Canad. Math. Bull. Vol. 35 (3), 1992 pp. 410–415

bv_0-NORMS FOR SOME TRIANGULAR MATRICES

B. E. RHOADES

ABSTRACT. In this paper we obtain the bv_0-norms for several well-known classes of matrix operators.

In a series of papers [14], [15], and [7] the author obtained the norms of generalized Hausdorff matrices, considered as bounded operators on ℓ^p spaces, $1 < p < \infty$. The ℓ^∞ and ℓ^1 norms have been known for some time. In this paper we focus on the last remaining common sequence space—$bv_0 := \{x : \lim_n x_n = 0 \text{ and } \sum |x_n - x_{n+1}| < \infty\}$. An equivalent norm on the space is $\sum |x_n - x_{n-1}|$. However, we shall use the former one, since it enjoys a greater symmetry and is the one commonly used. We shall also use the corresponding form for the bv_0 norm of linear operators. Necessary and sufficient conditions for an infinite matrix A to be a bounded operator on bv_0 are that A have null columns and that

$$\sup_r \sum_n \left| \sum_{k=0}^{r} a_{nk} - a_{n+1,k} \right| < \infty, \tag{1}$$

and (1) is the norm. See, *e.g.* [16].

Since all norms considered in this paper will be in bv_0, we shall simply write the bv_0 norm as $\|\cdot\|$, rather than $\|\cdot\|_{bv_0}$.

Our first norm result will be established for certain lower triangle matrices.

THEOREM 1. *Let A be a lower triangular matrix with non-negative entries, zero column limits, decreasing row sums, and satisfying*

$$\sum_{k=0}^{r} (a_{nk} - a_{n+1,k}) \geq 0 \textit{ for each } 0 \leq r \leq n, \quad n = 0, 1, 2, \ldots.$$

Then $\|A\| = a_{00}$.

Received by the editors September 14, 1990; revised May 28, 1991 and February 5, 1992.

AMS subject classification: 47A30, 40G05.

PROOF. For any $N = 0, 1, \ldots,$

$$\begin{aligned}
\sum_{n=0}^{N}\left|\sum_{k=0}^{r}(a_{nk} - a_{n+1,k})\right| &= \sum_{n=0}^{N}\sum_{k=0}^{r}(a_{nk} - a_{n+1,k}) \\
&= \sum_{k=0}^{r}\sum_{n=0}^{N}(a_{nk} - a_{n+1,k}) \\
&= \sum_{k=0}^{r}(a_{0k} - a_{N+1,k}) \\
&= a_{00} - \sum_{k=0}^{r} a_{N+1,k} \\
&\leq a_{00},
\end{aligned}$$

and $||A|| \leq a_{00}$.

On the other hand,

$$||A|| \geq \sum_{n=0}^{\infty}\left|\sum_{k=0}^{0}(a_{nk} - a_{n+1,k})\right| = \sum_{n=0}^{\infty}(a_{n0} - a_{n+1,0}) = a_{00}.$$

As a corollary of Theorem 1 we obtain the norm result for certain Nörlund matrices. A Nörlund matrix is a lower triangular matrix with non-zero entries of the form $a_{nk} = p_{n-k}/P_n$, where $\{p_n\}$ is a real or complex sequence such that $P_n := \sum_{k=0}^{n} p_k \neq 0$ for each n.

COROLLARY 1. *Let A be a Nörlund matrix generated by a positive monotone decreasing sequence $\{p_n\}$. Then $||A|| = 1$.*

PROOF. A is a triangle with row sums 1. Since $\{p_n\}$ is monotone decreasing, $P_n \geq (n+1)p_n$ and $p_n/P_n \longrightarrow 0$. Thus $a_{nk} = p_{n-k}/P_n \leq p_{n-k}/P_{n-k}$ and A has zero column limits.

$$\begin{aligned}
\sum_{k=0}^{r}(a_{nk} - a_{n+1,k}) &= \sum_{k=0}^{r}\left(\frac{p_{n-k}}{P_n} - \frac{p_{n+1-k}}{P_{n+1}}\right) \\
&\geq \frac{1}{P_{n+1}}\sum_{k=0}^{r}(p_{n-k} - p_{n+1-k}) \geq 0,
\end{aligned}$$

and the result follows from Theorem 1.

