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FAMILIES OF SURFACES IN $\mathbb{R}^4$

J. W. BRUCE[1] AND F. TARI[2]

[1] *Department of Pure Mathematics, The University of Liverpool, PO Box 147, Liverpool L69 3BX, UK* (jwbruce@liv.ac.uk)

[2] *Universidade de São Paulo, Instituto de Matemática e de Computação, Caixa Postal 668, CEP 13560-970, São Carlos (SP), Brasil* (tari@icmsc.sc.usp.br)

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Abstract We study the geometry of surfaces in $\mathbb{R}^4$ associated to contact with hyperplanes. We list all possible transitions that occur on the parabolic and so-called A_3-set, and analyse the configurations of the asymptotic curves and their bifurcations in generic one-parameter families.

Keywords: surfaces; singularities; parabolic set; asymptotic curves; bifurcations

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1. Introduction

In this paper we study the local geometry of surfaces in $\mathbb{R}^4$ associated to contact with hyperplanes. We investigate the transitions that occur on some robust curves associated to this contact when the surface changes in generic one-parameter families.

The contact of a surface with hyperplanes is described by the family of height functions. The geometry of this family of functions is dealt with in [**21**]. Two robust sets are of interest: the parabolic set and the set of A_3 singularities of the height functions. These generally form smooth curves away from inflection points. We describe in § 3 all the possible transitions of the parabolic and A_3 sets that can occur in generic one-parameter families of surfaces. We follow the approach used in [**9**] and [**11**], first introduced in [**1**], for dealing with families of surfaces in $\mathbb{R}^3$. The main tools are transversality theorems on the so-called Monge–Taylor map and a version of stratified Morse theory in [**2**].

The family of height functions also determines curves on the surface, namely the integral curves of the asymptotic direction field. These are integral curves of a special type of implicit differential equation, called binary differential equations (BDEs). We establish in § 4 the stable configurations of these integral curves as well as their generic bifurcations in one-parameter families. A topological result about compact orientable surfaces in $\mathbb{R}^4$ is also deduced via the singularities of the BDE of the asymptotic directions.

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2. Surfaces in $\mathbb{R}^4$ and the height function

We give here a brief review and establish some notation concerning a smooth surface M in $\mathbb{R}^4$. The main references are [**5**], [**20**] and [**21**]. There is a discussion in [**21**] of the local quadratic geometry in terms of the curvature ellipse. This ellipse is the image by a pair of quadratic forms (Q_1, Q_2) of the unit circle in the tangent plane T_pM at a point p in the normal plane N_pM. This pair of quadratics is the 2-jet of the 1-flat (without constant or linear terms) map $F : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$, whose graph (in orthogonal coordinates) is locally the surface M. However, the flat geometry of surfaces is affine invariant [**9**], and the curvature ellipse is not. A different approach to the geometry of surfaces in $\mathbb{R}^4$ is given in [**5**]. This is via the pencil of the binary forms determined by the pair (Q_1, Q_2). Each point on the surface determines a pair of quadratics:

$$(Q_1, Q_2) = (lx^2 + 2mxy + ny^2, ax^2 + 2bxy + cy^2).$$

Representing a binary form $Ax^2 + 2Bxy + Cy^2$ by its coefficients $(A, B, C) \in \mathbb{R}^3$, there is a cone $B^2 - AC = 0$ representing the perfect squares. If the forms Q_1 and Q_2 are independent, then they determine a line in the projective plane $\mathbb{R}P^2$ and the cone a conic. This line meets the conic in 0, 1, 2 points according as

$$\delta(p) = (an - cl)^2 - 4(am - bl)(bn - cm) < 0, \ = 0, \ > 0.$$

A point p is said to be elliptic/parabolic/hyperbolic if $\delta < 0/ = 0/ > 0$. The set of points (x, y) where $\delta = 0$ is called the *parabolic set* of M and is denoted by Δ. If Q_1 and Q_2 are dependent, the rank of the matrix

$$\begin{pmatrix} a & b & c \\ l & m & n \end{pmatrix}$$

is 1 (provided either of the forms is non-zero); the corresponding points on the surface are referred to as umbilics. The pencil determines a point in $\mathbb{R}P^2$ which lies inside, on or outside the cone. There is an action of $\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})$ on pairs of binary forms. The orbits are as follows (see, for example, [**19**]):

(x^2, y^2)	hyperbolic point,
$(xy, x^2 - y^2)$	elliptic point,
(x^2, xy)	parabolic point,
$(x^2 \pm y^2, 0)$	umbilic (or inflexion) point,
$(x^2, 0)$	degenerate umbilic point,
$(0, 0)$	degenerate umbilic point,

We can see, for example, that the image of the unit circle by the map (x^2, y^2) is the line segment joining $(0, 1)$ to $(1, 0)$. So one cannot use the curvature ellipse to study the flat geometry of surfaces in $\mathbb{R}^4$.

