We simply write r for $r(n)$, and similarly for other notation. We assume that for infinitely many n,

$$r \text{ divides } \sum_{j\in[n]} d_j. \tag{1.1}$$

All asymptotics in the paper are as n tends to infinity, along values for which (1.1) holds. Define $\mathcal{H}_r(\boldsymbol{d})$ to be the set of simple r-uniform hypergraphs with vertex set $V = \{1, 2, \ldots, n\}$ and degree sequence $\boldsymbol{d}$. Write $e(\boldsymbol{d}) := \frac{1}{r}\sum_{j\in[n]} d_j$ for the number of edges and $d := d(\boldsymbol{d}) = \frac{1}{n}\sum_{j\in[n]} d_j$ for the average degree.

Our first aim is to find an asymptotic expression for $H_r(\boldsymbol{d}) = |\mathcal{H}_r(\boldsymbol{d})|$ for degree sequences $\boldsymbol{d}$ which are neither too dense nor too sparse.

Our approach to hypergraph enumeration is based on the complex-analytical method. The answer is expressed in terms of high-dimensional integrals resulting from Fourier inversion applied to a multivariable generating function. Then, these integrals are approximated using multidimensional variants of the saddle-point method; see Section 2 for more details. In the context of combinatorial enumeration, this method was pioneered by McKay and Wormald in 1990 [21]. Since then, many other applications of this method have appeared; see for example [4, 5, 20], and the many results cited in [13]. In particular, Kuperberg, Lovett and Peled [17] prove an asymptotic formula for the number of r-uniform d-regular hypergraphs on n vertices which holds when the number of edges in the hypergraph and its complement are each at least n^c (which implies that $r > c$) for some sufficiently large constant c which is not identified explicitly.

Recently, Isaev and McKay [13] developed a general theory based on complex martingales for estimating the high-dimensional integrals which arise from the complex-analytical method. In this paper, we apply tools from [13] in the hypergraph setting.

For a survey of enumeration results for graphs with given degrees, see Wormald [29]. Here we discuss only r-uniform hypergraphs with $r \geq 3$. Dudek, Frieze, Ruciński and Šileikis [7] gave an asymptotic formula for the number of d-regular r-uniform hypergraphs on n vertices when $r \geq 3$ is constant, assuming that $d = o(n^{1/2})$. Building on [2], Blinovsky and Greenhill [3, Corollary 2.3] gave an asymptotic formula for $H_r(\boldsymbol{d})$ that holds when the maximum degree $d_{\max}$ satisfies $r^4 d_{\max}^3 = o(nd)$. These results were obtained using the switching method.

By adapting the "degree switching and contraction mapping" approach of [18, 19], Kamčev, Liebenau and Wormald [15, Theorem 1.2] proved that the degree sequence of a randomly chosen r-uniform hypergraph with m edges is closely related to a random vector with entries chosen from suitable independent binomial distributions, conditioned on the entries of the vector having sum nd. More precisely, they prove that the ratio of the probabilities of a particular vector $\boldsymbol{d}$ in these two models is well-approximated by a simple function of r and $\boldsymbol{d}$. We will restate their theorem as Theorem 1.6 below. This result holds under the assumptions that the degrees do not vary too much, the edge size is not too large and the average degree is at most a sufficiently small constant times $\frac{1}{r}\binom{n-1}{r-1}$. Kamčev, Liebenau and Wormald also considered sparse degree sequences in [15, Theorem 1.3], which subsumes the enumeration results of [2, 7].

Our second aim is to apply our enumeration formula to study the degree sequence of random uniform hypergraphs with a given number of edges. In particular, we prove a companion result to [15, Theorem 1.2] which allows larger edge size, more edges and more variation between the degrees, when the average degree is large enough. Furthermore, we verify (for our range of parameters) a conjecture made in [15], showing that vectors of independent hypergeometric random variables, conditioned on having sum nd, closely match the degree sequence of a random uniform hypergraph with nd/r edges almost everywhere.

1.1 Notation, assumptions and our general results

Define the density λ as a function of n, r and the average degree d by

$$d = \lambda\binom{n-1}{r-1}. \tag{1.2}$$

Write $\mathcal{S}_r(n)$ to denote the set of all subsets of $[n]$ of size r. Given a vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, for all $W \in \mathcal{S}_r(n)$ define

$$\lambda_W(\boldsymbol{\beta}) := \frac{e^{\sum_{j\in W}\beta_j}}{1 + e^{\sum_{j\in W}\beta_j}}. \tag{1.3}$$

Note that $\lambda_W(\boldsymbol{\beta})$ is the probability that the edge W appears in the β-*model* for hypergraphs with given degrees, see for example [27]. Let $\lambda(\boldsymbol{\beta})$ be the average values of the $\lambda_W(\boldsymbol{\beta})$; that is,

$$\lambda(\boldsymbol{\beta}) := \binom{n}{r}^{-1} \sum_{W\in\mathcal{S}_r(n)} \lambda_W(\boldsymbol{\beta}).$$

Observe that $\lambda_W(\boldsymbol{\beta}),\ \lambda(\boldsymbol{\beta}) \in (0, 1)$.

Define the positive symmetric $n \times n$ matrix $A(\boldsymbol{\beta}) = (a_{jk})$ as follows:

$$a_{jk} := \begin{cases} \frac{1}{2}\displaystyle\sum_{W\ni j} \lambda_W(\boldsymbol{\beta})(1 - \lambda_W(\boldsymbol{\beta})), & \text{for } j = k \in [n]; \\ \frac{1}{2}\displaystyle\sum_{W\supset\{j,k\}} \lambda_W(\boldsymbol{\beta})(1 - \lambda_W(\boldsymbol{\beta})), & \text{for } j, k \in [n],\ \ j \neq k. \end{cases} \tag{1.4}$$

We use $|M|$ to denote the determinant of a matrix M.

Let $\boldsymbol{\beta}^* \in \mathbb{R}^n$ be a solution to the system of equations

$$\sum_{W\ni j} \lambda_W(\boldsymbol{\beta}^*) = d_j \qquad \text{for } j \in [n]. \tag{1.5}$$

Summing (1.5) over $j \in [n]$ gives

$$d = \frac{1}{n}\sum_{j\in[n]} d_j = \frac{r}{n}\sum_{W\in\mathcal{S}_r(n)} \lambda_W(\boldsymbol{\beta}^*) = \lambda(\boldsymbol{\beta}^*)\binom{n-1}{r-1}. \tag{1.6}$$

This shows that $\lambda(\boldsymbol{\beta}^*)$ equals the density λ defined in (1.2). Similarly, if we write λ_W or A without argument, we always mean that the argument is $\boldsymbol{\beta}^*$.

Our main enumeration result is the following.

Theorem 1.1. *Let $\boldsymbol{d} = \boldsymbol{d}(n) = (d_1, \ldots, d_n)$ be a degree sequence. Suppose that $r = r(n)$ satisfies $3 \leq r \leq n - 3$ and*

$$r^3(n-r)^3 \log n \ll \lambda(1-\lambda)n\binom{n}{r}. \tag{1.7}$$

Further assume that $\boldsymbol{\beta}^* = \big(\beta_1^*, \ldots, \beta_n^*\big)$ is a solution of (1.5) such that

$$\max_{j,k\in[n]} \left|\beta_j^* - \beta_k^*\right| = O\left(\frac{n}{r(n-r)}\right). \tag{1.8}$$

Let $\lambda_W = \lambda_W(\boldsymbol{\beta}^*)$ be defined as in (1.3), for all $W \in \mathcal{S}_r(n)$, and let $A = A(\boldsymbol{\beta}^*)$ be defined as in (1.4). Then

$$H_r(\boldsymbol{d}) = \frac{r\big(1 + O(\varepsilon)\big)}{2^n\, \pi^{n/2}\, |A|^{1/2}} \prod_{W\in\mathcal{S}_r(n)} \left(\lambda_W^{-\lambda_W}\,(1 - \lambda_W)^{-(1-\lambda_W)}\right),$$

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Table 1. This table shows how the degrees, average degree, solution to (1.5), values of the lambda parameters with $W \in \mathcal{S}_r(n)$, and determinant of the matrix, behave under the symmetries

$H_r(\boldsymbol{d})$	$H_{n-r}(\boldsymbol{d}')$	$H_r\big(\widetilde{\boldsymbol{d}}\big)$	$H_{n-r}\big(\widetilde{\boldsymbol{d}}'\big)$
d_j	$e(\boldsymbol{d}) - d_j$	$\binom{n-1}{r-1} - d_j$	$\binom{n-1}{r} - e(\boldsymbol{d}) + d_j$
d	$\frac{n-r}{r} d$	$\frac{1-\lambda}{\lambda} d$	$\frac{(1-\lambda)}{\lambda} \frac{(n-r)}{r} d$
β_j^*	$\frac{1}{n-r}\big(\sum_{k\in[n]} \beta_k^*\big) - \beta_j^*$	$-\beta_j^*$	$\beta_j^* - \frac{1}{n-r}\big(\sum_{k\in[n]} \beta_k^*\big)$
λ_W	$\lambda_{V\setminus W}(\boldsymbol{\beta}') = \lambda_W$	$\lambda_W(\widetilde{\boldsymbol{\beta}}) = 1 - \lambda_W$	$\lambda_{V\setminus W}\big(\widetilde{\boldsymbol{\beta}}'\big) = 1 - \lambda_W$
$\lvert A(\boldsymbol{\beta}^*)\rvert$	$\big(\frac{n-r}{r}\big)^2\, \lvert A(\boldsymbol{\beta}^*)\rvert$	$\lvert A(\boldsymbol{\beta}^*)\rvert$	$\big(\frac{n-r}{r}\big)^2\, \lvert A(\boldsymbol{\beta}^*)\rvert$

where

$$\varepsilon := \frac{r(n-r)n}{\lambda(1-\lambda)\binom{n}{r}} + \frac{\log^9 n}{n^2}\left(\frac{r^3(n-r)^3}{\lambda(1-\lambda)\binom{n}{r}}\right)^{3/2} + n^{-\Omega(\log n)} = o\big((\log n)^{-1}\big).$$

The implicit constant in the $O(\varepsilon)$ term depends only on the implicit constant in (1.8).

The enumeration problem has two natural symmetries: given a hypergraph, we may replace every edge by its complement, or we may take the complement of the edge set. These symmetries show that for a given degree sequence $\boldsymbol{d}$,

$$H_r(\boldsymbol{d}) = H_{n-r}(\boldsymbol{d}') = H_r\big(\widetilde{\boldsymbol{d}}\big) = H_{n-r}\big(\widetilde{\boldsymbol{d}}'\big) \tag{1.9}$$

where

$$\begin{aligned} \boldsymbol{d}' &:= \big(e(\boldsymbol{d}) - d_1, \ldots, e(\boldsymbol{d}) - d_n\big), \\ \widetilde{\boldsymbol{d}} &:= \left(\binom{n-1}{r-1} - d_1, \ldots, \binom{n-1}{r-1} - d_n\right), \\ \widetilde{\boldsymbol{d}}' &:= \left(\binom{n-1}{r} - e(\boldsymbol{d}) + d_1, \ldots, \binom{n-1}{r} - e(\boldsymbol{d}) + d_n\right). \end{aligned} \tag{1.10}$$

Using these symmetries, we may assume that

$$r \le n/2 \quad \text{and} \quad e(\boldsymbol{d}) \le \frac{1}{2}\binom{n}{r}.$$

When these inequalities are both satisfied, we say that $(r, \boldsymbol{d})$ belongs to the *first quadrant*.

The conditions in Theorem 1.1 are invariant under these two symmetries. It is true, but not obvious, that the asymptotic formula in Theorem 1.1 is also invariant under these symmetries. We prove this in Lemma 1.2 below.

Lemma 1.2. *Suppose that $\boldsymbol{\beta}^*$ is a solution to (1.5). Let $\boldsymbol{\beta}'$, $\widetilde{\boldsymbol{\beta}}$, $\widetilde{\boldsymbol{\beta}}'$ be vectors with entries β_j', $\widetilde{\beta}_j$, $\widetilde{\beta}_j'$ defined in the fourth row of Table 1 for all $j \in [n]$. Then $\boldsymbol{\beta}'$, $\widetilde{\boldsymbol{\beta}}$, $\widetilde{\boldsymbol{\beta}}'$ are solutions of (1.5) for the degree sequences $\boldsymbol{d}'$, $\widetilde{\boldsymbol{d}}$ and $\widetilde{\boldsymbol{d}}'$ defined in (1.10), respectively. Furthermore, the following relationships hold:*

$$\begin{aligned} \lambda_{V\setminus W}(\boldsymbol{\beta}') = \lambda_W, \quad \lambda_W(\widetilde{\boldsymbol{\beta}}) = 1 - \lambda_W, \quad \lambda_{V\setminus W}\big(\widetilde{\boldsymbol{\beta}}'\big) = 1 - \lambda_W \qquad &\text{for all } W \in \mathcal{S}_r(n); \\ \big|A(\boldsymbol{\beta}')\big| = \left(\frac{n-r}{r}\right)^2 |A(\boldsymbol{\beta}^*)|, \quad \big|A(\widetilde{\boldsymbol{\beta}})\big| = |A(\boldsymbol{\beta}^*)|, \quad \big|A\big(\widetilde{\boldsymbol{\beta}}'\big)\big| = \left(\frac{n-r}{r}\right)^2 |A(\boldsymbol{\beta}^*)|&; \\ \big|\beta_j' - \beta_k'\big| = \big|\widetilde{\beta}_j - \widetilde{\beta}_k\big| = \big|\widetilde{\beta}_j' - \widetilde{\beta}_k'\big| = \big|\beta_j^* - \beta_k^*\big| \qquad &\text{for all } j, k \in [n]. \end{aligned}$$

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For the reader's convenience, in Table 1 we summarise information about our parameters under these symmetries.

It follows from (1.9) and Lemma 1.2 that it suffices to prove Theorem 1.1 when $(r, \boldsymbol{d})$ belongs to the first quadrant. In this case, using (1.6) the assumptions of Theorem 1.1 become

$$3 \leq r \leq \frac{1}{2}n,\ nr^4 \log n \ll d \leq \frac{1}{2}\binom{n-1}{r-1} \text{ and } \max_{j,k\in[n]} |\beta_j^* - \beta_k^*| = O\big(r^{-1}\big), \tag{1.11}$$

and the error term becomes

$$O\left(\frac{nr^2}{d} + \frac{r^6 n \log^9 n}{d^{3/2}} + n^{-\Omega(\log n)}\right).$$

Here we use the fact that $\lambda(1-\lambda)\binom{n}{r}$ is a lower bound on the number of edges of any hypergraph in $\mathcal{H}_r(\boldsymbol{d})$ and its complement. The following lemma provides sufficient conditions on r and $\boldsymbol{d}$ which guarantee the existence of solutions to (1.5).

Lemma 1.3. *Let $(r, \boldsymbol{d})$ belong to the first quadrant. Assume that there exists $\Delta \geq 0$ such that for all $j \in [n]$,*

$$de^{-\Delta/r} \leq d_j \leq de^{\Delta/r}.$$

Further, assume that one of the following two conditions hold:

(i) $\Delta \leq \Delta_0$ for some sufficiently small constant $\Delta_0 > 0$;

(ii) $rd = o(1)\binom{n-1}{r-1}$, $r = o(n)$, and $\Delta = \Theta(1)$.

Then there exists $\boldsymbol{\beta}^*$ satisfying (1.5) such that $\max_{j,k\in[n]} \left|\beta_j^* - \beta_k^*\right| = O(\Delta/r)$.

Uniqueness is a feature of similar situations [1, Section 3.3.4], but we have not found a proof of uniqueness in our case in the literature. For completeness we provide a short proof.

Lemma 1.4. *For a given degree sequence $\boldsymbol{d}$, the solution $\boldsymbol{\beta}^*$ to (1.5) is unique if it exists.*

Even though (1.5) doesn't have an explicit solution in general, we can evaluate the formula in Theorem 1.1 accurately if we have a sufficiently precise estimate of $\boldsymbol{\beta}^*$. Stasi, Sadeghi, Rinaldo, Petrović and Fienberg [27] stated without proof a generalisation of an algorithm of [6] that gives geometric convergence to $\boldsymbol{\beta}^*$ if it exists. Though we didn't use the iteration from [27], we will demonstrate how the precision to which an estimate of $\boldsymbol{\beta}^*$ satisfies (1.5) can be used to validate the corresponding estimate of $H_r(\boldsymbol{d})$. Our example will be degree sequences that are not far from regular, which will allow us to investigate the degree sequences of random hypergraphs.

For $j \in [n]$ define $\delta_j := d_j - d$. Define $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_n)$ and $\delta_{\max} := \max\{\|\boldsymbol{\delta}\|_\infty, 1\}$. Also define $R_q := \sum_{j=1}^n \delta_j^q$ for $q \geq 0$ and note that $R_1 = 0$.

Recall the definition of λ from (1.2). We will find it convenient to write some quantities in terms of the parameter Q, which is invariant under the symmetries of (1.9):

$$Q := (1-\lambda)(n-r)\,d = \lambda(1-\lambda)\,\frac{r(n-r)}{n}\binom{n}{r}.$$

We continue to use the error term of Theorem 1.1, which in terms of Q is

$$\varepsilon = \frac{r^2(n-r)^2}{Q} + \frac{r^6(n-r)^6 \log^9 n}{n^{7/2}Q^{3/2}} + n^{-\Omega(\log n)}. \tag{1.12}$$

Our criterion for being "near-regular" is

$$\delta_{\max} = O\big(Q^{3/5}n^{-3/5}\big), \tag{1.13}$$

which in the first quadrant is equivalent to $\delta_{\max} = O(d^{3/5})$.

Theorem 1.5. *If* $3 \le r \le n-3$ *and assumptions (1.7) and (1.13) hold, then*

$$H_r(\boldsymbol{d}) = \left(\frac{r(n-r)(n-1)^{n-1}}{2^n\,\pi^n\,Q^n}\right)^{1/2} \left(\lambda^\lambda(1-\lambda)^{1-\lambda}\right)^{-\binom{n}{r}}$$
$$\times \exp\left(-\frac{(n-1)\,R_2}{2Q} + \frac{n^2\,R_2}{4Q^2} + \frac{(1-2\lambda)(n-2r)n\,R_3}{6Q^2} - \frac{n^3\,R_4}{12Q^3} + O(\hat{\varepsilon})\right),$$

where $\hat{\varepsilon} := \varepsilon + \delta_{\max} n^{3/5} Q^{-3/5}$ *and* ε *is defined in (1.12).*

1.2 Degree sequences of random uniform hypergraphs

Assumption (1.13) is weak enough to include the degree sequences of random hypergraphs with high probability. Following the notation of Kamčev, Liebenau and Wormald [15], we define three probability spaces of integer vectors. Formulas will be given in Section 7.

- $\mathcal{D}_r(n,m)$ is the probability space of degree sequences of uniformly random r-uniform hypergraphs with n vertices and m edges.
- $\mathcal{B}_r(n,m)$ is the result of conditioning n independent binomial variables $\mathrm{Bin}\left(\binom{n-1}{r-1}, p\right)$ on having sum nd. (This distribution is independent of p.)
- Note that each component of $\mathcal{D}_r(n,m)$ has a hypergeometric distribution. $\mathcal{T}_r(n,m)$ is the result of conditioning n independent copies of that distribution on having sum nd.

The most important previous result on the near-regular case was obtained by Kamčev, Liebenau and Wormald [15]. All the overlap between [15, Theorem 1.2] and Theorem 1.1 occurs in Theorem 1.5, so we restate their theorem here.

Theorem 1.6. ([15, Theorem. 1.2]). *Fix* $\varphi \in \left(\frac{4}{9}, \frac{1}{2}\right)$. *For some constant* $c > 0$ *and every* $C > 0$, *suppose that* $3 \le r < cn^{1/4}/\log n$, $r^3 d^{1-3\varphi} < c$ *and* $\log^C n \ll d < \frac{c}{r}\binom{n-1}{r-1}$. *Let* $\boldsymbol{d}$ *be a degree sequence with mean* d *and* $\delta_{\max} \le d^{1-\varphi}$. *Then*

$$\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d}) = \mathbb{P}_{\mathcal{B}_r(n,m)}(\boldsymbol{d})\ \exp\left(\frac{r-1}{2} - \frac{(r-1)R_2}{2(1-\lambda)(n-r)d} + O(\eta)\right),$$

where

$$\eta := \begin{cases} \dfrac{\log^2 n}{\sqrt{n}} + \dfrac{d^{2-4\varphi}}{n} + d^{1-3\varphi}, & \text{if } r = 3; \\[2ex] \dfrac{r^2 \log^2 n}{\sqrt{n}} + (\lambda n + r)r^2 d^{1-3\varphi}, & \text{if } r \ge 4. \end{cases}$$

The conditions of Theorem 1.6 allow for much lower average degree than Theorem 1.1, but at the cost of stricter upper bounds on the edge size, the number of edges, and the variation between the degrees.

As can be seen, the relation between $\mathcal{D}_r(n,m)$ and $\mathcal{B}_r(n,m)$ becomes rapidly more distant as r increases. Theorem 1.5 would allow a statement for all r, but we prefer a statement that is more easily compared to Theorem 1.6. Note that our formula agrees with the expression given in Theorem 1.6 if $r = o(n^{1/2})$, since then $((n-1)/(n-r))^{(n-1)/2} \sim e^{(r-1)/2}$.

Theorem 1.7. *Suppose that* $3 \le r \le cn$ *and* $0 < \lambda < c$ *for some fixed* $c < 1$. *If* $d \gg r^4 n \log n$ *and* $\delta_{\max} = O(d^{3/5})$ *then*

$$\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d}) = \mathbb{P}_{\mathcal{B}_r(n,m)}(\boldsymbol{d}) \left(\frac{n-1}{n-r}\right)^{(n-1)/2} \exp\left(-\frac{(r-1)R_2}{2(1-\lambda)(n-r)d} + O(\bar{\varepsilon})\right).$$

where $\bar{\varepsilon} := \varepsilon + \delta_{\max} d^{-3/5}$ *and* ε *is defined in (1.12).*

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As noted in [15], one can expect $\mathcal{T}_r(n,m)$ to be a better match to $\mathcal{D}_r(n,m)$, especially for large edge sizes. We prove this for the full range of our parameters.