THEOREM 2. *Let A be a lower triangular matrix with entries $a_{nk} := c_n t^{n-k}$ for $0 \leq k \leq n$, where $0 \leq t \leq 1$ and $\{c_n\}$ is a non-negative, decreasing sequence such that $\{(n+1)c_n\}$ is also decreasing. Then $||A|| = c_0$.*

PROOF. A has non-negative entries and zero column limits. For any $0 \leq r \leq n$,

$$\begin{aligned}
\sum_{k=0}^{r}(a_{nk} - a_{n+1,k}) &= \sum_{k=0}^{r}(c_n t^{n-k} - c_{n+1} t^{n+1,k}) \\
&= \sum_{k=0}^{r} t^{n-k}(c_n - t c_{n+1}) \\
&\geq \sum_{k=0}^{r} t^{n-k}(c_n - c_{n+1}) \geq 0.
\end{aligned}$$

https://doi.org/10.4153/CMB-1992-054-2 Published online by Cambridge University Press

The result will follow from Theorem 1 upon showing that the row sums are decreasing. For $0 \leq t < 1$,

$$\sum_{k=0}^{r} c_{nk} = \frac{t^{n+1}(1-t^{n+1})}{(n+1)(1-t)},$$

and it will be sufficient to show that $g(t) := c_n - c_{n+1} - c_n t^{n+1} + c_{n+1} t^{n+2} \geq 0$ for $0 \leq t \leq 1$. This is true since g is decreasing in t and $g(1) = 0$.

For $t = 1$ the row sums are $(n+1)c_n$, which decrease in n by hypothesis.

As corollaries of Theorem 2 we obtain norm results for three classes of matrices, introduced by Rhaly [11]–[13], which are generalizations of the Cesàro matrix of order 1.

The first family is referred to as the *Rhaly generalized Cesàro matrices*, and are lower triangular matrices with non-zero entries $a_{nk} = t^{n-k}/(n+1)$, $0 < t \leq 1$. We shall consider only the cases when $t < 1$, since the case $t = 1$ is the Cesàro matrix.

COROLLARY 2. *The Rhaly generalized Cesàro matrices have norm 1.*

COROLLARY 3. *The Rhaly p-Cesàro matrices have norm 1.*

These matrices are lower triangular matrices with non-zero entries

$$b_{nk} = \frac{1}{(n+1)^p}, \quad p > 1.$$

COROLLARY 4. *The Rhaly terraced matrices have norm* a_0.

The terraced matrices are lower triangular matrices with non-zero entries $d_{nk} = a_n$, where $\{a_n\}$ is a monotone decreasing sequence with limit 0 such that $\{(n+1)a_n\}$ is monotone decreasing.

Each corollary is proved by observing that the corresponding matrix satisfies the conditions of Theorem 2.

Our next class of matrices to be considered are generalized Hausdorff matrices. In 1921 Hausdorff [8] defined a class of matrices which form the commutant of C, the Cesàro matrix of order 1. In 1958, K. Endl [4] and A. Jakimovski [5] independently defined a generalization of the Hausdorff matrices. Hausdorff himself [9] had previously defined a different generalization, involving a sequence $\{\lambda_n\}$, satisfying the conditions that $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \infty$, and $\sum 1/\lambda_n = \infty$, in connection with a uniform approximation problem. Jakimovski [6], in 1959, extended the work of Hausdorff to the case in which $\lambda_0 > 0$. In 1981, in joint work D. Borwein [3], he obtained a further substantial generalization by removing the restriction that the λ_n be distinct.

We shall consider this latter class of generalized Hausdorff matrices, which are lower triangular with entries

$$\lambda_{nk} = -\lambda_{k+1} \cdots \lambda_n \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f(z)\,dz}{(\lambda_k - z) \cdots (\lambda_n - z)} + \delta_k,$$

when Ω is a simply connected region that contains every positive λ_n; Γ_n is a positively sensed Jordan contour lying in Ω and enclosing every $\lambda_n \in \Omega$; and $\delta_k = f(\lambda_0)$ if $k = 0$

https://doi.org/10.4153/CMB-1992-054-2 Published online by Cambridge University Press

and $\lambda_n \notin \Omega$, and $\delta_k = 0$ otherwise. We adopt the usual convention that $\lambda_{k+1} \cdots \lambda_n = 1$ when $k = n$. As in [1] we shall assume that the function f satisfies the conditions