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The geometrical characterization of points on M can also be described in terms of the singularity types of the height function. We recall that the family of height functions is defined by

$$h : M \times S^3 \to \mathbb{R}$$
$$(p, v) \mapsto h(p, v) = p.v$$

Proposition 2.1 (see [21]).

(1) *If* $\delta(p) > 0$*, then* p *is a non-degenerate critical point of* h_v *for any* $v \in N_pM$*.*

(2) *If* $\delta(p) < 0$*, then there are exactly two directions in* N_pM *such that* p *is a degenerate critical point of the corresponding height functions.*

(3) *If* $\delta(p) = 0$*, then there is a unique direction in* N_pM *such that* p *is a degenerate critical point of the corresponding height function.*

We next establish some notation that will be used throughout the paper. We choose local coordinates at p so that the surface is given in Monge form:

$$(x, y, f^1(x, y), f^2(x, y))$$

where (f^1, f^2) is 1-flat. Then the (modified) family of height functions around the normal direction (0,0,0,1) is given locally by

$$h(x, y, v) = v_1x + v_2y + v_3f^1(x, y) + f^2(x, y),$$

where $v = (v_1, v_2, v_3, 1)$. By rotating the coordinate axes in the normal plane we can assume that (0,0,0,1) is a degenerate direction and write

$$f^1(x, y) = lx^2 + 2mxy + ny^2 + r_0x^3 + r_1x^2y + r_2xy^3 + r_3y^3 + \cdots ,$$
$$f^2(x, y) = a_0x^2 + \sum_{i=0}^{i=3} b_ix^{3-i}y^i + \sum_{i=0}^{i=4} c_ix^{4-i}y^i + \sum_{i=0}^{i=5} d_ix^{5-i}y^i + \cdots .$$

Of course we can simplify the 2-jet of (f^1, f^2) further and consider it as one of the normal forms of the pairs of binary forms given above. The given form allows us to study all the cases together. (We observe that the origin is parabolic in the above setting if and only if $a_0n = 0$.)

It follows from a transversality result of Looijenga, or Montaldi [**22**], that for a generic surface in $\mathbb{R}^4$ the height functions $h_0 = f^2(x, y)$ have singularities of type $A_{\leqslant 4}$ or D_4. We have the following result.

Proposition 2.2. *With the above notation, the conditions for the generic singularities of the height function* h_0 *are as follows:*

A_2 : $a_0 \neq 0$, $b_3 \neq 0$,
A_3 : $a_0 \neq 0$, $b_3 = 0$, $4a_0c_4 - b_2^2 \neq 0$,
A_4 : $a_0 \neq 0$, $b_3 = 0$, $4a_0c_4 - b_2^2 = 0$, $4a_0^2d_5 - 2a_0b_2c_3 + b_1b_2^2 \neq 0$,
D_4 : $a_0 = 0$, $b_0x^3 + b_1x^2y + b_2xy^3 + b_3y^3$ *is non-degenerate.*

These singularities are versally unfolded by the family of height functions if and only if

A_2 *always,*
A_3 $n \neq 0$ *or* $b_2 \neq 0$,
A_4 $n(a_0c_3 - b_1b_2) - b_2(a_0r_3 - mb_2) \neq 0$,
D_4 $3b_0(nb_2 - 3mb_3) - b_1(nb_1 - mb_2) + l(3b_1b_3 - b_2^2) \neq 0$.

The proof follows from relatively straightforward calculations.

3. Generic transitions in the parabolic and A_3 sets

As pointed out in § 1, there are two curves of interest on the surface: namely, the parabolic set Δ, and the set of points where the height function has an A_3-singularity (the A_3-set). We are interested in the way in which these sets can change in a one-parameter family of surfaces. We first determine when the changes occur.