Theorem 1.8. *If* $3 \le r \le n-3$ *and assumptions (1.7) and (1.13) hold, then*

$$\begin{aligned}\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d}) &= \mathbb{P}_{\mathcal{T}_r(n,m)}(\boldsymbol{d}) \left(\frac{n-1}{n}\right)^{(n-1)/2} \exp\left(\frac{R_2}{2Q} + O(\hat{\varepsilon})\right),\\ &= \mathbb{P}_{\mathcal{T}_r(n,m)}(\boldsymbol{d})\ \exp\left(-\frac{1}{2} + \frac{R_2}{2Q} + O(n^{-1} + \hat{\varepsilon})\right),\end{aligned}$$

where $\hat{\varepsilon} := \varepsilon + \delta_{\max} n^{3/5} Q^{-3/5}$ *and* ε *is defined in (1.12).*

Kamčev, Liebenau and Wormald [15] conjectured that $\mathcal{D}_r(n,m)$ is asymptotically equal to $\mathcal{T}_r(n,m)$ almost everywhere.

Conjecture 1.9. ([15]). *Let* $2 \le r \le n-2$ *and* $\min\{m, \binom{n}{r} - m\} = \omega(\log n)$. *Then there exists a set* $\mathfrak{W}$ *that has probability* $1 - O(n^{-\omega(1)})$ *in both* $\mathcal{D}_r(n,m)$ *and* $\mathcal{T}_r(n,m)$, *such that uniformly for all* $\boldsymbol{d} \in \mathfrak{W}$,

$$\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d}) = \mathbb{P}_{\mathcal{T}_r(n,m)}(\boldsymbol{d})\,(1 + o(1)).$$

We prove their conjecture for our range of parameters.

Theorem 1.10. *If* $3 \le r \le n-3$ *and assumption (1.7) holds, then there exists a set* $\mathfrak{W}$ *that has probability* $1 - n^{-\Omega(\log n)}$ *in both* $\mathcal{D}_r(n,m)$ *and* $\mathcal{T}_r(n,m)$, *such that uniformly for all* $\boldsymbol{d} \in \mathfrak{W}$,

$$\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d}) = \big(1 + O\big(\varepsilon + n^{1/10} Q^{-1/10} \log n + n^{-1/2} \log^2 n\big)\big)\mathbb{P}_{\mathcal{T}_r(n,m)}(\boldsymbol{d}).$$

1.3 Structure of the paper

Having now stated our main results, we describe the overall structure of the paper. In Section 2, we outline how $H_r(\boldsymbol{d})$ can be expressed as an n-dimensional integral and state the lemmas which lead to its evaluation. In Section 3 we prove some necessary bounds concerning the quantities $\lambda_W(\boldsymbol{\beta})$ and $A(\boldsymbol{\beta})$, and then in Section 4 we apply them to evaluate the integral, completing the proof of our main enumeration result, Theorem 1.1. In Section 5 we address existence and uniqueness of solutions to (1.5), proving Lemma 1.3 and Lemma 1.4. Section 6 examines the near-regular case, proving Theorem 1.5. Then in Section 7 we prove our results about the degree sequence of random uniform hypergraphs, as stated in Section 1.2. Finally, Section 8 contains several technical proofs that have been deferred, including the proof of Lemma 1.2.

Some of the calculations in this paper are rather tedious, particularly in Sections 6 and 7. We carried out the worst of them first using the computer algebra package Maple and later checked them by hand. All infinite series are based on Taylor's theorem and so have clear-cut truncation criteria.

2. Proof outline for Theorem 1.1

We will take advantage of Lemma 1.2 to work in the first quadrant, where the conditions of Theorem 1.1 are given by (1.11).

The number $H_r(\boldsymbol{d})$ of simple r-uniform hypergraphs with degree sequence $\boldsymbol{d} = (d_1, \ldots, d_n)$ can be expressed using a generating function, where the power of variable x_j gives the degree of vertex j for $j \in [n]$. Each $W \in \mathcal{S}_r(n)$ will contribute a factor of $\prod_{j \in W} x_j$, if W is an edge in the hypergraph, or 1 if W is not an edge. Using $\left[x_1^{d_1} \cdots x_n^{d_n}\right]$ to denote coefficient extraction, this gives

$$\begin{aligned}H_r(\boldsymbol{d}) &= \left[x_1^{d_1}\cdots x_n^{d_n}\right] \prod_{W\in\mathcal{S}_r(n)} \left(1+\prod_{j\in W} x_j\right)\\ &= \frac{1}{(2\pi i)^n} \oint\cdots\oint \frac{\prod_{W\in\mathcal{S}_r(n)}\left(1+\prod_{j\in W} x_j\right)}{\prod_{j\in[n]} x_j^{d_j+1}}\, d\boldsymbol{x},\end{aligned}$$

using Cauchy's coefficient formula for the second line. Each integral is over a contour enclosing the origin. Recalling that $\boldsymbol{\beta}^*$ is a solution of (1.5), we choose the jth contour to be a circle of radius $e^{\beta_j^*}$, for $j\in[n]$. This choice leads to the expression

$$\begin{aligned}H_r(\boldsymbol{d}) &= (2\pi)^{-n} \exp\Bigg(-\sum_{j\in[n]} \beta_j^* d_j\Bigg) \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \frac{\prod_{W\in\mathcal{S}_r(n)}\left(1+\prod_{j\in W} e^{\beta_j^*+i\theta_j}\right)}{\exp\left(i\sum_{j\in[n]} d_j\theta_j\right)}\, d\boldsymbol{\theta}\\ &= P_r(\boldsymbol{\beta}^*) \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \frac{\prod_{W\in\mathcal{S}_r(n)}\left(1+\lambda_W\left(\exp(i\sum_{j\in W}\theta_j)-1\right)\right)}{\exp\left(i\sum_{j\in[n]} d_j\theta_j\right)}\, d\boldsymbol{\theta}, \end{aligned} \tag{2.1}$$

where the factor in front of the integral is given by

$$P_r(\boldsymbol{\beta}^*) := (2\pi)^{-n} \exp\Bigg(-\sum_{j\in[n]} \beta_j^* d_j\Bigg) \prod_{W\in\mathcal{S}_r(n)} \left(1+e^{\sum_{j\in W}\beta_j^*}\right). \tag{2.2}$$

Let $F(\boldsymbol{\theta})$ denote the integrand, that is,

$$F(\boldsymbol{\theta}) := \frac{\prod_{W\in\mathcal{S}_r(n)}\left(1+\lambda_W\left(\exp\left(i\sum_{j\in W}\theta_j\right)-1\right)\right)}{\exp\left(i\sum_{j\in[n]} d_j\theta_j\right)}. \tag{2.3}$$

As we will see in Lemma 4.1, our choice of $\boldsymbol{\beta}^*$ ensures that the linear term in the expansion of $\log F(\boldsymbol{\theta})$ vanishes.

The maximum value of $|F(\boldsymbol{\theta})|$ is 1, which is achieved if and only if $\sum_{j\in W}\theta_j \equiv 0 \pmod{2\pi}$ for all $W\in\mathcal{S}_r(n)$. If this condition holds then all θ_j must be equal modulo 2π, as can be seen by considering two r-subsets W, W' which differ in just one vertex and observing that such a pair of subsets exists for any pair of vertices. Hence there are precisely r points where $F(\boldsymbol{\theta})$ is maximised in $(-\pi,\pi]^n$, namely $\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(r)}$, where for $t\in[r]$ the point $\boldsymbol{\theta}^{(t)} = \left(\theta_1^{(t)},\ldots,\theta_n^{(t)}\right)$ is defined by

$$\theta_1^{(t)} = \theta_2^{(t)} = \cdots = \theta_n^{(t)} \equiv \frac{2\pi t}{r} \pmod{2\pi}.$$

We will estimate the value of the integral first in the regions close to $\boldsymbol{\theta}^{(t)}$, for some $t\in[r]$, then for the remainder of the domain. Write $U_n(\rho)$ for the ball of radius ρ around the origin, with respect to the infinity norm; that is,

$$U_n(\rho) := \left\{\boldsymbol{x}\in\mathbb{R}^n : |x_j|\le\rho \text{ for } j\in[n]\right\},$$

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and for $\rho > 0$ define the region $\mathcal{R}(\rho)$ as

$$\mathcal{R}(\rho) := U_n(\rho) \cap \left\{ \boldsymbol{\theta} \in \mathbb{R}^n : \left| \sum_{j\in[n]} \theta_j \right| \leq n r^{-1/2} \rho \right\}. \tag{2.4}$$

Evaluation of the integral proceeds by the following sequence of lemmas, whose proof is deferred to Section 4. The first two lemmas give an estimate of the value of the integral over $U_n\big(r^{-1}\big)$, by providing an estimate over $\mathcal{R}\big(d^{-1/2}\log n\big)$ and $U_n\big(r^{-1}\big) \setminus \mathcal{R}\big(d^{-1/2}\log n\big)$ respectively.

Lemma 2.1. *If assumptions (1.11) hold then*

$$\int_{\mathcal{R}(d^{-1/2}\log n)} F(\boldsymbol{\theta})\, d\boldsymbol{\theta} = (1 + O(\varepsilon)) \frac{\pi^{n/2}}{|A|^{1/2}},$$

where ε is given in (1.12).

Lemma 2.2. *If assumptions (1.11) hold then*

$$\int_{U_n\left(r^{-1}\right)\setminus\mathcal{R}(d^{-1/2}\log n)} |F(\boldsymbol{\theta})|\, d\boldsymbol{\theta} = n^{-\Omega(\log n)} \frac{\pi^{n/2}}{|A|^{1/2}}.$$

Define the regions $U^{(t)}$ for $t \in [r]$ by

$$U^{(t)} := \left\{ \boldsymbol{\theta}^{(t)} + \boldsymbol{\theta} \pmod{2\pi} : \boldsymbol{\theta} \in U_n\big(r^{-1}\big) \right\}. \tag{2.5}$$

Let $\mathcal{B} := \cup_{t\in[r]} U^{(t)}$. Since $F(\boldsymbol{\theta}^{(t)} + \boldsymbol{\theta}) = F(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in U_n(\pi)$, each of the regions $U^{(1)}, \ldots, U^{(r)}$ makes an identical contribution to the integral. Lemmas 2.1 and 2.2 imply that under assumptions (1.11) we have

$$\int_{\mathcal{B}} F(\boldsymbol{\theta})\, d\boldsymbol{\theta} = (1 + O(\varepsilon)) \frac{r\,\pi^{n/2}}{|A|^{1/2}}. \tag{2.6}$$

The integral in the region $U_n(\pi) \setminus \mathcal{B}$ is approximated in the next result.

Lemma 2.3. *If assumptions (1.11) hold then*

$$\int_{U_n(\pi)\setminus\mathcal{B}} |F(\boldsymbol{\theta})|\, d\boldsymbol{\theta} = n^{-\omega(n)} \frac{\pi^{n/2}}{|A|^{1/2}}.$$

Continuing with the proof of Theorem 1.1, by combining Lemma 2.3 and (2.6) we obtain

$$\int_{U_n(\pi)} F(\boldsymbol{\theta})\, d\boldsymbol{\theta} = (1 + O(\varepsilon)) \frac{r\,\pi^{n/2}}{|A|^{1/2}}. \tag{2.7}$$

We can express $P_r(\boldsymbol{\beta}^*)$ in a more convenient form, as follows:

$$\begin{aligned}
P_r(\boldsymbol{\beta}^*) &\overset{(1.5)}{=} (2\pi)^{-n} \frac{\prod_{W\in\mathcal{S}_r(n)} \left(1 + e^{\sum_{j\in W} \beta_j^*}\right)}{\exp\left(\sum_{j\in[n]} \beta_j^* \sum_{W\ni j} \lambda_W\right)} \\
&= (2\pi)^{-n} \frac{\prod_{W\in\mathcal{S}_r(n)} \left(1 + e^{\sum_{j\in W} \beta_j^*}\right)}{\exp\left(\sum_{W\in\mathcal{S}_r(n)} \lambda_W \sum_{j\in W} \beta_j^*\right)} \\
&= (2\pi)^{-n} \prod_{W\in\mathcal{S}_r(n)} \frac{1 + e^{\sum_{j\in W} \beta_j^*}}{\exp\left(\lambda_W \sum_{j\in W} \beta_j^*\right)}
\end{aligned}$$

$$= (2\pi)^{-n} \prod_{W\in\mathcal{S}_r(n)} \left(\frac{1+e^{\sum_{j\in W}\beta_j^*}}{e^{\sum_{j\in W}\beta_j^*}}\right)^{\lambda_W} \left(1+e^{\sum_{j\in W}\beta_j^*}\right)^{1-\lambda_W}$$
$$\overset{(1.3)}{=} (2\pi)^{-n} \prod_{W\in\mathcal{S}_r(n)} \left(\lambda_W^{-\lambda_W}(1-\lambda_W)^{-(1-\lambda_W)}\right). \tag{2.8}$$

The proof of Theorem 1.1 in the first quadrant is completed by substituting (2.7) and (2.8) into (2.1). The full statement of Theorem 1.1 then follows from Lemma 1.2.

3. Properties of A and other useful bounds

We will need to analyse the behaviour of $\lambda_W(\boldsymbol{\beta})$, $\lambda(\boldsymbol{\beta})$ and $A(\boldsymbol{\beta})$, not only when $\boldsymbol{\beta}$ is a solution of (1.5), but more generally. We also need

$$\Lambda(\boldsymbol{\beta}) := \binom{n}{r}^{-1} \sum_{W\in\mathcal{S}_r(n)} \lambda_W(\boldsymbol{\beta})(1-\lambda_W(\boldsymbol{\beta})).$$

Recall that the elements of $A(\boldsymbol{\beta})$ are sums of terms of the form $\lambda_W(\boldsymbol{\beta})(1-\lambda_W(\boldsymbol{\beta}))$. We start by establishing bounds on $\lambda_W(\boldsymbol{\beta})$ and $1-\lambda_W(\boldsymbol{\beta})$.

Lemma 3.1. *Denote by $f:\mathbb{R}^r\to\mathbb{R}$ the function*

$$f(\boldsymbol{x}) = \frac{e^{\sum_{j=1}^r x_j}}{1+e^{\sum_{j=1}^r x_j}}.$$

Let $\boldsymbol{x}$, $\boldsymbol{y}$ satisfy $|x_i-y_i|\le\delta/r$ for some constant $\delta\ge 0$, and define $p := \left|\{j : x_j\ne y_j\}\right|$. Then

$$e^{-\delta p/r} \le \frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} \le e^{\delta p/r}, \qquad e^{-\delta p/r} \le \frac{1-f(\boldsymbol{x})}{1-f(\boldsymbol{y})} \le e^{\delta p/r}.$$

Proof. First suppose that $p=1$ and without loss of generality assume $x_1\ne y_1$. Then if $y_1\le x_1$ we have

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} = \frac{e^{x_1+X}}{1+e^{x_1+X}}\cdot\frac{1+e^{y_1+X}}{e^{y_1+X}} \le e^{x_1-y_1} \le e^{\delta/r},$$

where $X=\sum_{j=2}^r x_j=\sum_{j=2}^r y_j$. Observe that $\frac{1+e^y}{1+e^x}\le e^{y-x}$ whenever $x\le y$. Therefore when $y_1>x_1$,

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} = \frac{e^{x_1+X}}{1+e^{x_1+X}}\cdot\frac{1+e^{y_1+X}}{e^{y_1+X}} \le \frac{1+e^{y_1+X}}{1+e^{x_1+X}} \le e^{y_1-x_1} \le e^{\delta/r}.$$

As $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary vectors in $\mathbb{R}^r$, by symmetry we also have

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} \ge e^{-\delta/r}.$$

Similarly,

$$\frac{1-f(\boldsymbol{x})}{1-f(\boldsymbol{y})} = \frac{1+e^{y_1+X}}{1+e^{x_1+X}} \le \max\{e^{y_1-x_1},1\} \le e^{\delta/r} \quad\text{and}\quad \frac{1-f(\boldsymbol{x})}{1-f(\boldsymbol{y})} \ge e^{-\delta/r}.$$

For arbitrary $\boldsymbol{x},\boldsymbol{y}$, let $\boldsymbol{z}_0,\ldots,\boldsymbol{z}_p$ be a sequence of elements of $\mathbb{R}^n$ with $\boldsymbol{z}_0=\boldsymbol{x}$, $\boldsymbol{z}_p=\boldsymbol{y}$ such that $\boldsymbol{z}_j$ and $\boldsymbol{z}_{j-1}$ differ in only one coordinate for $j=1,\ldots,p$. Then

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} = \prod_{j=1}^p \frac{f(\boldsymbol{z}_{j-1})}{f(\boldsymbol{z}_j)}, \qquad \frac{1-f(\boldsymbol{x})}{1-f(\boldsymbol{y})} = \prod_{i=1}^p \frac{1-f(\boldsymbol{z}_{j-1})}{1-f(\boldsymbol{z}_j)},$$

and the statement follows as there are exactly p factors. $\square$

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We will apply this lemma in two slightly different scenarios. First we compare $\lambda(\boldsymbol{\beta})$ to $\lambda(\widehat{\boldsymbol{\beta}})$ for two different vectors $\boldsymbol{\beta}$ and $\widehat{\boldsymbol{\beta}}$.

Lemma 3.2. *Let $\boldsymbol{\beta}$ and $\widehat{\boldsymbol{\beta}}$ satisfy* $\max_{j\in[n]} \left|\beta_j - \widehat{\beta}_j\right| \le \delta/r$ *for some nonnegative constant δ. Then*

$$e^{-\delta}\lambda(\widehat{\boldsymbol{\beta}}) \le \lambda(\boldsymbol{\beta}) \le e^{\delta}\lambda(\widehat{\boldsymbol{\beta}}).$$

Proof. By Lemma 3.1 we have for each $W \in \mathcal{S}_r(n)$ that $e^{-\delta}\lambda_W(\widehat{\boldsymbol{\beta}}) \le \lambda_W(\boldsymbol{\beta}) \le e^{\delta}\lambda_W(\widehat{\boldsymbol{\beta}})$. The result follows from the definition of $\lambda(\boldsymbol{\beta})$. □

In the second application, we consider the ratios of $\lambda_W(\boldsymbol{\beta})$ and $\lambda_{W'}(\boldsymbol{\beta})$ for $W, W' \in \mathcal{S}_r(n)$.

Lemma 3.3. *Let $\boldsymbol{\beta}$ satisfy* $\max_{j,k\in[n]} |\beta_j - \beta_k| \le \delta/r$ *for some nonnegative constant δ. Then*

$$e^{-\delta\,(1-|W\cap W'|/r)} \le \frac{\lambda_W(\boldsymbol{\beta})}{\lambda_{W'}(\boldsymbol{\beta})} \le e^{\delta\,(1-|W\cap W'|/r)},$$
$$e^{-\delta\,(1-|W\cap W'|/r)} \le \frac{1-\lambda_W(\boldsymbol{\beta})}{1-\lambda_{W'}(\boldsymbol{\beta})} \le e^{\delta\,(1-|W\cap W'|/r)}$$

for all $W, W' \in \mathcal{S}_r(n)$. Hence

$$e^{-\delta} \le \frac{\lambda_W(\boldsymbol{\beta})}{\lambda(\boldsymbol{\beta})} \le e^{\delta} \quad \text{and} \quad e^{-2\delta} \le \frac{\lambda_W(\boldsymbol{\beta})\big(1-\lambda_W(\boldsymbol{\beta})\big)}{\Lambda(\boldsymbol{\beta})} \le e^{2\delta}$$

for all $W \in \mathcal{S}_r(n)$.

Proof. Note that the β_j terms corresponding to $j \in W \cap W'$ appear in both $\lambda_W(\boldsymbol{\beta})$ and $\lambda_{W'}(\boldsymbol{\beta})$. Together with Lemma 3.1 this implies the first half of the statement. The bounds involving $\lambda(\boldsymbol{\beta})$ and $\Lambda(\boldsymbol{\beta})$ follow from the definitions of these quantities. □

We use the previous result to deduce that $\lambda(\boldsymbol{\beta})$ and $\Lambda(\boldsymbol{\beta})$ have the same order of magnitude when $\lambda(\boldsymbol{\beta})$ is small enough.

Lemma 3.4. *Let $\boldsymbol{\beta}$ satisfy* $\max_{j,k\in[n]} |\beta_j - \beta_k| \le \delta/r$ *for a given nonnegative constant δ. If $\lambda(\boldsymbol{\beta}) \le 7/8$ then*

$$\frac{e^{-\delta}}{256}\lambda(\boldsymbol{\beta}) \le \Lambda(\boldsymbol{\beta}) \le \lambda(\boldsymbol{\beta}).$$

Proof. The upper bound holds as

$$\binom{n}{r}\Lambda(\boldsymbol{\beta}) = \sum_{W\in\mathcal{S}_r(n)} \lambda_W(\boldsymbol{\beta})(1-\lambda_W(\boldsymbol{\beta})) \le \sum_{W\in\mathcal{S}_r(n)} \lambda_W(\boldsymbol{\beta}) = \binom{n}{r}\lambda(\boldsymbol{\beta}).$$

Now consider the set $S = \left\{W \in \mathcal{S}_r(n) : \lambda_W(\boldsymbol{\beta}) > \frac{15}{16}\right\}$. First assume that $|S| \le \frac{15}{16}\,|\mathcal{S}_r(n)|$. Then

$$\binom{n}{r}\Lambda(\boldsymbol{\beta}) \ge \sum_{W\in\mathcal{S}_r(n)\setminus S} \lambda_W(\boldsymbol{\beta})(1-\lambda_W(\boldsymbol{\beta})) \overset{L.3.3}{\ge} \sum_{W\in\mathcal{S}_r(n)\setminus S} e^{-\delta}\,\lambda(\boldsymbol{\beta})\frac{1}{16} \ge \frac{e^{-\delta}}{256}\lambda(\boldsymbol{\beta})\binom{n}{r}.$$

On the other hand if $|S| > \frac{15}{16}\,|\mathcal{S}_r(n)|$, then

$$\lambda(\boldsymbol{\beta})\binom{n}{r} = \sum_{W\in S} \lambda_W(\boldsymbol{\beta}) > \left(\frac{15}{16}\right)^2\binom{n}{r} > \frac{7}{8}\binom{n}{r},$$

contradicting our assumption. □

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Now we turn to the matrix $A(\boldsymbol{\beta})$ and establish that the diagonal entries are relatively close to each other, and similarly for the off-diagonal entries.