$$(-1)^r f^{(r)}(x) \geq 0 \text{ for } r = 0, 1, 2 \ldots \text{ and } x > c,$$

and the region Ω, in which f is holomorphic, satisfies the condition $\Omega \supset (c, \infty)$. In addition, we shall assume that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty,$$

that $c = 0$ if $\lambda_0 > 0$, and that $c = -\varepsilon$ for some $\varepsilon > 0$ if $\lambda_0 = 0$. Then H will have non-negative entries, zero column limits and the entries will satisfy the recursion formula (see, *e.g.* (18) of [3])

$$\lambda_{nk} - \lambda_{n+1,k} = (\lambda_{k+1}\lambda_{n+1,k+1} - \lambda_k \lambda_{n+1,k})/\lambda_{n+1}. \tag{2}$$

Let $\{s_n\}$ be any sequence, $a_n := s_{n-1}$, $s_{-1} = 0$. Then, for any integer $r \geq 0$, using (2),

$$\begin{aligned}
\sum_{k=0}^{r} \lambda_{n+1,k}\lambda_k a_k &= \sum_{k=0}^{r} \lambda_{n+1,k}\lambda_k s_k - \sum_{k=0}^{r} \lambda_{n+1,k}\lambda_k s_{k-1} \\
&= \sum_{k=0}^{r} \lambda_{n+1,k}\lambda_k s_k - \sum_{k=0}^{r-1} \lambda_{k+1}\lambda_{n+1,k+1} s_k \\
&= \lambda_r \lambda_{n+1,r} s_r + \sum_{k=0}^{r-1} \lambda_{n+1,k}\lambda_k s_k \\
&\qquad - \lambda_{n+1} \sum_{k=0}^{r-1} (\lambda_{nk} - \lambda_{n+1,k}) s_k - \sum_{k=0}^{r-1} \lambda_k \lambda_{n+1,k} s_k \\
&= \lambda_r \lambda_{n+1,r} s_r - \lambda_{n+1} \sum_{k=0}^{r-1} (\lambda_{nk} - \lambda_{n+1,k}) s_k.
\end{aligned}$$

Set $a_0 = 1$, $a_k = 0$ for $k > 0$. Then $s_n = 1$ for all n and the above equation becomes

$$\lambda_0 \lambda_{n+1,0} = \lambda_r \lambda_{n+1,r} - \lambda_{n+1} \sum_{k=0}^{r-1} (\lambda_{nk} - \lambda_{n+1,k}).$$

or, replacing r with $r + 1$ yields

$$\sum_{k=0}^{r} (\lambda_{nk} - \lambda_{n+1,k}) = \frac{1}{\lambda_{n+1}} [\lambda_{r+1}\lambda_{n+1,r+1} - \lambda_0 \lambda_{n+1,0}]. \tag{3}$$

If $\lambda_0 = 0$ and the generalized Hausdorff matrix has zero column limits, then it follows from (3) and (1) that

$$\|H\| = \sup_{r>0} \left| \sum_{n=r}^{\infty} \frac{\lambda_r}{\lambda_n} |\lambda_{nr}| \right|$$

Thus for Hausdorff matrices $\|H\| = \mu_0$ and Lemma 2 of [10] is a special case of this fact.

THEOREM 3. *Let H be a generalized Hausdorff matrix satisfying the above conditions and the condition that there exists an $N \geq 0$ such that*

$$\lambda_r \lambda_{n+1,r} - \lambda_0 \lambda_{n+1,0} \begin{cases} < 0 & \text{for } 0 \leq n \leq N \text{ and each } r, \\ \geq 0 & \text{for } n > N \text{ and each } r. \end{cases} \tag{4}$$

Then $\|H\| = 2M_1 - \mu_0$,

$$M_1 := \int_0^1 |d\alpha(t)| - \delta|\alpha(0)|,$$

where $\delta = 0$ *if* $\lambda_0 = 0$ *and* $\delta = 1$ *if* $\lambda_0 > 0$.

PROOF. The conditions on H imply that it has zero column limits and non-negative entries.

Let N be the smallest positive integer for which (3) is satisfied, and let $S_n := \sum_{k=0}^{n} \lambda_{nk}$. From Theorems 1 and 2 of [3] $\{S_n\}$ is monotone increasing in n, and has limit M_1.