Let M be a given surface. We will be interested in families of embeddings $f : M \times I \to \mathbb{R}^4$, where I is some open, connected and finite interval. So for each $t \in I$ the set $f_t(M)$ is an embedded surface in $\mathbb{R}^4$, and will be denoted by M_t. We first consider the contact singularities occurring generically in such families, beyond those listed in Proposition 2.2. To establish a list, we need the following transversality theorem.

Theorem 3.1. *Let M be a compact surface, I an open interval. Let k be a positive integer and $\mathcal{S}$ an $\mathcal{A}$-invariant Whitney regular stratification of the multijet-space ${}_rJ^k(M,\mathbb{R})$. Let $\mathrm{Emb}^\infty(M,I,\mathbb{R}^4)$ denote the open subset of the space of smooth mappings $f : M \times \mathbb{R} \to \mathbb{R}^4$, with $f_t : M \to \mathbb{R}^4$ an embedding for each $t \in I$. Then the set of $f \in \mathrm{Emb}^\infty(M,I,\mathbb{R}^4)$ with the jet-extension*

$$ {}_rj_1^k h \circ f : M^{(r)} \times S^3 \times I \to {}_rJ^k(M,\mathbb{R}) $$

given by

$$ {}_rj_1^k h \circ f(p,u,t) = {}_rj^k(h_u \circ f_t)(p) $$

transverse to $\mathcal{S}$ is residual. (We can replace residual by open and dense if we ask for transversality over a compact subinterval J of I.)

Proof. The proof is a consequence of Theorem 1.1 in [**3**], which in turn follows easily from the main result of [**22**]. The key fact enabling us to apply this result to the present situation is that the family of height functions on the ambient space is $\mathcal{A}$-versal. This follows trivially from the definition. □

Note that in this paper we will largely ignore singularities arising from consideration of multi-jet spaces (semi-local singular phenomena). The key consequence that we require is the following result.

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Corollary 3.2. *Let M, I, J be as before (J a compact subinterval of I). Then for an open, dense set of mappings f in* $\mathrm{Emb}^\infty(M, I, \mathbb{R}^4)$ *there are finitely many points* $\{t(1), \dots, t(s)\}$ *in the interval J such that the following conditions are met.*

(i) *If $t \notin \{t(1), \dots, t(s)\}$, then the only singularities of the height functions h_u for the surface M_t are of type $A_{\leqslant 4}$ and D_4. Moreover, these singularities are versally unfolded by the family $h \circ f_t : M \times S^3 \to \mathbb{R}$.*

(ii) *If t is one of the $t(j)$, then either we have a singularity of type A_5 or D_5, or we have one listed in (i) above which is not versally unfolded by the family $h \circ f_{t(j)}$. All singularities (including these) are versally unfolded by the family $h \circ f : M \times S^3 \times I \to \mathbb{R}$, defined by $(h \circ f)(p, u, t) = h_u(f_t(p))$.*

Proof. The proof follows that of Corollary 3.2 in [**9**] for the family of height functions. □

So if f is a generic family of embeddings, then the flat geometry of the family of surfaces $f_t(M) = M_t$ for $t \in J$, as determined by the families $h \circ f_t$, has finitely many catastrophic events:

(i) some point of the surface $M_{t(j)}$ has contact with its tangent hyperplane which is more degenerate than an A_4 or D_4 singularity (an A_5 or D_5), or

(ii) the singularity is of type $A_{\leqslant 4}$ and D_4 but not versally unfolded by the family h of height functions alone.

We now determine the changes on the parabolic and A_3-set. We proceed as in [**9**] and [**11**], following the approach in [**1**].

Let p be a point on a surface, and choose two smooth independent vector fields in the normal plane and two smooth independent tangent vector fields in a neighbourhood U of p. This determines at each point near p a system of coordinates where the surface is given locally in Monge form: $(x, y, f^1(x, y), f^2(x, y))$, with f^1 and f^2 having no constant or linear terms.

Let V_k denote the set of polynomials in x, y of degree greater than or equal to 2 and less than or equal to k. We obtain a smooth map, the *Monge–Taylor map* $F : M, p \to V_k \times V_k$, which associates to each point q near p the k-jet of the pair of functions (f^1, f^2) defined above at the point q. The set $V_k \times V_k$ has a natural $\mathcal{G} = \mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})$-action given by linear change of coordinates in the tangent and normal plane. We showed in [**9**] that the flat geometry of smooth manifolds in a Euclidean space is affine invariant. A subset Z of $V_k \times V_k$ that is of any geometric significance will be $\mathcal{G}$-invariant. Moreover, if Z is furnished with a Whitney regular stratification, then for any generic M the map germ $M, p \to V_k \times V_k$ will be transverse to the strata of Z (see [**1**] for details).