Lemma 3.5. *Let $\boldsymbol{\beta}$ satisfy $\max_{j,k\in[n]} |\beta_j - \beta_k| \le \delta/r$ for some nonnegative constant δ. Then the entries of $A(\boldsymbol{\beta}) = (a_{jk})$ satisfy*

$$e^{-4\delta/r} \le \frac{a_{jk}}{a_{j'k'}} \le e^{4\delta/r}, \qquad e^{-4\delta/r} \le \frac{a_{jj}}{a_{kk}} \le e^{4\delta/r}$$

for any $j, k, j', k' \in [n]$ with $j \neq k$ and $j' \neq k'$. Furthermore,

$$\frac{1}{2}\, e^{-4\delta/r} \Lambda(\boldsymbol{\beta}) \binom{n-2}{r-2} \le a_{jk} \le \frac{1}{2}\, e^{4\delta/r} \Lambda(\boldsymbol{\beta}) \binom{n-2}{r-2},$$
$$\frac{1}{2}\, e^{-4\delta/r} \Lambda(\boldsymbol{\beta}) \binom{n-1}{r-1} \le a_{jj} \le \frac{1}{2}\, e^{4\delta/r} \Lambda(\boldsymbol{\beta}) \binom{n-1}{r-1}.$$

Proof. We start with the case when $j \neq k$ and $j' \neq k'$. Let $S_{jk} = \{W \in \mathcal{S}_r(n) : W \supset \{j,k\}\}$. Recall that

$$a_{jk} = \frac{1}{2} \sum_{W \in S_{jk}} \lambda_W(\boldsymbol{\beta})(1 - \lambda_W(\boldsymbol{\beta})) \quad \text{and} \quad a_{j'k'} = \frac{1}{2} \sum_{W' \in S_{j'k'}} \lambda_{W'}(\boldsymbol{\beta})(1 - \lambda_{W'}(\boldsymbol{\beta})).$$

Both S_{jk} and $S_{j'k'}$ contain exactly $\binom{n-2}{r-2}$ elements. We will show that there exists a bijection $\zeta : S_{j,k} \to S_{j'k'}$ such that for every pair (W, W') with $W' = \zeta(W)$, we have

$$e^{-4\delta/r} \lambda_{W'}(\boldsymbol{\beta})(1 - \lambda_{W'}(\boldsymbol{\beta})) \le \lambda_W(\boldsymbol{\beta})(1 - \lambda_W(\boldsymbol{\beta})) \le e^{4\delta/r} \lambda_{W'}(\boldsymbol{\beta})(1 - \lambda_{W'}(\boldsymbol{\beta})).$$

By Lemma 3.3, this follows if $|W \cap \zeta(W)| \ge r - 2$ for all $W \in S_{jk}$.

We can assume that either $\{j,k\} \cap \{j',k'\} = \emptyset$ or $j = j'$. Now consider the function $b : V \to V$, which is the identity for every vertex in $V \setminus \{j,k,j',k'\}$ and switches j with j' and k with k'. This function can be extended to a function $\zeta : S_{jk} \to S_{j'k'}$ by assigning to each set $W \in S_{jk}$ the set $\{b(j) : j \in W\}$. Clearly b is a bijection and so is ζ, and $|W \cap \zeta(W)| \ge r - 2$ for all $W \in S_{jk}$, as required.

The remaining results follow as

$$a_{jj} = \frac{1}{r-1} \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{jk} \quad \text{and} \quad \Lambda(\boldsymbol{\beta}) = \frac{1}{n} \sum_{j=1}^{n} a_{jj},$$

completing the proof. $\square$

We also establish an upper bound on the determinant of $A(\boldsymbol{\beta})$. It follows easily from the Matrix Determinant Lemma (see for example [22, equation (6.2.3)]) that for any real numbers a, b,

$$|aI + bJ| = a^{n-1}(a + bn), \tag{3.1}$$

where I is the $n \times n$ identity matrix and J is the $n \times n$ matrix with every entry equal to one.

Lemma 3.6. *Let $\boldsymbol{\beta}$ satisfy $\max_{j,k\in[n]} |\beta_j - \beta_k| \le \delta/r$ for some nonnegative constant δ. Then*

$$|A(\boldsymbol{\beta})| = \exp\left(O(n) \log\left(\Lambda(\boldsymbol{\beta}) \binom{n-1}{r-1} \right) \right).$$

Proof. Note that for any $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$\begin{aligned}\boldsymbol{x}^{\mathrm{t}} A(\boldsymbol{\beta})\, \boldsymbol{x} &= \frac{1}{2} \sum_{W \in \mathcal{S}_r(n)} \lambda_W(\boldsymbol{\beta})(1-\lambda_W(\boldsymbol{\beta})) \left(\sum_{j \in W} x_j \right)^2 \\ &\overset{L.3.3}{\leq} \frac{1}{2} e^{2\delta} \Lambda(\boldsymbol{\beta}) \sum_{W \in \mathcal{S}_r(n)} \left(\sum_{j \in W} x_j \right)^2 = \boldsymbol{x}^{\mathrm{t}} A' \boldsymbol{x},\end{aligned}$$

where $A' = \frac{1}{2} e^{2\delta} \Lambda(\boldsymbol{\beta}) \big(\binom{n-2}{r-1} I + \binom{n-2}{r-2} J \big)$. (Here $\boldsymbol{x}^{\mathrm{t}}$ is the transpose of $\boldsymbol{x}$.) Therefore, by the min-max theorem, the k-th largest eigenvalue of $A(\boldsymbol{\beta})$ is at most the k-th largest eigenvalue of A'. Since $A(\boldsymbol{\beta})$ is positive semidefinite, all its eigenvalues are non-negative, implying that $|A(\boldsymbol{\beta})| \leq |A'|$. Using (3.1), we have

$$|A'| = \exp\left(O(n) \log\left(\Lambda(\boldsymbol{\beta}) \binom{n-1}{r-1} \right) \right)$$

and the result follows. $\square$

3.1 Inverting $A(\beta)$

Next we bound the entries of $A(\boldsymbol{\beta})^{-1}$ and find a change of basis matrix T which transforms $A(\boldsymbol{\beta})$ to the identity matrix. For $p \in \{1, 2, \infty\}$, we use the notation $\|\cdot\|_p$ for the standard vector norms and the corresponding induced matrix norms (see for example [12, Section 5.6]). In particular, for an $n \times n$ matrix $M = (m_{ij})$,

$$\|M\|_1 = \max_{j \in [n]} \sum_{i \in [n]} |m_{ij}|, \qquad \|M\|_\infty = \max_{i \in [n]} \sum_{j \in [n]} |m_{ij}|.$$

The proof of this lemma is given in Section 8.2.

Lemma 3.7. *Let δ be a nonnegative constant. For every $\boldsymbol{\beta}$ such that $\max_{j,k \in [n]} |\beta_j - \beta_k| \leq \delta / r$ the following holds.*

Let $A(\boldsymbol{\beta})^{-1} = (\sigma_{jk})$ be the inverse of $A(\boldsymbol{\beta})$. There exists a constant C, independent of δ, such that for $n \geq 16e^{4\delta}$ we have

$$|\sigma_{jk}| \leq \begin{cases} \dfrac{Ce^{35\delta}}{\Lambda(\boldsymbol{\beta}) \binom{n-1}{r-1}}, & \text{if } j = k; \\[2ex] \dfrac{Ce^{35\delta}}{\Lambda(\boldsymbol{\beta}) \binom{n-1}{r-1} n}, & \text{otherwise.} \end{cases} \tag{3.2}$$

In addition, there exists a matrix $T = T(\boldsymbol{\beta})$ such that $T^{\mathrm{t}} A(\boldsymbol{\beta}) T = I$ with

$$\|T\|_1, \|T\|_\infty = O\left(\Lambda(\boldsymbol{\beta})^{-1/2} \binom{n-1}{r-1}^{-1/2} \right).$$

Furthermore, for any $\rho > 0$ there exists $\rho_1, \rho_2 = \Theta\Big(\rho\, \Lambda(\boldsymbol{\beta})^{1/2} \binom{n-1}{r-1}^{1/2}\Big)$ such that

$$T\big(U_n(\rho_1)\big) \subseteq \mathcal{R}(\rho) \subseteq T\big(U_n(\rho_2)\big),$$

where $\mathcal{R}(\rho)$ is defined in (2.4).

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4. Evaluating the integral

In this section we prove Lemmas 2.1–2.3. We have already seen that these lemmas establish Theorem 1.1.

Throughout this section we assume that (1.11) holds and thus $\lambda = \binom{n-1}{r-1}^{-1} d \le \frac{1}{2}$. Therefore, by Lemma 3.4, for $\Lambda := \Lambda(\boldsymbol{\beta}^*)$ we have

$$\Lambda \binom{n-1}{r-1} = \Theta\left(\lambda \binom{n-1}{r-1}\right) = \Theta(d). \tag{4.1}$$

4.1 Proof of Lemma 2.1

First, we will estimate the integral of $F(\boldsymbol{\theta})$ over $\mathcal{R}(d^{-1/2} \log n)$. For $\xi \in [0, 1]$ and $x \in [-1, 1]$, $|\xi(e^{ix} - 1)|$ is bounded below 1 and the fifth derivative of $\log\bigl(1 + \xi(e^{ix} - 1)\bigr)$ with respect to x is uniformly $O(\xi)$. Using the principal branch of the logarithm in this domain, we have by Taylor's theorem that uniformly

$$\log\bigl(1 + \xi\,(e^{ix} - 1)\bigr) = \sum_{p=1}^{4} i^p c_p(\xi)\, x^p + O(\xi)\, |x|^5, \tag{4.2}$$

where the coefficients are

$$c_1(\xi) := \xi, \qquad c_2(\xi) := \frac{1}{2}\xi(1-\xi), \qquad c_3(\xi) := \frac{1}{6}\xi(1-\xi)(1-2\xi),$$
$$c_4(\xi) := \frac{1}{24}\xi(1-\xi)\bigl(1 - 6\xi + 6\xi^2\bigr).$$

Lemma 4.1. *Let $\rho := d^{-1/2} \log n$. Then, for $\boldsymbol{\theta} \in U_n(\rho)$, we have*

$$\log F(\boldsymbol{\theta}) = -\boldsymbol{\theta}^{\mathrm{t}} A \boldsymbol{\theta} + \sum_{p=3}^{4} \sum_{W \in \mathcal{S}_r(n)} i^p c_p(\lambda_W) \Bigl(\sum_{j\in W} \theta_j\Bigr)^p + O\Bigl(\frac{n r^4 \log^5 n}{d^{3/2}}\Bigr).$$

Proof. Recall that $\lambda_W \in (0, 1)$ for all W, and note that (1.11) implies that $r\rho = o(1)$. Hence, recalling (2.3), we can apply (4.2) for each $W \in \mathcal{S}_r(n)$, taking $\xi = \lambda_W$ and $x = \sum_{j\in W} \theta_j$. The linear term of $\log F(\boldsymbol{\theta})$ (which includes terms from the denominator of $F(\boldsymbol{\theta})$), is

$$i \sum_{j\in[n]} \theta_j \biggl(\Bigl(\sum_{W\ni j} \lambda_W\Bigr) - d_j\biggr),$$

which equals zero by (1.5). In addition, for the quadratic term,

$$\sum_{W\in\mathcal{S}_r(n)} \frac{1}{2}\lambda_W(1-\lambda_W)\Bigl(\sum_{j\in W}\theta_j\Bigr)^2 = \sum_{j,k\in[n]} \sum_{W\supset\{j,k\}} \frac{1}{2}\lambda_W(1-\lambda_W)\theta_j\theta_k = \boldsymbol{\theta}^{\mathrm{t}} A\boldsymbol{\theta}.$$

Now $\lambda_W = O(\lambda)$ for all $W \in \mathcal{S}_r(n)$, by Lemma 3.3, so the combined error term is

$$O\Bigl(\lambda \binom{n}{r} r^5 d^{-5/2} \log^5 n\Bigr) \overset{(4.1)}{=} O\Bigl(\frac{n r^4 \log^5 n}{d^{3/2}}\Bigr).$$

□

Recall that for a complex variable Z, the *variance* is defined by

$$\operatorname{Var} Z = \mathbb{E}|Z - \mathbb{E}Z|^2 = \operatorname{Var}\Re Z + \operatorname{Var}\Im Z,$$

while the *pseudovariance* is

$$\mathbb{V} Z = \mathbb{E}(Z - \mathbb{E}Z)^2 = \operatorname{Var}\Re Z - \operatorname{Var}\Im Z + 2i\operatorname{Cov}(\Re Z, \Im Z).$$

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The following is a special case of [13, Theorem 4.4] that is sufficient for our current purposes.

Theorem 4.2. *Let A be an $n\times n$ positive definite symmetric real matrix and let T be a real matrix such that $T^{\mathrm{t}}AT=I$. Let Ω be a measurable set and let $f:\mathbb{R}^n\to\mathbb{C}$ and $h:\Omega\to\mathbb{C}$ be measurable functions. Make the following assumptions for some ρ_1,ρ_2,ϕ:*

(a) $T(U_n(\rho_1))\subseteq\Omega\subseteq T(U_n(\rho_2))$, *where* $\rho_1,\rho_2=\Theta(\log n)$.

(b) *For* $\boldsymbol{x}\in T(U_n(\rho_2))$, $2\rho_2\,\|T\|_1\,\left|\frac{\partial f}{\partial x_j}(\boldsymbol{x})\right|\le\phi n^{-1/3}\le\frac{2}{3}$ *for* $1\le j\le n$ *and*

$$4\rho_2^2\,\|T\|_1\,\|T\|_\infty\,\|H\|_\infty\le\phi n^{-1/3},$$

where $H=(h_{jk})$ is the matrix with entries defined by

$$h_{jk}=\sup_{\boldsymbol{x}\in T(U_n(\rho_2))}\left|\frac{\partial^2 f}{\partial x_j\,\partial x_k}(\boldsymbol{x})\right|.$$

(c) $|f(\boldsymbol{x})|\le n^{O(1)}e^{O(1/n)\,\boldsymbol{x}^{\mathrm{t}}A\boldsymbol{x}}$ *uniformly for* $\boldsymbol{x}\in\mathbb{R}^n$.

Let $\boldsymbol{X}$ be a Gaussian random vector with density $\pi^{-n/2}|A|^{1/2}\,e^{-\boldsymbol{x}^{\mathrm{t}}A\boldsymbol{x}}$. Then, provided $\mathbb{V}f(\boldsymbol{X})$ is finite and h is bounded in Ω,

$$\int_\Omega e^{-\boldsymbol{x}^{\mathrm{t}}A\boldsymbol{x}+f(\boldsymbol{x})+h(\boldsymbol{x})}\,d\boldsymbol{x}=(1+K)\pi^{n/2}|A|^{-1/2}e^{\mathbb{E}f(\boldsymbol{X})+\frac{1}{2}\mathbb{V}f(\boldsymbol{X})},$$

where, for sufficiently large n,

$$|K|\le e^{\frac{1}{2}\,Var\Im f(\boldsymbol{X})}\left(3e^{\phi^3+e^{-\rho_1^2/2}}-3+\sup_{\boldsymbol{x}\in\Omega}|e^{h(\boldsymbol{x})}-1|\right).$$

Now we will prove Lemma 2.1.

Proof of Lemma 2.1. Let $\rho=d^{-1/2}\log n$. Applying Lemma 4.1 gives

$$\int_{\mathcal{R}(\rho)}F(\boldsymbol{\theta})\,d\boldsymbol{\theta}=\int_{\mathcal{R}(\rho)}\exp\bigl(-\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta}+f(\boldsymbol{\theta})+h(\boldsymbol{\theta})\bigr)\,d\boldsymbol{\theta},$$

where

$$\begin{aligned}f(\boldsymbol{\theta})&=\sum_{W\in\mathcal{S}_r(n)}\sum_{p=3}^{4}i^p c_p(\lambda_W)\left(\sum_{j\in W}\theta_j\right)^p,\\ h(\boldsymbol{\theta})&=O\bigl(nr^4d^{-3/2}\log^5 n\bigr)\overset{(1.11)}{=}O\bigl(nr^2/d\bigr).\end{aligned}\tag{4.3}$$

We will apply Theorem 4.2 with $\Omega=\mathcal{R}(\rho)$. Let T,ρ_1,ρ_2 be as in Lemma 3.7. Then $T(U_n(\rho_1))\subseteq\mathcal{R}(\rho)\subseteq T(U_n(\rho_2))$. Observe that $\rho_1,\rho_2=\Theta\bigl(\rho d^{1/2}\bigr)=\Theta(\log n)$, by (4.1). Clearly $\rho_1\le\rho_2$ and thus condition (a) in Theorem 4.2 is satisfied.

Now for $j\in[n]$,

$$\begin{aligned}\frac{\partial f}{\partial\theta_j}(\boldsymbol{\theta})=&\ \frac{1}{6}\sum_{W\ni j}\lambda_W(1-\lambda_W)\bigl(1-6\lambda_W+6\lambda_W^2\bigr)\left(\sum_{\ell\in W}\theta_\ell\right)^3\\ &-\frac{i}{2}\sum_{W\ni j}\lambda_W(1-\lambda_W)(1-2\lambda_W)\left(\sum_{\ell\in W}\theta_\ell\right)^2.\end{aligned}$$

Thus, for all $\boldsymbol{\theta} \in T(U_n(\rho_2))$ and all $j \in [n]$ we have

$$\left|\frac{\partial f}{\partial \theta_j}(\boldsymbol{\theta})\right| = O\left(\Lambda\binom{n-1}{r-1} r^2 \, \|\boldsymbol{\theta}\|_\infty^2\right) = O\left(\Lambda\binom{n-1}{r-1} r^2 \rho^2\right), \tag{4.4}$$

by Lemmas 3.3 and 3.4 and using the fact that $r\rho = o(1)$. Hence, by (4.4) and Lemma 3.7,

$$2\rho_2 \, \|T\|_1 \left|\frac{\partial f}{\partial \theta_j}(\boldsymbol{\theta})\right| = O\left(\log n \cdot \Lambda^{-1/2}\binom{n-1}{r-1}^{-1/2} \Lambda\binom{n-1}{r-1} r^2 \rho^2\right) \overset{(4.1)}{=} O\left(\frac{r^2 \log^3 n}{d^{1/2}}\right) \tag{4.5}$$

for every $\boldsymbol{\theta} \in T(U_n(\rho_2))$ and $j \in [n]$. Also for all $j, k \in [n]$ (including $j = k$),

$$\begin{aligned}\frac{\partial^2 f}{\partial \theta_j \, \partial \theta_k}(\boldsymbol{\theta}) &= \frac{1}{2} \sum_{W \supset \{j,k\}} \lambda_W (1-\lambda_W)\left(1 - 6\lambda_W + 6\lambda_W^2\right)\left(\sum_{\ell \in W} \theta_\ell\right)^2 \\ &\quad - i \sum_{W \supset \{j,k\}} \lambda_W (1-\lambda_W)(1-2\lambda_W)\left(\sum_{\ell \in W} \theta_\ell\right).\end{aligned}$$

Arguing as above, if $\boldsymbol{\theta} \in T(U_n(\rho_2))$ then

$$\left|\frac{\partial^2 f}{\partial \theta_j \, \partial \theta_k}(\boldsymbol{\theta})\right| = \begin{cases} O\left(\Lambda\binom{n-1}{r-1} r \, \|\boldsymbol{\theta}\|_\infty\right), & \text{if } j = k; \\[2ex] O\left(\Lambda\binom{n-2}{r-2} r \, \|\boldsymbol{\theta}\|_\infty\right), & \text{otherwise.} \end{cases} \tag{4.6}$$

Then (4.6) and Lemma 3.7 imply that

$$\begin{aligned} &4\rho_2^2 \, \|T\|_1 \, \|T\|_\infty \, \|H\|_\infty \\ &\quad = O\left(\log^2 n \frac{1}{\Lambda\binom{n-1}{r-1}} \Lambda\left(\binom{n-1}{r-1} + (n-1)\binom{n-2}{r-2}\right) r\rho\right) = O\left(\frac{r^2 \log^3 n}{d^{1/2}}\right). \end{aligned} \tag{4.7}$$

By (4.5) and (4.7) there exists

$$\phi = O\left(\frac{r^2 \, n^{1/3} \log^3 n}{d^{1/2}}\right) \tag{4.8}$$

such that the left side of both (4.5) and (4.7) are at most $\phi n^{-1/3}$.

Recall that the 2-norm of the real symmetric matrix A^{-1} equals the largest eigenvalue of A^{-1}. Using this we obtain

$$\begin{aligned} f(\boldsymbol{\theta}) &= O\left(\left(r\|\boldsymbol{\theta}\|_\infty + r^2\|\boldsymbol{\theta}\|_\infty^2\right) \, \boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta}\right) = O\left(\left(1 + r^2\|\boldsymbol{\theta}\|_2^2\right) \, \boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta}\right) \\ &= O\left(\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta} + n^2(\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta})^2\|A^{-1}\|_2\right) \\ &\overset{(3.2)}{=} O\left(\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta} + \frac{n^2(\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta})^2}{\Lambda\binom{n-1}{r-1}}\right) \overset{(1.7)}{=} O\left(\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta} + n\left(\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta}\right)^2\right) \\ &= O\left(n^3 e^{\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta}/n}\right), \end{aligned}$$

so condition (c) is satisfied.

By Theorem 4.2 we have

$$\int_{U_n(\rho)} F(\boldsymbol{\theta}) \, d\boldsymbol{\theta} = (1+K)\frac{\pi^{n/2}}{|A|^{1/2}} \, \exp\left(\mathbb{E}f(\boldsymbol{X}) + \frac{1}{2}\mathbb{V}f(\boldsymbol{X})\right),$$

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where

$$K \le e^{\mathrm{Var}(\Im f(\boldsymbol{X}))/2}\left(O\left(\frac{nr^2}{d}\right)+3e^{\phi^3+e^{-\rho_1^2/2}}-3\right)=O\left(\frac{nr^2}{d}+\phi^3+e^{-\rho_1^2/2}\right)\,e^{\mathrm{Var}(\Im f(\boldsymbol{X}))/2}.$$

In the last step we use the fact that $\phi = o(1)$ and $e^{-\rho_1^2/2}=o(1)$. The nr^2/d term inside the $O(\cdot)$ is the bound on h from (4.3). To complete an estimate of K, it remains to bound

$$\mathrm{Var}\big(\Im f(\boldsymbol{X})\big)=\mathrm{Var}\left(\frac{1}{6}\sum_{W\in\mathcal{S}_r(n)}\lambda_W(1-\lambda_W)(1-2\lambda_W)\left(\sum_{j\in W}X_j\right)^3\right).$$

We will rely heavily on Isserlis' theorem (also called Wick's formula) in order to establish bounds for the variance of $\Im f(\boldsymbol{X})$ and later for the pseudovariance of $f(\boldsymbol{X})$. Isserlis' theorem states that the expected value of a product of jointly Gaussian random variables, each with zero mean, can be obtained by summing over all partitions of the variables into pairs, where the term corresponding to a partition is just the product of the covariances of each pair. See for example [23, Theorem 1.1].