For $r \leq N$,

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \sum_{k=0}^{r} (\lambda_{nk} - \lambda_{n+1,k}) \right| &= \left(\sum_{n=0}^{r-1} + \sum_{n=r}^{N} + \sum_{n=N+1}^{\infty} \right) \left| \sum_{k=0}^{r} (\lambda_{nk} - \lambda_{n+1,k}) \right| \\ &= \sum_{n=0}^{r-1} |S_n - S_{n+1}| + \sum_{n=r}^{N} \sum_{k=0}^{r} (\lambda_{n+1,k} - \lambda_{nk}) \\ &\quad + \sum_{n=N+1}^{\infty} \sum_{k=0}^{r} (\lambda_{nk} - \lambda_{n+1,k}) \\ &= S_r - S_0 + \sum_{k=0}^{r} \lambda_{N+1,k} - \sum_{k=0}^{r} \lambda_{rk} + \sum_{k=0}^{r} \lambda_{N+1,k} \\ &\leq 2S_{N+1} - S_0. \end{aligned}$$

For $r > N$,

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \sum_{k=0}^{r} (\lambda_{nk} - \lambda_{n+1,k}) \right| &= \sum_{n=0}^{r-1} \left| \sum_{k=0}^{r} (\lambda_{nk} - \lambda_{n+1,k}) \right| + \sum_{n=r}^{\infty} \left| \sum_{k=0}^{r} (\lambda_{nk} - \lambda_{n+1,k}) \right| \\ &= S_r - S_0 + S_r \\ &= 2S_r - S_0. \end{aligned}$$

Since S_r is monotone increasing in r and $S_0 = \mu_0$, the result follows.

If, in (4) it is the case that $\lambda_r \lambda_{n+1,0} - \lambda_0 \lambda_{n+1,0} \geq 0$ for all $r, n \geq 0$ then the result remains valid.

The author takes this opportunity to thank the referee for his careful reading of the manuscript and for providing Theorems 1 and 2, which yield more efficient proofs of Corollaries 1–4.

References

1. D. Borwein, F. Peter Cass and John E. Syare, *On generalized Hausdorff matrices*, J. Approx. Theory **48**(1986), 354–360.
2. D. Borwein and A. Jakimovski, *Matrix operators on l^p*, Rocky Mountain J. Math. **9**(1979), 453–477.
3. ———, *Generalization of the Hausdorff moment problem*, Canad. J. Math. **33**(1981), 946–960.
4. K. Endl, *Untersuchungen über Momentprobleme bei Verfahren vom Hausdorffschen Typus*, Math. Annal. **139**(1960), 403–432.
5. A. Jakimovski, *The product of summability methods; part 2*, Tech. Report (8), Jerusalem (1959).
6. ———, *The product of summability methods; new classes of transformations and their properties*, Tech. note (4) AF **61(052)-187**(1959).
7. A. Jakimovski, Billy E. Rhoades and Jean Tzimbalario, *Hausdorff matrices as bounded operators over ℓ^p*, Math Z. **138**(1974), 173–181.
8. F. Hausdorff, *Summationsmethoden und Momentfolgen, I*, Math. Z. **9**(1921), 74–109.
9. ———, *Summationsmethoden und Momentfolgen, II*, Math. Z. **9**(1921), 280–299.
10. J. I. Okutoyi, *On the spectrum of C_1 as an operator on bv_0*, J. Austral. Math. Soc. **48**(1990), 79–86.
11. H. C. Rhaly, Jr., *Discrete generalized Cesàro operators*, Proc. Amer. Math. Soc. **86**(1982), 405–409.
12. ———, *p-Cesàro matrices*, Houston J. Math. **15**(1989), 137–146.
13. ———, *Terraced matrices*, Bull. London Math. Soc. **21**(1989), 399–406.
14. B. E. Rhoades, *Spectra of some Hausdorff operators*, Acta Sci. Math. **32**(1971), 91–100.
15. ———, *Generalized Hausdorff matrices bounded on ℓ^p and c*, Acta Sci. Math. **43**(1981), 333–345.
16. M. Stieglitz and H. Tietz, *Matrixtransformationen von Folgenraumen Eine Ergebnisübersicht*, Math. Z. **154**(1977), 1–16.

Department of Mathematics
Indiana University
Bloomington, Indiana 47405
U.S.A.