Given a point p on our surface and a germ of a family of embeddings $i : M \times \mathbb{R}, (p, 0) \to \mathbb{R}^4$ we obtain a family of Monge–Taylor maps $\tilde{F} : M \times \mathbb{R}, (p, 0) \to V_k \times V_k$. This will be transverse to the A_3 (or $\varDelta$) stratum, say Z, in $V_k \times V_k$ for a generic family of embeddings. We then determine the diffeomorphism type of the inverse image

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$\tilde{F}^{-1}(Z)$ at $(p,0)$. For a generic family of embeddings we expect the natural projection $\pi : \tilde{F}^{-1}(Z),(p,0) \hookrightarrow M\times\mathbb{R},\ (p,0)\to\mathbb{R}$ to be generic, in the sense that it will be a stratified Morse function [**2**]. Usually, however, we can construct the module of vector fields on $M\times\mathbb{R}$ (locally just $\mathbb{R}^3$) tangent to the germ $\tilde{F}^{-1}(Z)$. We can then make a classification of smooth functions $\mathbb{R}^3,0\to\mathbb{R},0$ up to diffeomorphisms in the source preserving $\tilde{F}^{-1}(Z)$. (We also allow arbitrary changes of coordinate in the target.) We expect our projection to be a stable (or the least degenerate) germ in the classification. This is established by computing the conditions for the projection to be non-Morse (or non-stable, etc.), and showing that the resulting set of embeddings can be avoided in one-parameter families.

To carry out the calculations explicitly in $V_k\times V_k$ we need to compute the tangent space to the $\mathcal{G}$-orbit of (f^1,f^2) and the generators of the image of dF. It is not difficult to show that the tangent space to the $\mathcal{G}$-orbit of (f^1,f^2) is generated by (xf^1_x,xf^2_x), (yf^1_x,yf^2_x), (xf^1_y,xf^2_y), (yf^1_y,yf^2_y), $(f_1,0)$, $(0,f_1)$, $(f_2,0)$, $(0,f_2)$.

Proposition 2.2 in [**1**] can trivially be extended to cover the case of surfaces in $\mathbb{R}^4$ and give the generators u_1 and u_2 of the image of dF:

$$\begin{aligned} u_1 = (&-f^1_{xx}(0,0)x - f^1_{xy}(0,0)y + f^1_x(x,y) - f^1_{xx}(0,0)f^1_x(x,y)f^1(x,y)\\ &- f^1_{xy}(0,0)f^1_y(x,y)f^1(x,y) - f^2_{xx}(0,0)x - f^2_{xy}(0,0)y + f^2_x(x,y)\\ &- f^2_{xx}(0,0)f^2_x(x,y)f^2(x,y) - f^2_{xy}(0,0)f^2_y(x,y)f^2(x,y)),\end{aligned}$$

$$\begin{aligned} u_2 = (&-f^1_{xy}(0,0)x - f^1_{yy}(0,0)y + f^1_y(x,y) - f^1_{yy}(0,0)f^1_y(x,y)f^1(x,y)\\ &- f^1_{xy}(0,0)f^1_x(x,y)f^1(x,y) - f^2_{xy}(0,0)x - f^2_{yy}(0,0)y + f^2_y(x,y)\\ &- f^2_{yy}(0,0)f^2_y(x,y)f^2(x,y) - f^2_{xy}(0,0)f^2_x(x,y)f^2(x,y)).\end{aligned}$$

3.1. Changes on the A_3-set away from Δ

Let (f^1,f^2) be as in § 2, with the origin being an $A_{\geqslant 3}$ singularity of the height function $h_0=f^2$ away from the parabolic set. We can write $j^3(f^1,f^2) = (lx^2+2mxy+ny^2+f^1_3, a_0x^2+f^2_3)$, where f^1_3, f^2_3 are cubics in (x,y) and $a_0n\neq 0$. Note that we can in fact reduce the 2-jets to the pairs discussed in § 2, but by considering this normal form we can do several cases together. A complete transversal to the $\mathcal{G}$-orbit of (f^1,f^2) in $V_3\times V_3$ is given by

$$(lx^2+2mxy+ny^2+f^1_3+\overline{f^1_3},\ a_0x^2+f^2_3+\overline{f^2_3}),$$

where $\overline{f^1_3}$ and $\overline{f^2_3}$ are general cubics.