In particular, for a normally distributed random vector (Y_1, Y_2) with expected value $(0, 0)$, we have

$$\begin{aligned}\mathbb{E}\big(Y_1^3\big)&=0,\,\mathbb{E}\big(Y_1^4\big)=3\mathrm{Cov}(Y_1,Y_1),\\ \mathbb{E}\big(Y_1^3\,Y_2^3\big)&=9\mathrm{Cov}(Y_1,Y_1)\,\mathrm{Cov}(Y_2,Y_2)\,\mathrm{Cov}(Y_1,Y_2)+6\mathrm{Cov}(Y_1,Y_2)^3,\\ \mathbb{E}\big(Y_1^4\,Y_2^4\big)&=9\mathrm{Cov}(Y_1,Y_1)^2\mathrm{Cov}(Y_2,Y_2)^2+72\mathrm{Cov}(Y_1,Y_1)\mathrm{Cov}(Y_2,Y_2)\mathrm{Cov}(Y_1,Y_2)^2\\ &\qquad+24\mathrm{Cov}(Y_1,Y_2)^4.\end{aligned}$$

Since the sum of components of a normally distributed random vector is also normally distributed, we can apply Isserlis' theorem to sums involving the random variables X_j, $j\in[n]$. Then for any $W\in\mathcal{S}_r(n)$ we have

$$\mathbb{E}\left[\left(\sum_{j\in W}X_j\right)^3\right]=0, \tag{4.9}$$

and so

$$\mathrm{Var}\big(\Im f(\boldsymbol{X})\big)=\sum_{W\in\mathcal{S}_r(n)}\sum_{W'\in\mathcal{S}_r(n)}O\big(\Lambda^2\big)\,\mathbb{E}\left[\left(\sum_{j\in W}X_j\right)^3\left(\sum_{k\in W'}X_k\right)^3\right].$$

For $W, W'\in\mathcal{S}_r(n)$ let

$$\sigma(W,W'):=\mathrm{Cov}\left[\sum_{j\in W}X_j,\sum_{k\in W'}X_k\right].$$

Now $\mathrm{Cov}[X_j, X_k]$ equals the corresponding values of $(2A)^{-1}$ and hence, by Lemma 3.7 and (4.1),

$$\mathrm{Cov}\big[X_j,X_k\big]=\begin{cases}O\big(\frac{1}{d}\big), & \text{if } j=k;\\ O\big(\frac{1}{nd}\big), & \text{otherwise.}\end{cases}$$

Since covariance is additive, we have

$$\sigma(W,W')=O\left(\frac{r^2}{nd}+\frac{|W\cap W'|}{d}\right). \tag{4.10}$$

Using this together with Isserlis' theorem, for any pair W, W',

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{j\in W} X_j\right)^3\left(\sum_{k\in W'} X_k\right)^3\right] &= 9\,\sigma(W,W)\,\sigma(W',W')\,\sigma(W,W') + 6\,\sigma(W,W')^3\\ &= O\left(\frac{r^2}{d^2}\,\sigma(W,W')\right)\\ &= O\left(\frac{r^4}{nd^3} + \frac{r^2\,|W\cap W'|}{d^3}\right).\end{aligned}$$

The average value of $|W \cap W'|$ over pairs of r-sets is r^2/n, so we can sum over $W, W' \in \mathcal{S}_r(n)$ to obtain

$$\mathrm{Var}(\Im f(\boldsymbol{X})) = O\left(\Lambda^2\binom{n}{r}^2\left(\frac{r^4}{nd^3} + \frac{r^2\,(r^2/n)}{d^3}\right)\right) \overset{(4.1)}{=} O\left(\frac{nr^2}{d}\right).$$

By (1.11) this term tends to 0, implying that $K = O\big(nr^2/d + \phi^3 + e^{-\rho_1^2}\big)$.

All that is left is to establish bounds on $\mathbb{E}f(\boldsymbol{X})$ and $\mathbb{V}f(\boldsymbol{X})$. Due to (4.9), we have

$$\begin{aligned}\mathbb{E}f(\boldsymbol{X}) &= \frac{1}{24}\sum_{W\in\mathcal{S}_r(n)} \lambda_W(1-\lambda_W)\big(1-6\lambda_W+6\lambda_W^2\big)\,\mathbb{E}\left[\left(\sum_{j\in W} X_j\right)^4\right]\\ &= O\left(\Lambda\sum_{W\in\mathcal{S}_r(n)}\mathbb{E}\left[\left(\sum_{j\in W} X_j\right)^4\right]\right).\end{aligned}$$

Again using Isserlis' theorem, for any $W \in \mathcal{S}_r(n)$ we have

$$\mathbb{E}\left[\left(\sum_{j\in W} X_j\right)^4\right] = 3\sigma(W,W)^2 \overset{(4.10)}{=} O\left(\frac{r^2}{d^2}\right).$$

Hence by (4.1),

$$\mathbb{E}f(\boldsymbol{X}) = O\left(\frac{nr^2}{d}\right).$$

Now $\mathbb{V}f(\boldsymbol{X})$ satisfies

$$|\mathbb{V}f(\boldsymbol{X})| = |\mathbb{E}\,(f(\boldsymbol{X}) - \mathbb{E}f(\boldsymbol{X}))^2| \le \mathbb{E}\,|f(\boldsymbol{X}) - \mathbb{E}f(\boldsymbol{X})|^2 = \mathrm{Var}(\Re f(\boldsymbol{X})) + \mathrm{Var}(\Im f(\boldsymbol{X})).$$

Since we already established a bound on $\mathrm{Var}(\Im f(\boldsymbol{X}))$, we only need to consider $\mathrm{Var}(\Re f(\boldsymbol{X}))$. Note that

$$\mathrm{Var}(\Re f(\boldsymbol{X})) \le \sum_{W\in\mathcal{S}_r(n)}\ \sum_{W'\in\mathcal{S}_r(n)} c_4(\lambda_W)c_4(\lambda_{W'})\ \mathbb{E}\left[\left(\sum_{j\in W} X_j\right)^4\left(\sum_{k\in W'} X_k\right)^4\right].$$

By Isserlis' theorem, we have

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{j\in W} X_j\right)^4\left(\sum_{k\in W'} X_k\right)^4\right] &= 9\sigma(W,W)^2\sigma(W',W')^2 + 72\sigma(W,W)\sigma(W',W')\sigma(W,W')^2\\ &\quad + 24\sigma(W,W')^4.\end{aligned}$$

Since $\sigma(W,W') = O(r/d)$ from (4.10),

$$\mathrm{Var}(\Re(f(\boldsymbol{X}))) = O\left(\Lambda^2\binom{n}{r}^2\frac{r^4}{d^4}\right) \overset{(4.1)}{=} O\left(\frac{n^2r^2}{d^2}\right) \overset{(1.11)}{=} O\left(\frac{nr^2}{d}\right).$$

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Therefore $|\mathbb{V}(f(\boldsymbol{X}))| = O\big(nr^2/d\big)$ and so

$$\int_{\mathcal{R}(\rho)} F(\boldsymbol{\theta})\, d\boldsymbol{\theta} = \frac{\pi^{n/2}}{|A|^{1/2}} \exp\left(\mathbb{E}(f(\boldsymbol{X})) + \frac{1}{2}\mathbb{V}f(\boldsymbol{X}) + O\left(\frac{nr^2}{d} + \phi^3 + e^{-\rho_1^2}\right)\right)$$
$$= \left(1 + O\left(\frac{nr^2}{d} + \frac{r^6 n \log^9 n}{d^{3/2}} + n^{-\Omega(\log n)}\right)\right) \frac{\pi^{n/2}}{|A|^{1/2}},$$

using (4.8) and the definition of ρ_1. □

4.2 Proof of Lemma 2.2

In this section we evaluate the integral over the region $U_n\big(r^{-1}\big) \setminus \mathcal{R}(\rho)$. The following technical bound will be useful: for any $t \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$|1 + \lambda(e^{it} - 1)| \le \exp\left(-\frac{1}{2}\left(1 - \frac{t^2}{12}\right)\lambda(1-\lambda)t^2\right). \tag{4.11}$$

Proof of Lemma 2.2. We will show that for any $\hat{\rho}$ satisfying $(2r)^{-1} \ge \hat{\rho} \ge d^{-1/2} \log n$, we have

$$\int_{U_n(2\hat{\rho})\setminus\mathcal{R}(\hat{\rho})} |F(\boldsymbol{\theta})|\, d\boldsymbol{\theta} = n^{-\Omega(\log n)} \frac{\pi^{n/2}}{|A|^{1/2}}. \tag{4.12}$$

Observe that

$$U_n\big((2r)^{-1}\big) \setminus \mathcal{R}\big(d^{-1/2}\log n\big) \subseteq \bigcup_{\ell=0}^{L-1} \left(U_n\Big(2^{\ell+1} d^{-1/2}\log n\Big) \setminus \mathcal{R}\Big(2^{\ell} d^{-1/2}\log n\Big)\right)$$

for $L = \big\lceil \log_2\big((2r)^{-1}/\big(d^{-1/2}\log n\big)\big)\big\rceil = O(r\log n)$, and that

$$U_n\big(r^{-1}\big) \setminus \mathcal{R}\big(d^{-1/2}\log n\big) \subseteq \big(U_n\big(r^{-1}\big) \setminus \mathcal{R}\big((2r)^{-1}\big)\big) \cup \big(U_n\big((2r)^{-1}\big) \setminus \mathcal{R}\big(d^{-1/2}\log n\big)\big).$$

This expresses the region of integration in the lemma statement as a union of regions of the form given in (4.12), and the result follows.

It remains to prove (4.12). Using (4.11), for any such $\hat{\rho}$

$$\int_{U_n(2\hat{\rho})\setminus\mathcal{R}(\hat{\rho})} |F(\boldsymbol{\theta})|\, d\boldsymbol{\theta} \le \int_{\mathbb{R}^n\setminus\mathcal{R}(\hat{\rho})} e^{-(1-r^2\hat{\rho}^2/3)\,\boldsymbol{\theta}^{\mathrm{t}} A\boldsymbol{\theta}}\, d\boldsymbol{\theta}.$$

Let T be as in Lemma 3.7 and note that $|T| = |A|^{-1/2}$. Then by Lemma 3.7 and (4.1) there exists a $\hat{\rho}_1 = \Theta\big(\hat{\rho} d^{1/2}\big)$ such that $T(U_n(\hat{\rho}_1)) \subseteq \mathcal{R}(\hat{\rho})$. By taking $\hat{\rho}_1' = \big(1 - r^2\hat{\rho}^2/3\big)^{1/2}\hat{\rho}_1$ we find that $\big(1 - r^2\hat{\rho}^2/3\big)^{-1/2} U_n\big(\hat{\rho}_1'\big) = U_n(\hat{\rho}_1)$ and hence

$$\big(1 - r^2\hat{\rho}^2/3\big)^{-1/2}\, T(U_n(\hat{\rho}_1')) = T(U_n(\hat{\rho}_1)) \subseteq \mathcal{R}(\hat{\rho}).$$

Therefore, substituting $\boldsymbol{\theta} = \big(1 - r^2\hat{\rho}^2/3\big)^{-1/2} T\boldsymbol{x}$ gives

$$\int_{\mathbb{R}^n\setminus\mathcal{R}(\hat{\rho})} e^{-(1-r^2\hat{\rho}^2/3)\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta}}\, d\boldsymbol{\theta} \le \frac{\big(1 - r^2\hat{\rho}^2/3\big)^{-n/2}}{|A|^{1/2}} \int_{\mathbb{R}^n\setminus U_n(\hat{\rho}_1')} e^{-\boldsymbol{x}^{\mathrm{t}}\boldsymbol{x}}\, d\boldsymbol{x}.$$

Note that $\big(1 - r^2\hat{\rho}^2/3\big)^{-n/2} = \exp\big(O\big(r^2\hat{\rho}^2 n\big)\big)$. In addition we have $\hat{\rho}_1' = \Theta(\hat{\rho}_1) = \Theta\big(\hat{\rho} d^{1/2}\big)$ and thus

$$\int_{\mathbb{R}^n\setminus U_n(\hat{\rho}_1')} e^{-\boldsymbol{x}^{\mathrm{t}}\boldsymbol{x}}\, d\boldsymbol{x} \le n\exp\big(-\Omega\big(\hat{\rho}_1^2\big)\big) = n\exp\big(-\Omega\big(\hat{\rho}^2 d\big)\big).$$

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We deduce that

$$\int_{\mathbb{R}^n\setminus\mathcal{R}(\hat\rho)} e^{-\left(1-r^2\hat\rho^2/3\right)\boldsymbol{\theta}^{\mathrm{t}}A\boldsymbol{\theta}}\,d\boldsymbol{\theta} \le n\,\exp\!\left(O\!\left(r^2\hat\rho^2 n\right)-\Omega\!\left(\hat\rho^2 d\right)\right)\frac{1}{|A|^{1/2}} = n^{-\Omega(\log n)}\frac{\pi^{n/2}}{|A|^{1/2}},$$

as $d \gg r^2 n$, by (1.11), and $\hat\rho^2 d = \Omega\left(\log^2 n\right)$. □

4.3 Proof of Lemma 2.3

In this section we complete the evaluation of the integral by examining the values in the region $U_n(\pi)\setminus\mathcal{B}$. For $x\in\mathbb{R}$, define $|x|_{2\pi} = \min_{k\in\mathbb{Z}}|x-2k\pi|$ and note that $|1+\lambda(e^{ix}-1)|$ depends only on $|x|_{2\pi}$.

Proof of Lemma 2.3. Let $\boldsymbol{\theta}\in U_n(\pi)\setminus\mathcal{B}$. First suppose that $|\theta_a-\theta_b|_{2\pi} > (2r)^{-1}$ for some $a,b\in[n]$. For any $W_1, W_2\in\mathcal{S}_r(n)$ that $W_1 \triangle W_2 = \{a,b\}$, we have

$$\left|\sum_{j\in W_1}\theta_j - \sum_{j\in W_2}\theta_j\right|_{2\pi} > (2r)^{-1}.$$

So $\left|\sum_{j\in W_1}\theta_j\right|_{2\pi} > (4r)^{-1}$ or $\left|\sum_{j\in W_2}\theta_j\right|_{2\pi} > (4r)^{-1}$, or both. In any case, by Lemma 3.3 and (4.11) we have

$$\left|1+\lambda_{W_1}\left(e^{i\sum_{j\in W_1}\theta_j}-1\right)\right|\cdot\left|1+\lambda_{W_2}\left(e^{i\sum_{j\in W_2}\theta_j}-1\right)\right| \le e^{-\Omega\left(\Lambda/r^2\right)}. \tag{4.13}$$

Note that there are exactly $\binom{n-2}{r-1} = \Theta\left(\binom{n-1}{r-1}\right)$ pairs W_1, W_2 such that $W_1\triangle W_2 = \{a,b\}$. Furthermore, every $W\in\mathcal{S}_r(n)$ is contained in at most one such pair. Then, multiplying inequalities (4.13) for all such pairs, we obtain

$$|F(\boldsymbol{\theta})| = \exp\left(-\Omega\left(\Lambda\binom{n-1}{r-1}/r^2\right)\right) \overset{(4.1)}{=} e^{-\Omega\left(d/r^2\right)}.$$

By (1.11), $\frac{d}{r^2}\gg nr^2\log n$, while by Lemma 3.6 and because $d<n^r$, we have $|A| = e^{O(n\log d)} = e^{O(nr\log n)}$. Therefore, the total contribution to the integral from this case is at most

$$(2\pi)^n e^{-\Omega\left(d/r^2\right)} = e^{-\omega\left(nr^2\log n\right)} = n^{-\omega(n)}\frac{\pi^{n/2}}{|A|^{1/2}}.$$

All remaining points $\boldsymbol{\theta}\in U_n(\pi)\setminus\mathcal{B}$ satisfy $|\theta_a-\theta_b|_{2\pi}\le(2r)^{-1}$ for all $a,b\in[n]$ and $\min_{j\in[n],\,k\in[r]}|\theta_j-\frac{2\pi k}{r}|_{2\pi} > (2r)^{-1}$. These two conditions imply that for any such $\boldsymbol{\theta}$ there exists $k\in[r]$ such that for all $j\in[n]$ we have

$$\frac{2\pi k}{r}+\frac{1}{2r} < \theta_j < \frac{2\pi(k+1)}{r}-\frac{1}{2r}.$$

Summing the above over any $W\in\mathcal{S}_r(n)$ implies that $\frac12 \le \left|\sum_{j\in W}\theta_j\right|_{2\pi}\le\pi$. Hence (4.11) implies that

$$|F(\boldsymbol{\theta})| = \exp\left(-\Omega(\Lambda)\binom{n}{r}\right).$$

Again, multiplying by $(2\pi)^n$ for an upper bound, we see that the contribution of all such points $\boldsymbol{\theta}$ to the integral is at most

$$(2\pi)^n\,\exp\left(-\Omega(\Lambda)\binom{n}{r}\right) = \exp\left(-\Omega\left(\Lambda\binom{n-1}{r-1}\right)\right) = n^{-\omega(n)}\frac{\pi^{n/2}}{|A|^{1/2}},$$

completing the proof. □

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5. Solving the beta-system

We first prove that the solution to (1.5) is unique if it exists.

Proof of Lemma 1.4. Suppose $\boldsymbol{\beta}' \neq \boldsymbol{\beta}''$ both satisfy (1.5). For $y \in \mathbb{R}$ and $W \in \mathcal{S}_r(n)$ define $\xi_W(y) := (1-y)\lambda_W(\boldsymbol{\beta}') + y\lambda_W(\boldsymbol{\beta}'')$. Consider the entropy function

$$S(y) := \sum_{W \in \mathcal{S}_r(n)} \left(\xi_W(y) \log \frac{1}{\xi_W(y)} + (1 - \xi_W(y)) \log \frac{1}{1 - \xi_W(y)} \right).$$

The derivative of $S(y)$ at $y = 0$ is

$$\begin{aligned} S'(0) &= \sum_{W \in \mathcal{S}_r(n)} \left(\lambda_W(\boldsymbol{\beta}') - \lambda_W(\boldsymbol{\beta}'')\right) \log \frac{\lambda_W(\boldsymbol{\beta}')}{1 - \lambda_W(\boldsymbol{\beta}')} \\ &\overset{(1.3)}{=} \sum_{W \in \mathcal{S}_r(n)} \left(\lambda_W(\boldsymbol{\beta}') - \lambda_W(\boldsymbol{\beta}'')\right) \sum_{j \in W} \beta'_j \\ &= \sum_{j=1}^{n} \beta'_j \sum_{W \ni j} \left(\lambda_W(\boldsymbol{\beta}') - \lambda_W(\boldsymbol{\beta}'')\right) \overset{(1.5)}{=} 0. \end{aligned}$$

Similarly, the derivative of $S(y)$ at $y = 1$ is $S'(1) = 0$.

On the other hand, $\boldsymbol{\beta}' \neq \boldsymbol{\beta}''$ implies that $\lambda_W(\boldsymbol{\beta}') \neq \lambda_W(\boldsymbol{\beta}'')$ for at least one $W \in \mathcal{S}_r(n)$. The second derivative of $S(y)$ equals

$$- \sum_{W \in \mathcal{S}_r(n)} \left(\lambda_W(\boldsymbol{\beta}'') - \lambda_W(\boldsymbol{\beta}')\right)^2 \xi_W(y)^{-1} \left(1 - \xi_W(y)\right)^{-1},$$

and hence is strictly negative when $\boldsymbol{\beta}' \neq \boldsymbol{\beta}''$. Therefore, $S(y)$ is strictly concave and cannot have more than one stationary point. This completes the proof. □

To prove Lemma 1.3 we will employ the following lemma from [9].

Lemma 5.1. [9, Lemma 7.8] *Let* $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, $\eta > 0$, *and* $U = \{\boldsymbol{\beta} \in \mathbb{R}^n : \|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\| \leq \eta \|\Psi(\boldsymbol{\beta}^{(0)})\|\}$ *and* $\boldsymbol{\beta}^{(0)} \in \mathbb{R}^n$, *where* $\|\cdot\|$ *is any vector norm in* $\mathbb{R}^n$. *Assume that*

$$\Psi \text{ is analytic in } U \qquad \text{and} \qquad \sup_{\boldsymbol{x} \in U} \|J^{-1}(\boldsymbol{\beta})\| < \eta,$$

where J *denotes the Jacobian matrix of* Ψ *and* $\|\cdot\|$ *stands for the induced matrix norm. Then there exists* $\boldsymbol{\beta}^* \in U$ *such that* $\Psi(\boldsymbol{\beta}^*) = \mathbf{0}$.

In connection with the system of (1.5), we consider $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Psi_j(\boldsymbol{\beta}) = \sum_{W \ni j} \lambda_W(\boldsymbol{\beta}) - d_j. \tag{5.1}$$

Clearly, Ψ is analytic in $\mathbb{R}^n$. Observe that

$$\frac{d}{dx}\left(\frac{e^{x+X}}{1+e^{x+X}}\right) = \frac{e^{x+X}}{1+e^{x+X}}\left(1 - \frac{e^{x+X}}{1+e^{x+X}}\right)$$

and thus $J(\boldsymbol{\beta}) = 2A(\boldsymbol{\beta})$, where $J(\boldsymbol{\beta})$ is the Jacobian matrix of $\Psi(\boldsymbol{\beta})$ and $A(\boldsymbol{\beta})$ is defined by (1.4). We start by bounding $\|J^{-1}(\boldsymbol{\beta})\|_\infty$ as required for Lemma 5.1.

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Lemma 5.2. *Let $\boldsymbol{\beta}^{(0)} \in \mathbb{R}^n$ and real numbers $\delta_1, \delta_2 \geq 0$ satisfy $\max_{j,k\in[n]} \left|\beta_j^{(0)} - \beta_k^{(0)}\right| \leq \delta_1/r$ and $e^{\delta_2}\lambda(\boldsymbol{\beta}^{(0)}) \leq 7/8$. Suppose that $n \geq 16e^{4\delta_1+8\delta_2}$. Then for any $\boldsymbol{\beta} \in \mathbb{R}^n$ such that $\|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\|_\infty \leq \delta_2/r$, we have*

$$\|J^{-1}(\boldsymbol{\beta})\|_\infty = \|(2A(\boldsymbol{\beta}))^{-1}\|_\infty \leq 2^8 C \frac{e^{36\delta_1+73\delta_2}}{\binom{n-1}{r-1}\lambda\left(\boldsymbol{\beta}^{(0)}\right)},$$

where C is the constant from Lemma 3.7.

Proof. Let $\boldsymbol{\beta} \in \mathbb{R}^n$ satisfy $\|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\|_\infty \leq \delta_2/r$. Then

$$\max_{j,k\in[n]} |\beta_j - \beta_k| \leq \max_{j,k\in[n]} \left|\beta_j^{(0)} - \beta_k^{(0)}\right| + 2\|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\|_\infty \leq \frac{\delta_1 + 2\delta_2}{r}.$$

Applying Lemma 3.7 for $\boldsymbol{\beta}$ implies for all sufficiently large n that

$$\|(2A(\boldsymbol{\beta}))^{-1}\|_\infty \leq C \frac{e^{35\delta_1+70\delta_2}}{\Lambda(\boldsymbol{\beta})\binom{n-1}{r-1}}.$$

By Lemma 3.2 and our assumptions we have $\lambda(\boldsymbol{\beta}) \leq e^{\delta_2}\lambda(\boldsymbol{\beta}^{(0)}) \leq 7/8$. Therefore, the conditions of Lemma 3.4 are satisfied and we have

$$\|(2A(\boldsymbol{\beta}))^{-1}\|_\infty \leq 2^8 C \frac{e^{36\delta_1+72\delta_2}}{\binom{n-1}{r-1}\lambda(\boldsymbol{\beta})}.$$

The result follows as $\lambda(\boldsymbol{\beta}) \geq e^{-\delta_2}\lambda(\boldsymbol{\beta}^{(0)})$ by Lemma 3.2. □

Further, we explain how to carefully choose U and $\boldsymbol{\beta}^{(0)}$ depending on whether d is small relative to $\binom{n-1}{r-1}$ or not.

5.1 Proof of Lemma 1.3(i)

Recalling (1.2), define

$$\boldsymbol{\beta}^{(0)} := \left(\frac{1}{r}\log\frac{\lambda}{1-\lambda}, \ldots, \frac{1}{r}\log\frac{\lambda}{1-\lambda}\right)$$

and note that $\|\Psi(\boldsymbol{\beta}^{(0)})\|_\infty = \max_{j\in[n]} |d - d_j|$. Define

$$U := \left\{\boldsymbol{\beta} : \|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\|_\infty \leq \eta\|\Psi\left(\boldsymbol{\beta}^{(0)}\right)\|_\infty\right\} = \left\{\boldsymbol{\beta} : \|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\|_\infty \leq \eta \max_{j\in[n]} |d - d_j|\right\},$$

where $\eta = 2^{10}C/d$ and C is the constant from Lemma 5.2. Since $\max_{j,k\in[n]} \left|\beta_j^{(0)} - \beta_k^{(0)}\right| = 0$, we set $\delta_1 := 0$. Now assume that Δ is sufficiently small, in particular $\Delta \leq \Delta_0 := \min\{(2^{17}C)^{-1}, 1\}$. Then for any $\boldsymbol{\beta} \in U$,

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\|_\infty \leq \eta d\left(e^{\Delta/r} - 1\right) \leq 2\eta d\Delta/r = \frac{2^{11}C}{d} \cdot \frac{d\Delta}{r} \leq \frac{1}{64r}. \tag{5.2}$$

Hence we define $\delta_2 := 1/64$. Since

$$\lambda\left(\boldsymbol{\beta}^{(0)}\right) = d\binom{n-1}{r-1}^{-1} \overset{(1.11)}{\leq} \frac{1}{2},$$

we deduce that

$$\lambda\left(\boldsymbol{\beta}^{(0)}\right) e^{\delta_2} \leq e^{1/64}\lambda\left(\boldsymbol{\beta}^{(0)}\right) \leq e^{1/64}/2 \leq \frac{7}{8}.$$

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Therefore, the conditions of Lemma 5.2 are met for δ_1 and δ_2 as above, and we deduce for every $\boldsymbol{\beta} \in U$,

$$\|J^{-1}(\boldsymbol{\beta})\|_\infty = \|(2A(\boldsymbol{\beta}))^{-1}\|_\infty \le 2^8 C \frac{e^{73\delta_2}}{\lambda(\boldsymbol{\beta}^{(0)})\binom{n-1}{r-1}} < \frac{2^{10}C}{d} = \eta.$$

Hence all the conditions of Lemma 5.1 hold, and applying this lemma shows that there exists a solution $\boldsymbol{\beta}^*$ to (1.5). Finally note that (5.2) implies that $\max_{j,k\in[n]} \left|\beta_j^* - \beta_k^*\right| = O(1/r)$, completing the proof.

5.2 Proof of Lemma 1.3(ii)

For part (ii), we define $\boldsymbol{\beta}^{(0)} = \left(\beta_1^{(0)}, \ldots, \beta_n^{(0)}\right)^{\mathrm{t}}$ by

$$\beta_j^{(0)} := \log d_j - \frac{1}{r}\log S,$$

where

$$S := \frac{n-r+1}{n} \sum_{W\in\mathcal{S}_{r-1}(n)} \prod_{k\in W} d_k.$$

Note that $\max_{j,k\in[n]} \left|\beta_j^{(0)} - \beta_k^{(0)}\right| = \max_{j,k\in[n]} |\log d_j - \log d_k| \le 2\Delta/r$. Define

$$U := \left\{\boldsymbol{\beta} : \|\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}\|_\infty \le \Delta/r\right\}.$$

For any $W \in \mathcal{S}_r(n)$, using the assumptions of the lemma we have

$$\lambda_W\left(\boldsymbol{\beta}^{(0)}\right) = \frac{\exp\left(\sum_{k\in W}\beta_k^{(0)}\right)}{1+\exp\left(\sum_{k\in W}\beta_k^{(0)}\right)} = O(1)\,\frac{\prod_{k\in W} d_k}{S} = O(1)\,\frac{d^r}{S}.$$

Furthermore,

$$S = \Omega\left(\frac{n-r+1}{n}\binom{n}{r-1}d^{r-1}\right) = \Omega\left(\binom{n-1}{r-1}d^{r-1}\right),$$

and so, using our assumption on rd,

$$\lambda_W\left(\boldsymbol{\beta}^{(0)}\right) = O\left(\frac{d}{\binom{n-1}{r-1}}\right) = o\left(r^{-1}\right).$$

It follows that for all $j \in [n]$, Lemma 3.3 implies that $\lambda_W(\boldsymbol{\beta}^{(0)}) = \Theta\left(\lambda\left(\boldsymbol{\beta}^{(0)}\right)\right)$, and hence

$$\begin{aligned}\lambda_W(\boldsymbol{\beta}^{(0)}) = \frac{\exp\left(\sum_{k\in W}\beta_k^{(0)}\right)}{1+\exp\left(\sum_{k\in W}\beta_k^{(0)}\right)} &= \left(1+O\left(\lambda\left(\boldsymbol{\beta}^{(0)}\right)\right)\right)\frac{\prod_{k\in W} d_k}{S}\\ &= \left(1+o\left(r^{-1}\right)\right)\frac{\prod_{k\in W} d_k}{S}.\end{aligned}$$

It follows that for all $j \in [n]$,

$$\sum_{W\ni j}\lambda_W\left(\boldsymbol{\beta}^{(0)}\right) = d_j\left(1+o\left(r^{-1}\right)\right)\frac{\sum_{W\ni j}\prod_{k\in W-j} d_k}{S}.$$

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Next, we observe that the quantity $\sum_{W\ni j}\prod_{k\in W-j} d_k$ depends insignificantly on j. Indeed, by our assumptions we have

$$\sum_{W\ni \ell}\prod_{k\in W-\ell} d_k = \Theta(1)\binom{n-1}{r-1}d^{r-1}$$

for $\ell \in \{j, j'\}$, while

$$\begin{aligned}\sum_{W\ni j}\prod_{k\in W-j} d_k - \sum_{W\ni j'}\prod_{k\in W-j'} d_k &= \sum_{\substack{W\in \mathcal{S}_{r-2}(n)\\ j,j'\notin W}} (d_{j'}-d_j)\prod_{k\in W} d_k\\ &\le \binom{n-2}{r-2} d\big(e^{\Delta/r}-e^{-\Delta/r}\big)\, d^{r-2}e^{O(1)}\\ &= O(n^{-1})\binom{n-1}{r-1}d^{r-1}.\end{aligned}$$

The last line uses the fact that for any $x\in\mathbb{R}$ we have

$$e^{x/r}-1\le \frac{e^x}{r}. \tag{5.3}$$

This shows that for any $j, j' \in [n]$,

$$\frac{\sum_{W\ni j}\prod_{k\in W-j} d_k}{\sum_{W\ni j'}\prod_{k\in W-j'} d_k} = 1+O(n^{-1}).$$

Observe also that

$$\frac{1}{n}\sum_{j\in[n]}\sum_{W\ni j}\prod_{k\in W-j} d_k = \frac{n-r+1}{n}\sum_{W\in\mathcal{S}_{r-1}(n)}\prod_{k\in W} d_k = S.$$

Combining the above and using the assumptions, we conclude that for all $j\in[n]$,

$$\sum_{W\ni j}\lambda_W(\boldsymbol{\beta}^{(0)}) = \big(1+o\big(r^{-1}\big)+O(n^{-1})\big)d_j = \big(1+o\big(r^{-1}\big)\big)d_j. \tag{5.4}$$

Taking the average of (5.4) implies that

$$\lambda\big(\boldsymbol{\beta}^{(0)}\big)\binom{n-1}{r-1} = \Theta(d) \quad \text{and} \quad \lambda\big(\boldsymbol{\beta}^{(0)}\big)\, e^{\Delta} = o(1).$$

Applying Lemma 5.2 with $\delta_1 := 2\Delta$ and $\delta_2 := \Delta$, we conclude that for every $\boldsymbol{\beta}\in U$,

$$\|J^{-1}(\boldsymbol{\beta})\|_\infty = \|(2A(\boldsymbol{\beta}))^{-1}\|_\infty = O\big(d^{-1}\big).$$

By the definition of Ψ and our assumptions on d_j, it follows from (5.4) that $\|\Psi(\boldsymbol{\beta}^{(0)})\|_\infty = o(d/r)$. Hence we can apply Lemma 5.1 with $\eta := \Delta(r\|\Psi(\boldsymbol{\beta}^{(0)}\|_\infty)^{-1} = \omega(d^{-1})$, completing the proof.

6. The near-regular case

In this section we will prove Theorem 1.5. As mentioned at the end of Section 1, we have omitted some of the calculations in this and the following section. These calculations can be verified using the identities in Section 9. It will be convenient for us to begin the analysis in the first quadrant. By assumption (1.13), Lemma 1.3(i) guarantees the existence of a solution $\boldsymbol{\beta}^* = (\beta_1^*, \ldots, \beta_n^*)$ which satisfies (1.8), and by Lemma 1.4 this solution is unique. Therefore we are justified in applying Theorem 1.1.

Next, recalling (1.2), define $\boldsymbol{\gamma}^* = (\gamma_1^*, \ldots, \gamma_n^*)$ by

$$\beta_j^* = \frac{1}{r}\log\frac{\lambda}{1-\lambda} + \gamma_j^*, \qquad \text{for } j \in [n].$$

In the regular case, $\boldsymbol{\beta}^*$ satisfies (1.5) when $\boldsymbol{\gamma}^* = \mathbf{0}$. For $W \in \mathcal{S}_r(n)$, define $\gamma_W^* := \sum_{j\in W}\gamma_j^*$. In addition, for $W \in \mathcal{S}_r(n)$ and $s \in \mathbb{N}$, define $\Gamma_s = \Gamma_s(W) := \sum_{j\in W}\delta_j^s$.

Lemma 6.1. *Under assumptions (1.7) and (1.13) in the first quadrant, there is a solution of (1.5) with*

$$\gamma_j^* = \frac{(n-1)\,\delta_j}{(1-\lambda)(n-r)d} - \frac{(n-2\lambda n-2r)n\,\delta_j^2}{2(1-\lambda)^2(n-r)^2d^2} + \frac{\delta_j^3}{3d^3} - \frac{rR_2}{2(n-r)^2d^2} + O\big(r^{-1}n^{-1}d^{-3/5}\big)$$

uniformly for $j \in [n]$.

Proof. Equations (1.5) can be written as $\Phi(\boldsymbol{\gamma}) = \boldsymbol{\delta}$, where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\Phi_j(\boldsymbol{\gamma}) := \lambda(1-\lambda)\sum_{W\ni j}\frac{e^{\gamma_W}-1}{1+\lambda(e^{\gamma_W}-1)}$$

for $j \in [n]$. Consider $\bar{\boldsymbol{\gamma}} = (\bar\gamma_1, \ldots, \bar\gamma_n)$ defined by

$$\bar\gamma_j := \frac{(n-1)\delta_j}{(1-\lambda)(n-r)d} - \frac{(n-2\lambda n-2r)n\,\delta_j^2}{2(1-\lambda)^2(n-r)^2d^2} + \frac{\delta_j^3}{3d^3} - \frac{rR_2}{2(n-r)^2d^2} + \frac{R_2}{2n(n-r)d^2}.$$

The function $L(x) = (e^x-1)/(1+\lambda(e^x-1))$ has bounded fifth derivative for $\lambda \in [0,1]$, $x \in [-1,1]$, so by Taylor's theorem we have in that domain that

$$L(x) = x + \left(\frac{1}{2}-\lambda\right)x^2 + \left(\frac{1}{6}-\lambda+\lambda^2\right)x^3 + \left(\frac{1}{24}-\frac{7}{12}\lambda+\frac{3}{2}\lambda^2-\lambda^3\right)x^4 + O\big(|x|^5\big). \tag{6.1}$$

For $W \in \mathcal{S}_r(n)$, define $\bar\gamma_W := \sum_{j\in W}\bar\gamma_j$. Now

$$\bar\gamma_W = O\Big(d^{-1}\sum_{j\in W}\delta_j\Big) = O\big(\delta_{\max} r d^{-1}\big),$$

which implies that $(\bar\gamma_W)^5 = O\big(r^{-1}n^{-1}d^{-3/5}\big)$. Therefore, from (6.1) we have

$$\begin{aligned} L(\bar\gamma_W) = {} & \frac{(n-1)\Gamma_1}{(1-\lambda)(n-r)d} + \frac{(n^2-2\lambda n^2-2n+1)\Gamma_1^2}{2(1-\lambda)^2(n-r)^2d^2} + \frac{(n-3)n^2\,\Gamma_1^3}{6(n-r)^3d^3} \\ & + \frac{n^4\,\Gamma_1^4}{24(n-r)^4d^4} - \frac{n(n-2\lambda n-2r)\Gamma_2}{2(1-\lambda)^2(n-r)^2d^2} - \frac{(n-2r)n^2\,\Gamma_1\Gamma_2}{2(n-r)^3d^3} + \frac{\Gamma_3}{3d^3} \\ & - \frac{r(rn-n+r)\,R_2}{2(n-r)^2nd^2} - \frac{r^2n\,R_2\Gamma_1}{2(n-r)^3d^3} + O\big(r^{-1}n^{-1}d^{-3/5}\big). \end{aligned} \tag{6.2}$$

Summing (6.2) over the $\binom{n-1}{r-1} = d/\lambda$ sets W that include j, for each j, we verify that

$$\|\Phi(\bar{\boldsymbol{\gamma}}) - \boldsymbol{\delta}\|_\infty = O\big(r^{-1}n^{-1}d^{2/5}\big). \tag{6.3}$$

These calculations rely heavily on the identities given in Section 9.2.

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Define $C' := 2^{10}C$, where C is the constant from Lemma 5.2, and let

$$U(C') = \left\{ \boldsymbol{x} : \|\boldsymbol{x} - \bar{\boldsymbol{\gamma}}\|_\infty \leq \frac{C'}{d} \|\boldsymbol{\Phi}(\bar{\boldsymbol{\gamma}}) - \boldsymbol{\delta}\|_\infty \right\}.$$

Define the function $\nu : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\nu(\boldsymbol{x}) = \frac{1}{r} \log \frac{\lambda}{1-\lambda} (1, \ldots, 1)^{\mathrm{t}} + \boldsymbol{x}.$$

Let Ψ be the function defined in (5.1). Then for any $\boldsymbol{x} \in \mathbb{R}^n$ we have $\Psi(\nu(\boldsymbol{x})) = \boldsymbol{\Phi}(\boldsymbol{x}) - \boldsymbol{\delta}$. In particular this implies that $J_{\boldsymbol{\Phi}}^{-1}(\boldsymbol{x}) = J_{\Psi}^{-1}(\nu(\boldsymbol{x}))$ where $J_{\boldsymbol{\Phi}}(\boldsymbol{x})$ and $J_{\Psi}(\nu(\boldsymbol{x}))$ denote the Jacobians of $\boldsymbol{\Phi}(\boldsymbol{x})$ and $\Psi(\nu(\boldsymbol{x}))$ respectively.

We wish to apply Lemma 5.2. Then

$$\delta_1 := r \max_{j,k \in [n]} |\nu(\bar{\gamma})_j - \nu(\bar{\gamma})_k| = r \max_{j,k \in [n]} |\bar{\gamma}_j - \bar{\gamma}_k| = o(1).$$

Next, using (6.3), we have that

$$\delta_2 := \frac{r\,C'}{d} \|\boldsymbol{\Phi}(\bar{\boldsymbol{\gamma}}) - \boldsymbol{\delta}\|_\infty = o(1).$$

Finally, since $\lambda(\nu(\mathbf{0})) = \lambda \leq 1/2$ and $\max_{j \in [n]} |\bar{\gamma}_j| = o(1/r)$, Lemma 3.2 implies that

$$e^{\delta_2}\, \lambda(\nu(\bar{\gamma})) = (1 + o(1))\lambda \leq \frac{7}{8}.$$

Hence Lemma 5.2 implies that for every $\boldsymbol{x} \in U(C')$, we have

$$\|J_{\boldsymbol{\Phi}}^{-1}(\boldsymbol{x})\|_\infty = \|J_{\Psi}^{-1}(\nu(\boldsymbol{x}))\|_\infty \leq \frac{2^8 C\, e^{o(1)}}{(1+o(1))\, d} < \frac{C'}{d}.$$

Therefore, by Lemma 5.1 there exists $\boldsymbol{x} \in U(C')$ such that $\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{\delta}$. Setting $\boldsymbol{\gamma}^* = \boldsymbol{x}$ proves the lemma, since $\|\boldsymbol{x} - \bar{\boldsymbol{\gamma}}\|_\infty = O(r^{-1}n^{-1}d^{-3/5})$ and the last term of $\bar{\gamma}_j$ is $O\big(r^{-1}n^{-1}d^{-3/5}\big)$. □

Now we can calculate the values of the quantities that appear in Theorem 1.1.