An element in this transversal has an A_3 singularity if for some λ close to zero the height function along the normal $(0,0,\lambda,1)$ given by

$$\lambda(lx^2+2mxy+ny^2+f^1_3+\overline{f^1_3})+a_0x^2+f^2_3+\overline{f^2_3}$$

has an A_3 singularity. This occurs when the quadratic part is degenerate, i.e. $Q=L^2$ and L divides the cubic; this occurs if $\lambda=0$ and $\bar{b}_3=0$. Therefore, the A_3-stratum is given by $\bar{b}_3=0$ in the transversal to the $\mathcal{G}$-orbit, and the A_3-stratum is a smooth set of codimension 1 in $V_3\times V_3$.

For a generic embedding of M the Monge–Taylor map is transverse to the A_3-stratum, so the pre-image of this set on M is locally a smooth curve. The Monge–Taylor map fails to be transverse to the A_3-stratum if and only if the tangent space to the A_3-stratum in the transversal together with the tangent vectors to the $\mathcal{G}$-orbit and the generators u_1 and u_2 of the image of the Monge–Taylor map fail to generate $V_3 \times V_3$. This occurs if and only if

$$4a_0c_4 - b_2^2 = 0 \quad \text{and} \quad n(a_0c_3 - b_1b_2) - b_2(a_0r_3 - mb_2) = 0.$$

Equivalently, the origin is an A_4 and the family of height functions fails to versally unfold this singularity. In a generic one-parameter family the associated Monge–Taylor map is transverse to the A_3-stratum and a three-dimensional transversal yields a smooth surface. So the family of the A_3-sets in the source is a smooth surface. One can show that generically the projection to the time parameter yields Morse transitions (that is max/min or saddles) on the A_3-sets. In the case of surfaces in $\mathbb{R}^3$, where we studied transitions of the parabolic set [**9**], we showed that a versal family of the height function implies that the projection to the time parameter is generic. However, this is not the case here. The condition for a generic projection is distinct from that of the versality of the family. Both conditions, being open, are satisfied for generic families of embeddings of the surface.

We can also expect changes at an A_5-singularity. It is clear from above and the condition in Proposition 2.2 that in general the A_3 set is smooth in this situation. In fact what occurs here is that the Monge–Taylor map fails to be transverse to the A_4-stratum. In a generic one-parameter family we obtain a birth of two A_4-points on a smooth A_3-set.

Proposition 3.3. *In a generic one-parameter family the A_3-set changes as follows away from the parabolic set.*

(1) *Morse transitions at a non-transverse A_3. This occurs at an A_4 singularity where the family of height functions fails to be a versal unfolding.*

(2) *The birth/annihilation of a pair of A_4 points on a smooth A_3-set at an A_5 transition.*

3.2. Changes on Δ and the A_3 sets away from umbilics

In the setting of § 2, the origin is a parabolic point if and only if $n = 0$. As we are away from umbilic points, $m \neq 0$ and a transversal to the $\mathcal{G}$-orbit of (f^1, f^2) in $V_3 \times V_3$ is given by

$$(lx^2 + 2mxy + f_3^1 + \overline{f_3^1},\ a_0x^2 + \overline{a}_2y^2 + f_3^2 + \overline{f_3^2}),$$

where $\overline{f_3^1}$ and $\overline{f_3^2}$ are general cubics and $\overline{a}_2 \in \mathbb{R}$.

In this transversal the parabolic stratum is given by $\overline{a}_2 = 0$. At points on Δ where the height function has an $A_{\leqslant 2}$-singularity ($b_3 \neq 0$) the Monge–Taylor map is always transverse to the parabolic stratum so on the surface the parabolic set is locally a smooth curve.