Lemma 6.2. *Under assumptions (1.7) and (1.13), we have in the first quadrant that*

$$\prod_{W \in \mathcal{S}_r(n)} \lambda_W^{\lambda_W} (1-\lambda_W)^{1-\lambda_W}$$
$$= \big(\lambda^\lambda (1-\lambda)^{1-\lambda}\big)^{\binom{n}{r}} \exp\left(\frac{(n-1)\,R_2}{2(1-\lambda)(n-r)d} - \frac{(1-2\lambda)\,R_3}{6(1-\lambda)^2 d^2} + \frac{R_4}{12 d^3} + O\big(\delta_{\max}\, d^{-3/5}\big) \right).$$

Proof. Define z_W by $\lambda_W = \lambda(1 + z_W)$ and

$$\begin{aligned}
\eta(z) &= \log \frac{(\lambda(1+z))^{\lambda(1+z)}(1-\lambda(1+z))^{1-\lambda(1+z)}}{\lambda^\lambda(1-\lambda)^{1-\lambda}} - \lambda z \log \frac{\lambda}{1-\lambda} \\
&= \log\left((1+z)^{\lambda(1+z)} \left(1 - \frac{\lambda z}{1-\lambda}\right)^{1-\lambda(1+z)} \right) \\
&= \sum_{j=2}^{\infty} \left(\left(\frac{\lambda}{1-\lambda}\right)^{j-1} + (-1)^j \right) \frac{\lambda}{(j-1)j}\, z^j.
\end{aligned} \tag{6.4}$$

Recall that $\sum_{W \in \mathcal{S}_r(n)} z_W = 0$, therefore,

$$\prod_{W \in \mathcal{S}_r(n)} \lambda_W^{\lambda_W}(1-\lambda_W)^{1-\lambda_W} = \big(\lambda^\lambda(1-\lambda)^{1-\lambda}\big)^{\binom{n}{r}} \exp\left(\sum_{W \in \mathcal{S}_r(n)} \eta(z_W) \right). \tag{6.5}$$

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Lemma 6.1 implies that $\gamma_W^* = \bar\gamma_W + O\big(n^{-1}d^{-3/5}\big)$. Recalling (6.1), this implies that $L(\gamma_W^*) = L(\bar\gamma_W) + O\big(n^{-1}d^{-3/5}\big)$, as $\gamma_W^* = o(1)$. Using (6.2), we have

$$\begin{aligned} z_W &= \frac{(1-\lambda)(e^{\gamma_W^*}-1)}{1+\lambda(e^{\gamma_W^*}-1)} = (1-\lambda)L_W(\gamma_W^*) \\ &= \frac{(n-1)\Gamma_1}{(n-r)d} + \frac{n(n-2\lambda n-2)\Gamma_1^2}{2(1-\lambda)(n-r)^2d^2} + \frac{n^3\,\Gamma_1^3}{6(n-r)^3d^3} - \frac{(n-2\lambda n-2r)n\,\Gamma_2}{2(1-\lambda)(n-r)^2d^2} \\ &\quad - \frac{\Gamma_1\Gamma_2}{2d^3} + \frac{\Gamma_3}{3d^3} - \frac{r^2\,R_2}{2(n-r)^2d^2} + O\big(n^{-1}d^{-3/5}\big). \end{aligned} \tag{6.6}$$

The coefficients of the Taylor expansion of $\eta(z)$ are uniformly $O(\lambda)$, as shown in (6.4). Also note that $z_W = O(\delta_{\max} rd^{-1}) = O(d^{-1/5})$. This gives

$$\begin{aligned} \eta(z_W) &= \frac{\lambda(n-1)^2\,\Gamma_1^2}{2(1-\lambda)(n-r)^2d^2} + \frac{\lambda(n-2\lambda n-3)n^2\,\Gamma_1^3}{3(1-\lambda)^2(n-r)^3d^3} + \frac{\lambda n^4\,\Gamma_1^4}{8(n-r)^4d^4} + \frac{\lambda\,\Gamma_2^2}{8d^4} + \frac{\lambda\,\Gamma_1\Gamma_3}{3d^4} \\ &\quad - \frac{\lambda(n-2\lambda n-2r)n^2\,\Gamma_1\Gamma_2}{2(1-\lambda)^2(n-r)^3d^3} - \frac{\lambda\,\Gamma_1^2\Gamma_2}{2d^4} - \frac{\lambda r^2 n\,R_2\,\Gamma_1}{2(n-r)^3d^3} + O\big(\lambda r\delta_{\max} n^{-1}d^{-8/5}\big). \end{aligned}$$

Using the identities in Section 9.1, we can sum over all $W \in \mathcal{S}_r(n)$:

$$\sum_{W\in\mathcal{S}_r(n)} \eta(z_W) = \frac{(n-1)\,R_2}{2(1-\lambda)(n-r)d} - \frac{(1-2\lambda)\,R_3}{6(1-\lambda)^2d^2} + \frac{R_4}{12d^3} + O\big(\delta_{\max} d^{-3/5}\big). \tag{6.7}$$

The lemma now follows from (6.5) and (6.7). □

Let A_0 be the matrix A in the case that $\boldsymbol{d} = (d, d, \dots, d)$. That is,

$$A_0 = \frac{(1-\lambda)(n-r)d}{2(n-1)}\,I + \frac{(1-\lambda)(r-1)d}{2(n-1)}\,J.$$

Then

$$\begin{aligned} A_0^{-1} &= \frac{2(n-1)}{(1-\lambda)(n-r)d}\,I - \frac{2(r-1)}{(1-\lambda)r(n-r)d}\,J, \\ |A_0| &= \frac{(1-\lambda)^n r(n-r)^{n-1}d^n}{2^n(n-1)^{n-1}} = \frac{r\,Q^n}{2^n(n-r)(n-1)^{n-1}}, \end{aligned} \tag{6.8}$$

where the determinant follows from (3.1).

Lemma 6.3. *Under assumptions (1.7) and (1.13), we have in the first quadrant that*

$$|A| = |A_0|\,\exp\Big(-\frac{R_2}{2d^2} + O\big(\delta_{\max} d^{-3/5}\big)\Big).$$

Proof. Define the matrix E by $A = A_0 + E$. Then

$$\begin{aligned} A &= A_0(I-D)^{-1}(I+M), \quad \text{where} \\ D &:= \operatorname{diag}\Big(\frac{(1-2\lambda)\delta_1}{(1-\lambda)d}, \dots, \frac{(1-2\lambda)\delta_n}{(1-\lambda)d}\Big) \quad \text{and} \\ M &:= -D + (I-D)A_0^{-1}E. \end{aligned}$$

For $W \in \mathcal{S}_r(n)$ we have $\lambda_W = \lambda(1+z_W)$, where z_W is given by (6.6). This gives

$$\frac{1}{2}\lambda_W(1-\lambda_W) = \frac{1}{2}\lambda(1-\lambda) + \frac{\lambda(1-2\lambda)\Gamma_1}{2d} + \frac{\lambda\,\Gamma_1^2}{4d^2} - \frac{\lambda\,\Gamma_2}{4d^2} + O\big(\lambda\delta_{\max} n^{-1}d^{-3/5}\big).$$

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Summing over $W \ni j$ and $W \ni j, k$, using Sections 9.2 and 9.3, we have $E = (e_{jk})$, where

$$e_{jk} = \begin{cases} \dfrac{1}{2}(1-2\lambda)\delta_j + O\big(\delta_{\max} n^{-1} d^{2/5}\big), & \text{if } j = k; \\[2ex] \dfrac{(1-2\lambda)(r-1)(\delta_j+\delta_k)}{2n} + \dfrac{(r-1)\delta_j\delta_k}{2nd} + O\big(\delta_{\max} r n^{-2} d^{2/5}\big), & \text{if } j \neq k. \end{cases}$$

This implies that $A_0^{-1}E = (e'_{jk})$, where

$$e'_{jk} = \begin{cases} \dfrac{(1-2\lambda)\delta_j}{(1-\lambda)d} + O\big(\delta_{\max} n^{-1} d^{-3/5}\big), & \text{if } j = k; \\[2ex] \dfrac{(1-2\lambda)(r-1)\delta_j}{(1-\lambda)nd} + \dfrac{(r-1)\delta_j\delta_k}{nd^2} + O\big(\delta_{\max} r n^{-2} d^{-3/5}\big), & \text{if } j \neq k. \end{cases}$$

Finally, we have $M = (m_{jk})$, where

$$m_{jk} = \begin{cases} -\dfrac{\delta_j^2}{d^2} + O\big(\delta_{\max} n^{-1} d^{-3/5}\big), & \text{if } j = k; \\[2ex] \dfrac{(1-2\lambda)(r-1)\delta_j}{(1-\lambda)nd} - \dfrac{(r-1)\delta_j^2}{nd^2} + \dfrac{(r-1)\delta_j\delta_k}{nd^2} + O\big(\delta_{\max} r n^{-2} d^{-3/5}\big), & \text{if } j \neq k. \end{cases}$$

To complete the proof, note that

$$|(I-D)^{-1}| = \prod_{j=1}^{n} \left(1 - \frac{(1-2\lambda)\,\delta_j}{(1-\lambda)d}\right)^{-1} = \exp\left(\frac{R_2}{2d^2} + O\big(\delta_{\max} d^{-3/5}\big)\right)$$

and, since $\|M\|_2 \le \sqrt{\|M\|_1 \|M\|_\infty} = o(1)$,

$$\begin{aligned} |I+M| &= \prod_{j=1}^{n} (1+\mu_j) = \exp\left(\sum_{j=1}^{n} \big(\mu_j + O(|\mu_j|^2)\big)\right) = \exp\big(\mathrm{tr} M + O(\|M\|_F^2)\big) \\ &= \exp\left(-\frac{R_2}{d^2} + O\big(\delta_{\max} d^{-3/5}\big)\right), \end{aligned}$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of M and $\|M\|_F$ is the Frobenius norm. The penultimate equality follows by [30, equation (3.71)], which states that $\sum_{j=1}^{n} |\mu_j|^2 \le \|M\|_F^2$. □

Corollary 6.4. *Under assumptions (1.7) and (1.13), we have in the first quadrant that*

$$\begin{aligned} H_r(\boldsymbol{d}) = {} & \frac{r}{2^n\, \pi^{n/2}\, |A_0|^{1/2}} \big(\lambda^\lambda (1-\lambda)^{1-\lambda}\big)^{-\binom{n}{r}} \\ & \times \exp\left(-\frac{(n-1)\,R_2}{2(1-\lambda)(n-r)d} + \frac{R_2}{4d^2} + \frac{(1-2\lambda)\,R_3}{6(1-\lambda)^2 d^2} - \frac{R_4}{12d^3} + O(\bar{\varepsilon})\right), \end{aligned}$$

where $\bar{\varepsilon} = \varepsilon + \delta_{\max} d^{-3/5}$ and $|A_0|$ is given by (6.8).

Proof. This follows by substituting Lemmas 6.2 and 6.3 into Theorem 1.1. □

Finally, Theorem 1.5 removes the assumption of being in the first quadrant.

Proof of Theorem 1.5. Since the formula is invariant under the symmetries and matches Corollary 6.4 within the error term in the first quadrant, it is true in all quadrants. To see this, observe that under either of our two symmetries, R_3 becomes $-R_3$ and $(1-2\lambda)(n-2r)$ becomes $-(1-2\lambda)(n-2r)$. □

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7. Degrees of random uniform hypergraphs

We now show how to apply the results of Section 6 to analyse the degree sequence of a random uniform hypergraph with a given number of edges. Define $B(K,x)=\binom{K}{\lambda K+x}$ where K, $\lambda K+x$ are integers. The following lemma is a consequence of Stirling's expansion for the gamma function.

Lemma 7.1. *Let K,x,λ be functions of n such that, as $n\to\infty$, $\lambda\in(0,1)$, $\lambda(1-\lambda)K\to\infty$ and $x=o\big(\lambda(1-\lambda)K\big)$. Then*

$$\begin{aligned}B(K,x)&=\frac{\lambda^{-\lambda K-x-1/2}\,(1-\lambda)^{-(1-\lambda)K+x-1/2}}{\sqrt{2\pi K}}\\&\quad\times\exp\biggl(-\frac{x^2}{2\lambda(1-\lambda)K}-\frac{(1-2\lambda)x}{2\lambda(1-\lambda)K}-\frac{1-\lambda+\lambda^2}{12\lambda(1-\lambda)K}+\frac{(1-2\lambda)x^3}{6\lambda^2(1-\lambda)^2K^2}\\&\quad+\frac{(1-2\lambda+2\lambda^2)x^2}{4\lambda^2(1-\lambda)^2K^2}+\frac{(1-2\lambda)x}{12\lambda^2(1-\lambda)^2K^2}-\frac{(1-3\lambda+3\lambda^2)x^4}{12\lambda^3(1-\lambda)^3K^3}\\&\quad+O\Bigl(\frac{|x|^3+1}{\lambda^3(1-\lambda)^3K^3}+\frac{|x|^5}{\lambda^4(1-\lambda)^4K^4}\Bigr)\biggr).\end{aligned}$$

Proof. This follows from Stirling's expansion for the factorial, which we use in the form

$$N!=\sqrt{2\pi}\,N^{N+1/2}e^{-N}\,\exp\Bigl(\frac{1}{12N}+O(N^{-3})\Bigr).$$

From this we obtain

$$\begin{aligned}B(K,x)&=\frac{K^{K+1/2}}{\sqrt{2\pi}\,(\lambda K+x)^{\lambda K+x+1/2}((1-\lambda)K-x)^{(1-\lambda)K-x+1/2}}\\&\quad\times\exp\Bigl(\frac{1}{12K}-\frac{1}{12(\lambda K+x)}-\frac{1}{12((1-\lambda)K-x)}+O\Bigl(\frac{1}{\lambda^3(1-\lambda)^3K^3}\Bigr)\Bigr).\end{aligned}$$

Now write

$$(\lambda K+x)^{\lambda K+x+1/2}=(\lambda K)^{\lambda K+x+1/2}\exp\Bigl(\Bigl(K+x+\frac12\Bigr)\log\Bigl(1+\frac{x}{\lambda K}\Bigr)\Bigr)$$

and similarly for $((1-\lambda)K-x)^{(1-\lambda)K-x+1/2}$. Expanding the logarithms gives the desired result. □

Proof of Theorem 1.7. For some $p\in(0,1)$, let $X_1,\dots,X_n$ be iid random variables with the binomial distribution $\mathrm{Bin}\bigl(\binom{n-1}{r-1},p\bigr)$. Then $\mathcal{B}_r(n,m)$ is the distribution of $(X_1,\dots,X_n)$ conditioned on the sum being nd. Since the sum has distribution $\mathrm{Bin}\bigl(n\binom{n-1}{r-1},p\bigr)$, we find that the conditional probability is independent of p:

$$\mathbb{P}_{\mathcal{B}_r(n,m)}(\boldsymbol{d})=\binom{n\binom{n-1}{r-1}}{nd}^{-1}\prod_{j=1}^{n}\binom{\binom{n-1}{r-1}}{d_j}.$$

Consequently,

$$\frac{\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d})}{\mathbb{P}_{\mathcal{B}_r(n,m)}(\boldsymbol{d})}=\frac{B\bigl(n\binom{n-1}{r-1},0\bigr)H_r(\boldsymbol{d})}{B\bigl(\binom{n}{r},0\bigr)\prod_{j=1}^{n}B\bigl(\binom{n-1}{r-1},\delta_j\bigr)}.$$

Now use Theorem 1.5 for $H_r(\boldsymbol{d})$ and Lemma 7.1 for the other factors. □

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Let $Z_1, \ldots, Z_n$ be iid random variables having the hypergeometric distribution with parameters $\binom{n}{r}$, m, $\binom{n-1}{r-1}$, where $m = e(\boldsymbol{d})$. That is,

$$\mathbb{P}(Z_j = k) = \binom{\binom{n}{r}}{m}^{-1} \binom{\binom{n-1}{r-1}}{k} \binom{\binom{n}{r} - \binom{n-1}{r-1}}{m-k}. \tag{7.1}$$

Note that Z_1 has precisely the distribution of the degree of one vertex in a uniformly random r-uniform hypergraph with n vertices and m edges. Now let $\mathcal{T}_r(n, m)$ be the distribution of $Z_1, \ldots, Z_n$ when conditioned on having sum nd. If $P := \mathbb{P}(Z_1 + \cdots + Z_n = nd)$, for which there seems to be no closed formula, we have

$$\mathbb{P}_{\mathcal{T}_r(n,m)}(\boldsymbol{d}) = P^{-1} \binom{\binom{n}{r}}{m}^{-n} \prod_{j=1}^{n} \left(\binom{\binom{n-1}{r-1}}{d_j} \binom{\binom{n}{r} - \binom{n-1}{r-1}}{m - d_j} \right). \tag{7.2}$$

Lemma 7.2. *Let $Z_1, \ldots, Z_n$ be independent hypergeometric variables with distribution given by (7.1) and let $X_1, \ldots, X_n$ be the same conditioned on $\sum_{j=1}^n Z_j = nd$. Then*

(a) Each Z_j and X_j has mean d. Also, Z_j has variance

$$\sigma^2 = \frac{(1-\lambda)(n-r)d^2}{nd - \lambda r} = \frac{Q}{n} \left(1 - \binom{n}{r}^{-1} \right)^{-1}. \tag{7.3}$$

(b) For $t \geq 0$, we have for any j that

$$\mathbb{P}(|Z_j - d| \geq t) \leq 2 \exp\left(-\frac{t^2}{2(d + t/3)} \right) \leq \begin{cases} 2 \exp\left(-\frac{t^2}{4d} \right), & 0 \leq t \leq 3d; \\ 2e^{-3t/4}, & t \geq 3d. \end{cases}$$

(c) If $nd + y$ is an integer in $[0, mn]$, then

$$\mathbb{P}\Big(\sum_{j=1}^{n} Z_j = nd + y \Big) = \frac{1}{\sigma \sqrt{2\pi n}} \exp\left(-\frac{y^2}{2n\sigma^2} \right) + O\big(n^{-1}\sigma^{-2}\big),$$

where the implicit constant in the error term is bounded absolutely.

(d) For every nonnegative integer y, $\mathbb{P}(X_1 = y) = C(y)\mathbb{P}(Z_1 = y)$, where uniformly

$$C(y) = \frac{\mathbb{P}\big(\sum_{j=2}^n Z_j = nd - y\big)}{\mathbb{P}\big(\sum_{j=1}^n Z_j = nd\big)} = \big(1 + O(n^{-1})\big) \exp\left(-\frac{(y-d)^2}{2(n-1)\sigma^2} \right) + O\Big(n^{-1/2}\sigma^{-1}\Big).$$

(e) If $\sigma^2 \geq 1$ then for $t > 0$,

$$\begin{aligned} \mathbb{E} \min\big\{(Z_1 - d)^2, t^2\big\} &= \sigma^2 + O\big(e^{-t^2/(4d)} d + e^{-9d/4} d\big), \\ \mathbb{E} \min\big\{(X_1 - d)^2, t^2\big\} &= \big(1 + O(n^{-1})\big)\, \sigma^2 + O\big(e^{-t^2/(4d)} d + e^{-9d/4} d\big). \end{aligned}$$

Proof. Part (a) is standard theory of the hypergeometric distribution. For parts (b) and (c), we note that Vatutin and Michailov [28] proved that Z_j can be expressed as the sum of m independent Bernoulli random variables (generally with different means). Inequality (b) is now standard (see [14, Theorem 2.1]), while (c) was proved by Fountoulakis, Kang and Makai [8, Theorem 6.3].

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For part (d), the standard formula for conditional probability implies that the expression for $\mathbb{P}(X_1 = y)$ holds with $C(y) = \frac{\mathbb{P}\left(\sum_{j=2}^{n} Z_j = nd - y\right)}{\mathbb{P}\left(\sum_{j=1}^{n} Z_j = nd\right)}$. Then by part (c) we have

$$\mathbb{P}\left(\sum_{j=2}^{n} Z_j = nd - y\right) = \frac{1}{\sigma\sqrt{2\pi(n-1)}} \exp\left(\frac{-(y-d)^2}{2(n-1)\sigma^2}\right) + O\left(n^{-1}\sigma^{-2}\right),$$
$$\mathbb{P}\left(\sum_{j=1}^{n} Z_j = nd\right) = \frac{1}{\sigma\sqrt{2\pi n}}\left(1 + O\left(n^{-1/2}\sigma^{-1}\right)\right),$$

and dividing the first expression by the second gives the stated approximation for $C(y)$.

For (e), we have

$$\mathbb{E}\min\left\{(Z_1 - d)^2, t^2\right\} = \sigma^2 - \sum_{|\ell|>t} \left(\ell^2 - t^2\right) \mathbb{P}(Z_1 = d + \ell),$$

where the sum is restricted to integer $d + \ell$. We will consider the upper tail, noting that the lower tail is much the same:

$$\begin{aligned}\sum_{\ell>t} \left(\ell^2 - t^2\right) \mathbb{P}(Z_1 = d + \ell) &= \sum_{\ell>t} (\ell^2 - t^2)\big(\mathbb{P}(Z_1 \geq d + \ell) - \mathbb{P}(Z_1 \geq d + \ell + 1)\big) \\ &\leq (2t+1)\mathbb{P}(Z_1 \geq d + t) + \sum_{\ell>t} (2\ell + 1)\, \mathbb{P}(Z_1 \geq d + \ell + 1).\end{aligned}$$

Now we can use the first case of part (b) to obtain the bound $O(e^{-t^2/(4d)}d)$ and the second case to obtain the bound $O(e^{-9d/4}d)$.

For the second part of (e), we have

$$\begin{aligned}\mathbb{E}\left((X_1 - d)^2\right) &= \sigma^2 + \sum_j \left(C(j) - 1\right) \mathbb{P}(Z_1 = j)\, (j - d)^2 \\ &= \sigma^2 + \sum_j \left(\exp\left(-\frac{(j-d)^2}{2(n-1)\sigma^2}\right) - 1 + O(1/n)\right) \mathbb{P}(Z_1 = j)\, (j-d)^2 \\ &= \sigma^2\left(1 + O(n^{-1})\right) + O\left(\frac{\mathbb{E}\left((Z_1 - d)^4\right)}{n\sigma^2}\right).\end{aligned}$$

Since $\sigma^2 \geq 1$, the fourth central moment of Z_1 satisfies $\mathbb{E}\left((Z_1 - d)^4\right) = O(\sigma^4)$, as follows from the exact expression given in [16, equation (5.55)]. Therefore

$$\mathbb{E}\left((X_1 - d)^2\right) = \sigma^2\left(1 + O\left(n^{-1}\right)\right).$$

Then the effect of truncation at t can be bounded as before, using the fact that $C(\ell) = O(1)$. $\square$

Proof of Theorem 1.8. From the definitions of $\mathcal{D}_r(n, m)$ and (7.2), we have

$$\frac{\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d})}{\mathbb{P}_{\mathcal{T}_r(n,m)}(\boldsymbol{d})} = \frac{B\left(\binom{n}{r}, 0\right)^{n-1} P\, H_r(\boldsymbol{d})}{\prod_{j=1}^{n}\left(B\left(\binom{n-1}{r-1}, \delta_j\right) B\left(\binom{n}{r} - \binom{n-1}{r-1}, -\delta_j\right)\right)}.$$

Now use Theorem 1.5 for $H_r(\boldsymbol{d})$, Lemma 7.2(c) for P, and Lemma 7.1 for the other factors. $\square$

For the proof of Theorem 1.10 we need a concentration lemma.

Lemma 7.3. *Let $f(x_1, \ldots, x_K) : \{0,1\}^K \to \mathbb{R}$ be a function such that $|f(\boldsymbol{x}) - f(\boldsymbol{x}')| \le a$ whenever $\boldsymbol{x}, \boldsymbol{x}'$ differ in only one coordinate. Let $\boldsymbol{Z} = (Z_1, \ldots, Z_K)$ be independent Bernoulli variables (not necessarily identical), conditioned on having constant sum S. Then, for any $t \ge 0$,*

$$\mathbb{P}\big(|f(\boldsymbol{Z}) - \mathbb{E}f(\boldsymbol{Z})| > t\big) \le 2\exp\left(-\frac{t^2}{8a^2 S}\right).$$

Proof. According to Pemantle and Peres [25, Example 5.4], the measure defined by independent Bernoulli variables conditioned on a fixed sum has the "strong Rayleigh" property. The proof is completed by applying [25, Theorem 3.1]. □

Proof of Theorem 1.10. Probabilities in the hypergeometric distribution are symmetric under the two operations (that is, replacing r by $n-r$, or replacing m by $\binom{n}{r} - m$). Since the error term given in the theorem is also symmetric under these operations, it suffices to assume that $(r, \boldsymbol{d})$ belongs to the first quadrant.

Define

$$R_2(\boldsymbol{d}) := \sum_{j=1}^{n} (d_j - d)^2 \quad \text{and} \quad R_2'(\boldsymbol{d}) := \sum_{j=1}^{n} \min\big\{(d_j - d)^2, d\log^2 n\big\},$$

and

$$\mathfrak{W} := \left\{\boldsymbol{d} : \delta_{\max} \le d^{1/2}\log n \text{ and } |R_2(\boldsymbol{d}) - n\sigma^2| \le n^{1/2}\sigma^2\log^2 n\right\}.$$

Let $Z_1, \ldots, Z_n$ be iid random variables with distribution (7.1). The distribution $\mathcal{T}_r(n,m)$ is that of $(Z_1, \ldots, Z_n)$ conditioned on $\sum_{j=1}^{n} Z_j = nd$.

By the union bound, we have

$$\mathbb{P}_{\mathcal{T}_r(n,m)}\big(|R_2(\boldsymbol{d}) - n\sigma^2| > n^{1/2}\sigma^2\log^2 n\big) \le \mathbb{P}_{\mathcal{T}_r(n,m)}\big(R_2(\boldsymbol{d}) \ne R_2'(\boldsymbol{d})\big) + \mathbb{P}_{\mathcal{T}_r(n,m)}\Big(|R_2'(\boldsymbol{d}) - \mathbb{E}R_2'(\boldsymbol{d})| > n^{1/2}\sigma^2\log^2 n - |n\sigma^2 - \mathbb{E}R_2'(\boldsymbol{d})|\Big).$$

Since always $C(i) = O(1)$, Lemma 7.2(b,d) and the union bound give

$$\mathbb{P}_{\mathcal{T}_r(n,m)}\big(R_2(\boldsymbol{d}) \ne R_2'(\boldsymbol{d})\big) \le n \sum_{i:|i-d|>d^{1/2}\log n} \mathbb{P}_{\mathcal{T}_r(n,m)}(Z_1 = i)\, C(i) = n^{-\Omega(\log n)}.$$

Next, note that in $\mathcal{T}_r(n,m)$ we have $|n\sigma^2 - \mathbb{E}R_2'(\boldsymbol{d})| = O(\sigma^2) = O(d)$ by Lemma 7.2(e); for later use note that this only relies on the condition $\delta_{\max} \le d^{1/2}\log n$. Recall that each Z_j is the sum of m independent Bernoulli variables, so $R_2'(\boldsymbol{d})$ is a function of mn independent Bernoulli variables conditioned on fixed sum nd. Changing one of the Bernoulli variables changes the corresponding d_j by one and changes d by $1/n$. Overall, this changes the value of $R_2'(\boldsymbol{d})$ by at most $2 + 4d^{1/2}\log n$. Applying Lemma 7.3, we have

$$\mathbb{P}_{\mathcal{T}_r(n,m)}\big(|R_2'(\boldsymbol{d}) - \mathbb{E}R_2'(\boldsymbol{d})| > n^{1/2}\sigma^2\log^2 n - |n\sigma^2 - \mathbb{E}R_2'(\boldsymbol{d})|\big) = n^{-\Omega(\log n)}. \tag{7.4}$$

Therefore, $\mathbb{P}_{\mathcal{T}_r(n,m)}(\mathfrak{W}) = 1 - n^{-\Omega(\log n)}$. Now we can apply Theorem 1.8 to obtain

$$\mathbb{P}_{\mathcal{D}_r(n,m)}(\boldsymbol{d}) = \big(1 + O(\varepsilon + n^{1/10}Q^{-1/10}\log n + n^{-1/2}\log^2 n)\big)\mathbb{P}_{\mathcal{T}(n,m)}(\boldsymbol{d})$$

for $\boldsymbol{d} \in \mathfrak{W}$. Here, ε and $n^{1/10}Q^{-1/10}\log n$ come from the error terms in Theorem 1.8, while $n^{-1/2}\log^2 n$ comes from the term R_2/Q in Theorem 1.8 since $n\sigma^2 = Q(1 + O(n^{-1/2}\log^2 n))$ in $\mathfrak{W}$, by the definition of $\mathfrak{W}$ and (7.3).

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Now consider the probability space $\mathcal{D}_r(n, m)$. Since the distribution of each individual degree is the same as the distribution of Z_1, using a union bound and applying Lemma 7.2(b) gives $\mathbb{P}_{\mathcal{D}_r(n,m)}(\delta_{\max} > d^{1/2} \log n) = n^{-\Omega(\log n)}$ and hence

$$\mathbb{P}_{\mathcal{D}_r(n,m)}\big(R_2(\boldsymbol{d}) \neq R_2'(\boldsymbol{d})\big) = n^{-\Omega(\log n)}.$$

In [15], concentration of $R_2(\boldsymbol{d})$ in $\mathcal{D}_r(n, m)$ is shown using a lemma on functions of random subsets. However, that approach (at least, using the same concentration lemma) apparently only works for $r = o(n/\log n)$, so we will adopt a different approach.

By the same argument as used to prove (7.4),

$$\mathbb{P}_{\mathcal{T}_r(n,m)}\big(|R_2(\boldsymbol{d}) - n\sigma^2| > kn^{1/2} d \log^2 n \mid \delta_{\max} \le d^{1/2} \log n\big) \le e^{-Ck^2 \log^2 n}$$

for any positive integer k and some constant $C > 0$ independent of k. (Similarly to before, we have used $|n\sigma^2 - \mathbb{E}R_2'(\boldsymbol{d})| = O(d)$.)

If $R_2(\boldsymbol{d}) \le (k+1)\, n^{1/2}\sigma^2 \log^2 n$ then $-\frac{1}{2} + \frac{R_2(\boldsymbol{d})}{2Q} \le \frac{(k+1)\log^2 n}{2n^{1/2}} + o(1)$ and so applying Theorem 1.8 gives

$$\begin{aligned}\mathbb{P}_{\mathcal{D}_r(n,m)}\big(kn^{1/2}\sigma^2 \log^2 n < |R_2(\boldsymbol{d}) - n\sigma^2| \le (k+1)n^{1/2}\sigma^2 \log^2 n \mid \delta_{\max} \le d^{1/2} \log n\big)& \\ \le \exp\Big(-Ck^2 \log^2 n + \frac{(k+1)\log^2 n}{2n^{1/2}} + o(1)\Big)&.\end{aligned}$$

Summing over $k \ge 1$, we have

$$\mathbb{P}_{\mathcal{D}_r(n,m)}\big(|R_2(\boldsymbol{d}) - n\sigma^2| > n^{1/2}\sigma^2 \log^2 n \mid \delta_{\max} \le d^{1/2} \log n\big) = n^{-\Omega(\log n)},$$

and therefore $\mathbb{P}_{\mathcal{D}_r(n,m)}(\mathfrak{W}) = 1 - n^{-\Omega(\log n)}$, completing the proof. □

8. Deferred proofs

8.1 Proof of Lemma 1.2

We begin with the operation of replacing each edge by its complement in V, which sends d_j to $d_j' = e(\boldsymbol{d}) - d_j$ for each j. Recall that

$$\beta_j' = \frac{1}{n-r}\Big(\sum_{\ell\in[n]} \beta_\ell^*\Big) - \beta_j^*$$

and note that for all $j, k \in [n]$,

$$\big|\beta_j' - \beta_k'\big| = \left|\frac{1}{n-r}\Big(\sum_{\ell\in[n]} \beta_\ell^*\Big) - \beta_j^* - \frac{1}{n-r}\Big(\sum_{\ell\in[n]} \beta_\ell^*\Big) + \beta_k^*\right| = \big|\beta_j^* - \beta_k^*\big|.$$

In addition, for any $W \in \mathcal{S}_r(n)$ we have

$$\sum_{j\in V\setminus W} \beta_j' = \frac{n-r}{n-r}\Big(\sum_{\ell\in[n]} \beta_\ell^*\Big) - \sum_{j\in V\setminus W} \beta_j^* = \sum_{j\in W} \beta_j^*.$$

Therefore for any $W \in \mathcal{S}_r(n)$ we have

$$\lambda_{V\setminus W}(\boldsymbol{\beta}') = \frac{e^{\sum_{k\in V\setminus W} \beta_k'}}{1 + e^{\sum_{k\in V\setminus W} \beta_k'}} = \frac{e^{\sum_{k\in W} \beta_k^*}}{1 + e^{\sum_{k\in W} \beta_k^*}} = \lambda_W(\boldsymbol{\beta}^*). \tag{8.1}$$

Note that summing (1.5) over j each edge is counted r times, so $\sum_{W\in\mathcal{S}_r(n)}\lambda_W(\boldsymbol{\beta}^*)=e(\boldsymbol{d})$. Hence

$$\sum_{\substack{W\ni j\\ W\in\mathcal{S}_{n-r}(n)}}\lambda_W(\boldsymbol{\beta}')\overset{(8.1)}{=}\sum_{\substack{W\not\ni j\\ W\in\mathcal{S}_r}}\lambda_W(\boldsymbol{\beta}^*)=\sum_{W\in\mathcal{S}_r(n)}\lambda_W(\boldsymbol{\beta}^*)-\sum_{\substack{W\ni j\\ W\in\mathcal{S}_r}}\lambda_W(\boldsymbol{\beta}^*)=e(\boldsymbol{d})-d_j,$$

proving that $(\boldsymbol{d}',\boldsymbol{\beta}')$ satisfies (1.5). It only remains to show that

$$\frac{|A(\boldsymbol{\beta}')|}{(n-r)^2}=\frac{|A(\boldsymbol{\beta}^*)|}{r^2}. \tag{8.2}$$

For $W\subseteq[n]$, define the $n\times n$ matrix Ξ_W by

$$(\Xi_W)_{jk}=\begin{cases}1, & \text{if } j,k\in W;\\ 0, & \text{otherwise.}\end{cases}$$

Then,

$$A(\boldsymbol{\beta}^*)=\sum_{W\in\mathcal{S}_r(n)}\lambda_W(\boldsymbol{\beta}^*)(1-\lambda_W(\boldsymbol{\beta}^*))\,\Xi_W.$$

Now note that $(I-\frac{1}{r}J)\Xi_W(I-\frac{1}{r}J)=\Xi_{V\setminus W}$ for any $W\in\mathcal{S}_r(n)$. (The case $W=\{1,\dots,r\}$ is representative and easy to check.) Together with (8.1), this proves that

$$\left(I-\frac{1}{r}J\right)A(\boldsymbol{\beta}^*)\left(I-\frac{1}{r}J\right)=A(\boldsymbol{\beta}').$$

Finally, $|I-\frac{1}{r}J|=-\frac{n-r}{r}$ by (3.1), which proves (8.2).

Next, consider the operation that complements the edge set, sending d_j to $\widetilde{d}_j=\binom{n-1}{r-1}-d_j$ without changing the edge size. Recall that $\widetilde{\beta}_j=-\beta^*_j$ for each j. Then $|\widetilde{\beta}_j-\widetilde{\beta}_k|=|\beta^*_j-\beta^*_k|$ for all j,k. Note that for any $W\in\mathcal{S}_r(n)$ we have

$$\lambda_W(\widetilde{\boldsymbol{\beta}})=\frac{e^{\sum_{k\in W}\widetilde{\beta}_k}}{1+e^{\sum_{k\in W}\widetilde{\beta}_k}}=\frac{e^{-\sum_{k\in W}\beta^*_k}}{1+e^{-\sum_{k\in W}\beta^*_k}}=1-\lambda_W(\boldsymbol{\beta}^*),$$

which implies that $A(\widetilde{\boldsymbol{\beta}})=A(\boldsymbol{\beta}^*)$. In addition,

$$\sum_{W\ni j}\lambda_W(\widetilde{\boldsymbol{\beta}})=\binom{n-1}{r-1}-\sum_{W\ni j}\lambda_W(\boldsymbol{\beta}^*)=\binom{n-1}{r-1}-d_j=\widetilde{d}_j,$$

proving that $(\widetilde{\boldsymbol{d}},\widetilde{\boldsymbol{\beta}})$ satisfies (1.5).

The third operation, which complements both the edges and the edge set simultaneously, is just the composition of the first two in either order. Hence the result for this operation follows immediately, completing the proof.

8.2 Proof of Lemma 3.7

The following lemmas will be useful.

Lemma 8.1. ([11, (1.13)]). *For $p\in\mathbb{R}$, define*

$$\alpha_p(x):=\frac{(1+x^2)^p-1}{x^2}.$$

Then, for $\boldsymbol{x}\in\mathbb{R}^n$,

$$(I+\boldsymbol{x}\boldsymbol{x}^{\mathrm{t}})^p=I+\alpha_p(\|\boldsymbol{x}\|_2)\,\boldsymbol{x}\boldsymbol{x}^{\mathrm{t}}.$$

Also, for $x\geq 0$, $|\alpha_{-1/2}(x)|\leq x^{-2}$ and $|\alpha_{1/2}(x)|\leq x^{-1}$.

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For a matrix $X=(x_{jk})$, $\|X\|_{\max}:=\max_{j,k}|x_{jk}|$ is a matrix norm that is not submultiplicative. The following is a special case of a lemma in [13].

Lemma 8.2. ([13, Lemma 4.9]) *Let M be a real symmetric positive definite $n\times n$ matrix with*

$$\|M-I\|_{\max}\le\frac{\kappa}{n}\quad\text{and}\quad \boldsymbol{x}^{\mathrm t}M\boldsymbol{x}\ge\gamma\,\boldsymbol{x}^{\mathrm t}\boldsymbol{x}$$

for some $1\ge\gamma>0$, $\kappa>0$ and all $\boldsymbol{x}\in\mathbb{R}^n$. Then the following are true.

(a)

$$\|M^{-1}-I\|_{max}\le\frac{(\kappa+\gamma)\kappa}{\gamma n}.$$

(b) There exists a real matrix T such that $T^{\mathrm t}MT=I$ and

$$\|T\|_1,\|T\|_\infty\le\frac{\kappa+\gamma^{1/2}}{\gamma^{1/2}},\qquad \|T^{-1}\|_1,\|T^{-1}\|_\infty\le\frac{(\kappa+1)(\kappa+\gamma^{1/2})}{\gamma^{1/2}}.$$

The next result will be used to find a change of basis matrix to invert $A(\boldsymbol{\beta})$.

Lemma 8.3. *Let $\bar A=D+\boldsymbol{s}\boldsymbol{s}^{\mathrm t}+X$ be a symmetric positive definite real matrix of order n, where D is a positive diagonal matrix and $\boldsymbol{s}\in\mathbb{R}^n$. Define these quantities:*

$$\begin{aligned}
\gamma &:= \text{a value in } (0,1) \text{ such that } \boldsymbol{x}^{\mathrm t}\bar A\boldsymbol{x}\ge\gamma\,\boldsymbol{x}^{\mathrm t}(D+\boldsymbol{s}\boldsymbol{s}^{\mathrm t})\boldsymbol{x} \text{ for all } \boldsymbol{x}\in\mathbb{R}^n,\\
D_{\min},D_{\max} &:= \text{the minimum and maximum diagonal entries of } D,\\
B &:= 1+D_{\max}D_{\min}^{-1}\|\boldsymbol{s}\|_1\|\boldsymbol{s}\|_\infty\|\boldsymbol{s}\|_2^{-2},\\
\kappa &:= B^2D_{\min}^{-1}\,n\,\|X\|_{\max}.
\end{aligned}$$

Then there is a real $n\times n$ matrix T such that $T^{\mathrm t}\bar AT=I$ and the following are true:

(a)

$$\|\bar A^{-1}-(D+\boldsymbol{s}\boldsymbol{s}^{\mathrm t})^{-1}\|_{\max}\le\frac{B^2\kappa(\kappa+1)}{D_{\min}\gamma n},\quad\text{where}$$
$$(D+\boldsymbol{s}\boldsymbol{s}^{\mathrm t})^{-1}=D^{-1}-\frac{D^{-1}\boldsymbol{s}\boldsymbol{s}^{\mathrm t}D^{-1}}{1+\|D^{-1/2}\boldsymbol{s}\|_2^2};$$

(b)

$$\|T\|_1,\|T\|_\infty\le BD_{\min}^{-1/2}\gamma^{-1/2}(\kappa+1);$$

(c) For any $\rho>0$, define

$$\mathcal{Q}(\rho):=\ U_n(\rho)\cap\left\{\boldsymbol{x}\in\mathbb{R}^n:|\boldsymbol{s}^{\mathrm t}\boldsymbol{x}|\le\frac{D_{\max}\|\boldsymbol{s}\|_1}{D_{\min}^{1/2}\|\boldsymbol{s}\|_2}\rho\right\}.$$

Then

$$T\big(U_n(\rho_1)\big)\subseteq\mathcal{Q}(\rho)\subseteq T\big(U_n(\rho_2)\big),$$

where

$$\rho_1:=\frac{1}{B}\,D_{\min}^{1/2}\,\gamma^{1/2}\,(\kappa+1)^{-1}\rho,\qquad \rho_2:=BD_{\max}^{1/2}\,\gamma^{-1/2}\,(\kappa+1)^2\rho.$$

Proof. Define $\boldsymbol{s}_1:=D^{-1/2}\boldsymbol{s}$, $X_1:=D^{-1/2}XD^{-1/2}$, $T_1:=(I+\boldsymbol{s}_1\boldsymbol{s}_1^{\mathrm t})^{-1/2}$ and $X_2:=T_1^{\mathrm t}X_1T_1$. By Lemma 8.1, we have

$$T_1=I+\alpha_{-1/2}(\|\boldsymbol{s}_1\|_2)\boldsymbol{s}_1\boldsymbol{s}_1^{\mathrm t},\tag{8.3}$$

and note that T_1 is symmetric, that is, $T_1 = T_1^{\mathrm{t}}$. Therefore

$$\bar{A} = D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}} + X = D^{1/2}\left(I + \boldsymbol{s}_1\boldsymbol{s}_1^{\mathrm{t}} + X_1\right)D^{1/2} = D^{1/2}T_1^{-1}\left(I + X_2\right)T_1^{-1}D^{1/2}. \tag{8.4}$$

Recall that by Lemma 8.1 we have $|\alpha_{-1/2}(\|\boldsymbol{s}_1\|_2)| \le \|\boldsymbol{s}_1\|_2^{-2}$, so by (8.3),

$$\|T_1\|_1,\ \|T_1\|_\infty \le 1 + \frac{\|\boldsymbol{s}_1\|_1\|\boldsymbol{s}_1\|_\infty}{\|\boldsymbol{s}_1\|_2^2} \le 1 + \frac{D_{\max}\|\boldsymbol{s}\|_1\|\boldsymbol{s}\|_\infty}{D_{\min}\|\boldsymbol{s}\|_2^2} = B. \tag{8.5}$$

Next we apply Lemma 8.2 with $M = I + X_2$. By (8.4), $\boldsymbol{x}^{\mathrm{t}}\bar{A}\boldsymbol{x} \ge \gamma\boldsymbol{x}^{\mathrm{t}}(D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})\boldsymbol{x}$ is equivalent to

$$(T_1^{-1}D^{1/2}\boldsymbol{x})^{\mathrm{t}}\,(I + X_2)T_1^{-1}D^{1/2}\boldsymbol{x} \ge \gamma\,(T_1^{-1}D^{1/2}\boldsymbol{x})^{\mathrm{t}}\,T_1^{-1}D^{1/2}\boldsymbol{x}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. Also

$$\|X_2\|_{\max} \le D_{\min}^{-1}\|T_1\|_\infty^2\|X\|_{\max} \overset{(8.5)}{\le} B^2D_{\min}^{-1}\|X\|_{\max} = \frac{\kappa}{n}.$$

Therefore, M, γ, κ satisfy the conditions of Lemma 8.2. Consequently, there exists a transformation T_2 such that $T_2^{\mathrm{t}}(I + X_2)T_2 = I$. This, together with (8.4) implies that $T = D^{-1/2}T_1T_2$ satisfies $T^{\mathrm{t}}\bar{A}T = I$. In addition, by Lemma 8.2(b), we have

$$\|T_2\|_1, \|T_2\|_\infty \le \gamma^{-1/2}(\kappa + 1), \qquad \|T_2^{-1}\|_1, \|T_2^{-1}\|_\infty \le \gamma^{-1/2}(\kappa + 1)^2. \tag{8.6}$$

Together with (8.5) and $\|D^{-1/2}\|_1, \|D^{-1/2}\|_\infty \le D_{\min}^{-1/2}$, this proves part (b).

Next we prove the first inclusion of part (c). Let $\boldsymbol{x} \in U_n(\rho_1)$, that is, $\|\boldsymbol{x}\|_\infty \le \rho_1$. Then $\|T\boldsymbol{x}\|_\infty \le \|T\|_\infty\,\rho_1 \le \rho$ by part (b), so $T\boldsymbol{x} \in U_n(\rho)$. Next

$$|\boldsymbol{s}^{\mathrm{t}}T\boldsymbol{x}| = |\boldsymbol{s}_1^{\mathrm{t}}T_1T_2\boldsymbol{x}| \le \|T_1\boldsymbol{s}_1\|_1\|T_2\boldsymbol{x}\|_\infty.$$

From (8.6), $\|T_2\boldsymbol{x}\|_\infty \le \gamma^{-1/2}(\kappa + 1)\rho_1$. Also (8.3) gives $T_1\boldsymbol{s}_1 = (1 + \|\boldsymbol{s}_1\|_2^2)^{-1/2}\boldsymbol{s}_1$, so

$$\|T_1\boldsymbol{s}_1\|_1 \le \|\boldsymbol{s}_1\|_1\|\boldsymbol{s}_1\|_2^{-1} \le D_{\max}^{1/2}\|\boldsymbol{s}\|_1D_{\min}^{-1/2}\|\boldsymbol{s}\|_2^{-1}.$$

Combining these bounds proves the inclusion, as $B \ge 1$.

For the second inclusion of part (c), consider $\boldsymbol{x} \in \mathcal{Q}(\rho)$. Lemma 8.1 implies that $T_1^{-1} = I + \alpha_{1/2}(\|\boldsymbol{s}_1\|_2)\boldsymbol{s}_1\boldsymbol{s}_1^{\mathrm{t}}$, and hence

$$\|T^{-1}\boldsymbol{x}\|_\infty = \|T_2^{-1}T_1^{-1}D^{1/2}\boldsymbol{x}\|_\infty \le \|T_2^{-1}\|_\infty\left\|D^{1/2}\boldsymbol{x} + \alpha_{1/2}(\|\boldsymbol{s}_1\|_2)\boldsymbol{s}_1\boldsymbol{s}^{\mathrm{t}}\boldsymbol{x}\right\|_\infty.$$

Now apply (8.6) to $\|T_2^{-1}\|_\infty$, the first part of the definition of $\mathcal{Q}(\rho)$ to $\|D^{1/2}\boldsymbol{x}\|_\infty$, the second part of the definition of $\mathcal{Q}(\rho)$ to $|\boldsymbol{s}^{\mathrm{t}}\boldsymbol{x}|$, and recall from Lemma 8.1 that $|\alpha_{1/2}(\|\boldsymbol{s}_1\|_2)| \le \|\boldsymbol{s}_1\|_2^{-1}$. Then we have

$$\begin{aligned}\|T^{-1}\boldsymbol{x}\|_\infty &\le \gamma^{-1/2}(\kappa + 1)^2\left(D_{\max}^{1/2}\rho + \frac{\|\boldsymbol{s}_1\|_\infty}{\|\boldsymbol{s}_1\|_2}\cdot\frac{D_{\max}\|\boldsymbol{s}\|_1}{D_{\min}^{1/2}\|\boldsymbol{s}\|_2}\rho\right)\\ &\le \gamma^{-1/2}(\kappa + 1)^2D_{\max}^{1/2}\,\rho\left(1 + \frac{D_{\max}\|\boldsymbol{s}\|_1\|\boldsymbol{s}\|_\infty}{D_{\min}\|\boldsymbol{s}\|_2^2}\right) = \rho_2.\end{aligned}$$

Finally, we prove part (a). Define $X_3 := (I + X_2)^{-1} - I$. By (8.4) and since $T_1 = T_1^{\mathrm{t}}$ we have $T_1^{\mathrm{t}}D^{-1/2}\bar{A}D^{-1/2}T_1 = I + X_2$. Together with $T_1 = (I + \boldsymbol{s}_1\boldsymbol{s}_1^{\mathrm{t}})^{-1/2}$, this implies

$$\bar{A}^{-1} = D^{-1/2}\,T_1(I + X_2)^{-1}\ \ T_1D^{-1/2} = D^{-1/2}\,T_1X_3T_1^{\mathrm{t}}\,D^{-1/2} + (D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})^{-1}.$$

By Lemma 8.2(a), $\|X_3\|_{\max} \le \kappa(\kappa+1)\gamma^{-1}n^{-1}$ and thus using (8.5) we have

$$\|\bar{A}^{-1} - (D+\boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})^{-1}\|_\infty \le D_{\min}^{-1}\,\|T_1\|_1^2\,\|X_3\|_{\max} \le \frac{B^2\kappa(\kappa+1)}{D_{\min}\,\gamma n}.$$

The expression for $(D+\boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})^{-1}$ follows from the Sherman–Morrison theorem (see for example [22, equation (3.8.2)]). □

Proof of Lemma 3.7. Define $\check{\Lambda} := \Lambda(\boldsymbol{\beta})$ and

$$c := \sqrt{\frac{1}{2}\check{\Lambda}\binom{n-2}{r-2}}.$$

Then let $\boldsymbol{s} := (c, c, \ldots, c)^{\mathrm{t}}$ and $D := \mathrm{diag}(a_{11}-c^2, \ldots, a_{nn}-c^2)$. We write $A(\boldsymbol{\beta}) = D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}} + X$.

First we show that the entries of X are small. Note that all the diagonal entries of X are exactly 0. By Lemma 3.5, the absolute value of any off-diagonal entry in X is at most

$$|a_{jk} - c^2| \le (e^{4\delta/r} - 1)\check{\Lambda}\binom{n-2}{r-2} \overset{(5.3)}{\le} \frac{e^{4\delta}}{r}\check{\Lambda}\binom{n-2}{r-2}. \tag{8.7}$$

In addition, Lemma 3.5 also implies that for any $1 \le j \le n$ we have

$$\begin{aligned} a_{jj} - c^2 &\ge \frac{1}{2}e^{-4\delta/r}\check{\Lambda}\binom{n-1}{r-1} - \frac{1}{2}\check{\Lambda}\binom{n-2}{r-2} = \frac{1}{2}e^{-4\delta/r}\check{\Lambda}\binom{n-1}{r-1}\left(1 - \frac{(r-1)e^{4\delta/r}}{n-1}\right) \\ &\overset{(5.3)}{\ge} \frac{1}{2}e^{-4\delta/r}\check{\Lambda}\binom{n-1}{r-1}\left(1 - \frac{(r-1)(e^{4\delta}+r)}{r(n-1)}\right) \ge \frac{1}{2}e^{-4\delta/r}\check{\Lambda}\binom{n-1}{r-1}\left(1 - \frac{e^{4\delta}+r}{n-1}\right) \\ &\ge \frac{1}{5}e^{-4\delta/r}\check{\Lambda}\binom{n-1}{r-1}, \end{aligned} \tag{8.8}$$

where in the last step we used $r \le n/2$ and $n \ge 16e^{4\delta}$.

Consider the value of γ as in Lemma 8.3. For any $\boldsymbol{y} \in \mathbb{R}^n$ we have

$$\begin{aligned} \boldsymbol{y}^{\mathrm{t}}A(\boldsymbol{\beta})\boldsymbol{y} &= \frac{1}{2}\sum_{W\in\mathcal{S}_r(n)} \lambda_W(\boldsymbol{\beta})(1-\lambda_W(\boldsymbol{\beta}))\Big(\sum_{j\in W} y_j\Big)^2 \overset{L.3.3}{\ge} \frac{1}{2}e^{-2\delta}\check{\Lambda}\sum_{W\in\mathcal{S}_r(n)}\Big(\sum_{j\in W} y_j\Big)^2 \\ &= \frac{1}{2}e^{-2\delta}\check{\Lambda}\boldsymbol{y}^{\mathrm{t}}\left(\binom{n-1}{r-1}I - \binom{n-2}{r-2}I + \binom{n-2}{r-2}J\right)\boldsymbol{y} \\ &= \frac{1}{2}e^{-2\delta}\check{\Lambda}\left(\binom{n-2}{r-1}\|\boldsymbol{y}\|_2^2 + \binom{n-2}{r-2}\Big(\sum_{j=1}^n y_j\Big)^2\right). \end{aligned}$$

On the other hand, by Lemma 3.5 we have

$$\begin{aligned} \boldsymbol{y}^{\mathrm{t}}(D + ss^{\mathrm{t}})\boldsymbol{y} &\le \frac{1}{2}e^{4\delta}\check{\Lambda}\left(\binom{n-1}{r-1}\|\boldsymbol{y}\|_2^2 + \binom{n-2}{r-2}\Big(\sum_{j=1}^n y_j\Big)^2\right) \\ &= \frac{1}{2}e^{4\delta}\check{\Lambda}\left(\frac{n-1}{n-r}\binom{n-2}{r-1}\|\boldsymbol{y}\|_2^2 + \binom{n-2}{r-2}\Big(\sum_{j=1}^n y_j\Big)^2\right) \\ &\le 2e^{4\delta}\check{\Lambda}\left(\binom{n-2}{r-1}\|\boldsymbol{y}\|_2^2 + \binom{n-2}{r-2}\Big(\sum_{j=1}^n y_j\Big)^2\right), \end{aligned}$$

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where the last inequality holds as $r \le n/2$. Therefore setting $\gamma := e^{-6\delta}/4$, we have for any $\boldsymbol{y} \in \mathbb{R}^n$ that $\boldsymbol{y}^{\mathrm{t}} A \boldsymbol{y} \ge \gamma\, \boldsymbol{y}^{\mathrm{t}}(D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})\boldsymbol{y}$. Let B be as in Lemma 8.3. Then

$$B = 1 + \frac{D_{\max} \|\boldsymbol{s}\|_1 \|\boldsymbol{s}\|_\infty}{D_{\min} \|\boldsymbol{s}\|_2^2} = 1 + \frac{D_{\max}}{D_{\min}} \le 4e^{8\delta/r}, \tag{8.9}$$

which follows from Lemma 3.5 and (8.8).

For κ as in Lemma 8.3, using (8.7), (8.8) and (8.9), we have

$$\kappa = B^2\, D_{\min}^{-1}\, n\, \|X\|_{\max} \le 80 e^{16\delta/r} \frac{e^{4\delta} r^{-1} \check{\Lambda} \binom{n-2}{r-2}}{e^{-4\delta/r} \check{\Lambda} \binom{n-1}{r-1}} n \le 80 e^{20\delta/r + 4\delta}. \tag{8.10}$$

Next we consider the matrix $(D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})^{-1}$. By Lemma 8.3 we have

$$(D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})^{-1} = D^{-1} - \frac{D^{-1} \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}} D^{-1}}{1 + \|D^{-1/2}\boldsymbol{s}\|_2^2},$$

and we are interested in an upper bound on the absolute value of the elements of this matrix. First consider the vector $D^{-1}\boldsymbol{s}$ and note that

$$D^{-1}\boldsymbol{s} = \begin{pmatrix} \frac{c}{a_{11} - c^2} \\ \vdots \\ \frac{c}{a_{nn} - c^2} \end{pmatrix}.$$

Together with (8.8) this implies that every element in the matrix $D^{-1}\boldsymbol{s}\boldsymbol{s}^{\mathrm{t}} D^{-1}$ has absolute value at most

$$16 e^{8\delta/r} \frac{\check{\Lambda} \binom{n-2}{r-2}}{\check{\Lambda}^2 \binom{n-1}{r-1}^2} \le 16 e^{8\delta/r} \frac{1}{\check{\Lambda} \binom{n}{r}}. \tag{8.11}$$

Similarly

$$D^{-1/2}\boldsymbol{s} = c \begin{pmatrix} (a_{11} - c^2)^{-1/2} \\ \vdots \\ (a_{nn} - c^2)^{-1/2} \end{pmatrix},$$

implying that

$$\|D^{-1/2}\boldsymbol{s}\|_2^2 = c^2 \sum_{j=1}^{n} \frac{1}{a_{jj} - c^2} \overset{L.3.5}{\ge} e^{-4\delta/r} \frac{\check{\Lambda} \binom{n-2}{r-2}}{\check{\Lambda} \binom{n-1}{r-1}} n \ge \frac{r}{2} e^{-4\delta/r},$$

and hence

$$1 + \|D^{-1/2}\boldsymbol{s}\|_2^2 \ge \frac{r}{2}\, e^{-4\delta/r}. \tag{8.12}$$

Therefore, by (8.11) and (8.12), every element of $\frac{D^{-1}\boldsymbol{s}\boldsymbol{s}^{\mathrm{t}} D^{-1}}{1 + \|D^{-1/2}\boldsymbol{s}\|_2^2}$ has absolute value at most

$$\frac{16}{1/2} \cdot \frac{e^{12\delta/r}}{r \check{\Lambda} \binom{n}{r}} = 32 \frac{e^{12\delta/r}}{\check{\Lambda} \binom{n-1}{r-1} n}, \tag{8.13}$$

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and thus so do the off-diagonal elements of $(D + \boldsymbol{s}\boldsymbol{s}^{\mathrm{t}})^{-1}$. As for the diagonal elements, by (8.8) and (8.13), each has absolute value at most

$$\frac{5e^{4\delta/r}}{\check{\Lambda}\binom{n-1}{r-1}} + \frac{32e^{12\delta/r}}{\check{\Lambda}\binom{n-1}{r-1}n} \leq \frac{8e^{12\delta/r}}{\check{\Lambda}\binom{n-1}{r-1}},$$

as $n \geq 16e^{4\delta} \geq 16$.

Now we have all the information needed to establish a bound on the absolute value of the elements in $A(\boldsymbol{\beta})^{-1}$ using Lemma 8.3(a). In particular, using (8.8), (8.9) and (8.10), the diagonal entries of $A(\boldsymbol{\beta})^{-1}$ have absolute value at most

$$\frac{8e^{12\delta/r}}{\check{\Lambda}\binom{n-1}{r-1}} + \frac{B^2\kappa(\kappa+1)}{D_{\min}\gamma n} \leq \frac{8e^{12\delta/r}}{\check{\Lambda}\binom{n-1}{r-1}} + \frac{\hat{C}e^{60\delta/r+14\delta}}{\check{\Lambda}\binom{n-1}{r-1}n} \leq (8+\hat{C})\frac{e^{60\delta/r+14\delta}}{\check{\Lambda}\binom{n-1}{r-1}},$$

for some sufficiently large constant $\hat{C}$. On the other hand, the off-diagonal entries have absolute value at most

$$\frac{32e^{12\delta/r}}{\check{\Lambda}\binom{n-1}{r-1}n} + \frac{B^2\kappa(\kappa+1)}{D_{\min}\gamma n} \leq \frac{32e^{12\delta/r}}{\check{\Lambda}\binom{n-1}{r-1}n} + \frac{\hat{C}e^{60\delta/r+14\delta}}{\check{\Lambda}\binom{n-1}{r-1}n} \leq (32+\hat{C})\left(\frac{e^{60\delta/r+14\delta}}{\check{\Lambda}\binom{n-1}{r-1}n}\right).$$

The first statement follows by setting $C = 32 + \hat{C}$ and using the fact that $r \geq 3$.

Now for the second statement. Substituting (8.8), (8.9) and (8.10) into Lemma 8.3(b) gives

$$\|T\|_1, \|T\|_\infty = O\left(\frac{1}{\check{\Lambda}^{1/2}\binom{n-1}{r-1}^{1/2}}\right),$$

as required.

Now for the last statement of the lemma. For any real $z \geq 0$, let

$$\hat{\rho}(z) = z\frac{n}{r^{1/2}}\frac{D_{\min}^{1/2}\|\boldsymbol{s}\|_2}{D_{\max}\|\boldsymbol{s}\|_1}c\rho.$$

Then

$$\mathcal{Q}\big(\hat{\rho}(z)\big) = \left\{\boldsymbol{x} \in U_n\big(\hat{\rho}(z)\big) : \left|\sum_{j\in[n]} x_j\right| \leq z\, nr^{-1/2}\rho\right\}.$$

Note that

$$\frac{n}{r^{1/2}}\frac{D_{\min}^{1/2}\|\boldsymbol{s}\|_2}{D_{\max}\|\boldsymbol{s}\|_1}c = \Theta\left(\frac{n}{r^{1/2}}\frac{\|\boldsymbol{s}\|_2}{D_{\min}^{1/2}\|\boldsymbol{s}\|_1}c\right) = \Theta\left(\frac{n}{r^{1/2}}\frac{1}{\check{\Lambda}^{1/2}\binom{n-1}{r-1}^{1/2}}\frac{n^{1/2}c}{nc}c\right)$$
$$= \Theta\left(\frac{n^{1/2}}{r^{1/2}}\frac{1}{\check{\Lambda}^{1/2}\binom{n-1}{r-1}^{1/2}}\check{\Lambda}^{1/2}\binom{n-2}{r-2}^{1/2}\right) = \Theta\left(\left(\frac{n(r-1)}{(n-1)r}\right)^{1/2}\right) = \Theta(1).$$

Therefore there exists $z_1 = \Omega(1)$ such that $\hat{\rho}(z_1) \leq \rho$ and $z_1 \leq 1$. Together with Lemma 8.3, this implies that

$$T\big(U_n(\rho_1)\big) \subseteq \mathcal{Q}\big(\hat{\rho}(z_1)\big) \subseteq \mathcal{R}(\rho),$$

where

$$\rho_1 = \frac{1}{B}D_{\min}^{1/2}\,\gamma^{1/2}\,(1+\kappa)^{-1}\hat{\rho}(z_1) = \Theta\left(\check{\Lambda}^{1/2}\binom{n-1}{r-1}^{1/2}\rho\right).$$

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Similarly there must exist $z_2 = O(1)$ such that $\hat{\rho}(z_2) \geq \rho$ and $z_2 \geq 1$. Then by Lemma 8.3 we have

$$T\big(U_n(\rho_2)\big) \supseteq \mathcal{Q}\big(\hat{\rho}(z_2)\big) \supseteq \mathcal{R}(\rho),$$

where

$$\rho_2 = BD_{\max}^{1/2}\, \gamma^{-1/2}\, (1+\kappa)^2 \hat{\rho}(z_2) = \Theta\left(\check{\Lambda}^{1/2} \binom{n-1}{r-1}^{1/2} \rho \right).$$

This completes the proof. □

9. Appendix: useful identities

In this appendix we provide summations that help for the calculations in Section 6. We use the notation

$$N = \binom{n-1}{r-1} = \frac{d}{\lambda}, \qquad \Gamma_s = \Gamma_s(W) = \sum_{\ell \in W} \delta_\ell^s, \qquad R_s = \sum_{\ell=1}^{n} \delta_\ell^s$$

and recall that $R_1 = 0$. We provide approximations for some expressions, assuming that $(r, \boldsymbol{d})$ belongs to the first quadrant and $\delta_{\max} = O(d^{3/5})$. The error bounds are good enough for our applications but are not necessarily tight.

9.1 Summations over all $W \in \mathcal{S}_r(n)$

$$\frac{1}{N} \sum_{W \in \mathcal{S}_r(n)} \Gamma_\ell = R_\ell \quad (\ell \geq 1),$$

$$\frac{1}{N} \sum_{W \in \mathcal{S}_r(n)} \Gamma_1 \Gamma_\ell = \frac{(n-r) R_{\ell+1}}{n-1} \quad (\ell \geq 1),$$

$$\frac{1}{N} \sum_{W \in \mathcal{S}_r(n)} \Gamma_1^3 = \frac{(n-r)(n-2r) R_3}{(n-2)(n-1)} = R_3 + O\big(\delta_{\max}\, d^{7/5}\big),$$

$$\begin{aligned}\frac{1}{N} \sum_{W \in \mathcal{S}_r(n)} \Gamma_1^4 &= \frac{3(r-1)(n-r)(n-r-1) R_2^2}{(n-3)(n-2)(n-1)} + \frac{(n-r)(n^2 - 6rn + 6r^2 + n) R_4}{(n-3)(n-2)(n-1)} \\ &= \frac{3(r-1)R_2^2}{n} + R_4 + O\big(\delta_{max}\, d^{12/5}\big),\end{aligned}$$

$$\frac{1}{N} \sum_{W \in \mathcal{S}_r(n)} \Gamma_2^2 = \frac{(r-1)R_2^2}{n-1} + \frac{(n-r)R_4}{n-1} = \frac{(r-1)R_2^2}{n} + R_4 + O\big(\delta_{\max}\, d^{12/5}\big),$$

$$\begin{aligned}\frac{1}{N} \sum_{W \in \mathcal{S}_r(n)} \Gamma_1^2\, \Gamma_2 &= \frac{(r-1)(n-r)R_2^2}{(n-2)(n-1)} + \frac{(n-r)(n-2r)R_4}{(n-2)(n-1)} \\ &= \frac{(r-1)R_2^2}{n} + R_4 + O\big(\delta_{\max}\, d^{12/5}\big).\end{aligned}$$

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9.2 Summations over all $W \ni j$

$$\frac{1}{N}\sum_{W\ni j}\Gamma_\ell = \frac{(r-1)R_\ell}{n-1} + \frac{(n-r)\,\delta_j^\ell}{n-1} \quad (\ell \geq 1),$$

$$\frac{1}{N}\sum_{W\ni j}\Gamma_1\Gamma_\ell = \frac{(r-1)(n-r)\delta_j R_\ell}{(n-2)(n-1)} + \frac{(r-1)(n-r)R_{\ell+1}}{(n-2)(n-1)} + \frac{(n-r)(n-2r)\delta_j^{\ell+1}}{(n-2)(n-1)} \quad (\ell \geq 1),$$

$$\begin{aligned}\frac{1}{N}\sum_{W\ni j}\Gamma_1^3 &= \frac{3(r-1)(n-r)(n-r-1)\delta_j R_2}{(n-3)(n-2)(n-1)} + \frac{(r-1)(n-r)(n-2r+1)R_3}{(n-3)(n-2)(n-1)}\\ &\quad + \frac{(n-r)(n^2-6rn+6r^2+n)\delta_j^3}{(n-3)(n-2)(n-1)}\\ &= \frac{3(r-1)\delta_j R_2 + (r-1)R_3}{n} + \delta_j^3 + O\left(\frac{d^{12/5}}{rn}\right),\end{aligned}$$

$$\begin{aligned}\frac{1}{N}\sum_{W\ni j}\Gamma_1^4 &= \frac{3(r-2)(r-1)(n-r)(n-r-1)R_2^2}{(n-4)(n-3)(n-2)(n-1)}\\ &\quad + \frac{6(r-1)(n-r)(n-r-1)(n-2r)\delta_j^2 R_2}{(n-4)(n-3)(n-2)(n-1)}\\ &\quad + \frac{4(r-1)(n-r)(n-r-1)(n-2r)\delta_j R_3}{(n-4)(n-3)(n-2)(n-1)}\\ &\quad + \frac{(r-1)(n-r)(n^2-6rn+6r^2+5n-6r)R_4}{(n-4)(n-3)(n-2)(n-1)}\\ &\quad + \frac{(n-r)(n-2r)(n^2-12rn+12r^2+5n)\delta_j^4}{(n-4)(n-3)(n-2)(n-1)}\\ &= O\left(\frac{d^{17/5}}{rn}\right).\end{aligned}$$

9.3 Summations over all $W \supset \{j,k\}$

$$\begin{aligned}\frac{1}{N}\sum_{W\supset\{j,k\}}\Gamma_\ell &= \frac{(r-2)(r-1)R_\ell}{(n-2)(n-1)} + \frac{(r-1)(n-r)(\delta_j^\ell+\delta_k^\ell)}{(n-2)(n-1)}\\ &= \frac{(r-2)(r-1)R_\ell}{n^2} + \frac{(r-1)(\delta_j^\ell+\delta_k^\ell)}{n} + O\left(\frac{\delta_{\max} r\, d^{\ell-3/5}}{n^2}\right) \ (\ell \geq 1),\end{aligned}$$

$$\begin{aligned}\frac{1}{N}\sum_{W\supset\{j,k\}}\Gamma_1^2 &= \frac{(r-2)(r-1)(n-r)R_2}{(n-3)(n-2)(n-1)}\\ &\quad + \frac{(r-1)(n-r)\big((n-2r+1)(\delta_j^2+\delta_k^2)+2(n-r-1)\delta_j\delta_k\big)}{(n-3)(n-2)(n-1)}\\ &= \frac{(r-2)(r-1)R_2}{n^2} + \frac{(r-1)\big(\delta_j+\delta_k\big)^2}{n} + O\left(\frac{\delta_{\max} r\, d^{7/5}}{n^2}\right).\end{aligned}$$

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