When $b_3 = 0$, the height function has an A_3-singularity, so we expect an A_3-curve in a neighbourhood of the origin. To compute the A_3-stratum in the above transversal we

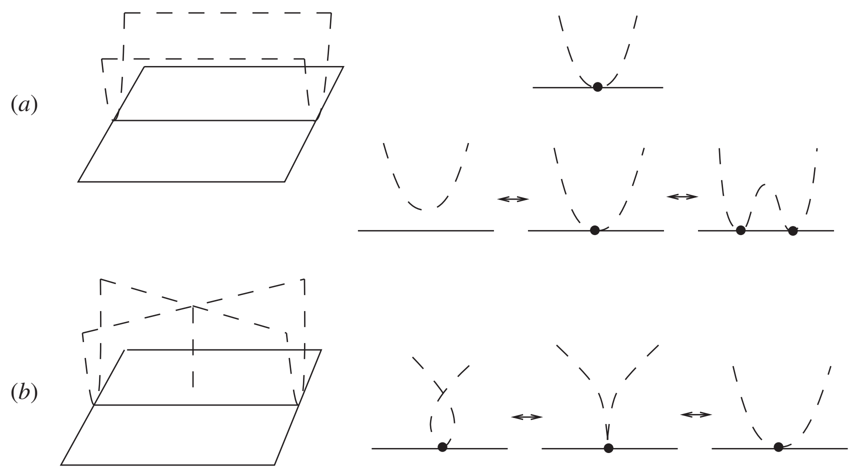

Figure 1. Changes on the A_3-set (dashed) on a smooth parabolic curve (continuous).

need to consider the height function in the normal direction $(0,0,\lambda,1)$, with λ close to zero. This is given by

$$(a_0+\lambda l)x^2+2m\lambda xy+\overline{a}_2y^2+f_3^2+\overline{f_3^2}+\lambda(f_3^1+\overline{f_3^1}).$$

The quadratic part of this function is degenerate if and only if

$$\overline{a}_2(a_0+\lambda l)-\lambda^2m^2=0,$$

and it has a factor in common with the cubic if and only if

$$(f_3^2+\overline{f_3^2}+\lambda(f_3^1+\overline{f_3^1}))(-\lambda m, a_0+\lambda l)=0.$$

From the above two equations, we can write $\overline{a}_2$ and $\overline{b}_3$ as functions of λ and the remaining variables in the transversal and give the A_3-stratum in a parametrized form. This set is a smooth hypersurface if and only if the coefficient of λ in $\overline{b}_3$ is not zero, if and only if $a_0r_3-mb_2\neq 0$. Suppose this is the case (we shall analyse the case $a_0r_3-mb_2=0$ later). Then the Δ and A_3 strata are tangential along their points of intersection $\overline{a}_2=\overline{b}_3=0$. It follows that when the Monge–Taylor map is transverse to the $A_3\cap\Delta$-stratum, the pre-images, on the surface, by this map are two tangential curves. This is not surprising, as A_3 singularities of the height function occur only in the non-elliptic region.

Proposition 3.4. *The parabolic curve and the A_3-set are generically two tangential curves at their (2-point) contact points (Figure 1a, top right).*

One can show that the Monge–Taylor map fails to be transverse to the $A_3\cap\Delta$ stratum if and only if

$$b_2(3a_0r_3^2-5mb_2r_3+8m^2c_4)=0.$$

(Note that this condition is distinct from that giving an A_4-singularity, so the Monge–Taylor map does not detect the higher-order singularities of the height function at

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parabolic points.) The condition $b_2 = 0$ implies that the Monge–Taylor map fails to be transverse to both the A_3 and Δ strata. The second condition $3a_0r_3^2 - 5mb_2r_3 + 8m^2c_4 = 0$ (but $b_2 \neq 0$) implies that the Monge–Taylor map is transverse to these strata but not to the $A_3 \cap \Delta$-stratum.

In a generic one-parameter family the Monge–Taylor map is transverse to the A_3, Δ and $A_3 \cap \Delta$ strata and a three-dimensional transversal is diffeomorphic to the variety $V = \{(u, v, w) \in \mathbb{R},\ 0 : w(w - v^2) = 0\}$ (Figure 1*a*, left). We seek to classify functions up to diffeomorphisms that preserve V in the source and any diffeomorphism in the target. The module of vector fields tangent to V is generated by ∂_u, $w\partial_v + 2vw\partial_w$, $v\partial_v + 2w\partial_w$ (where, for example, ∂_u is shorthand for $\partial/\partial u$). After some calculation it follows, from the complete transversal approach in [**10**], that the orbits of interest are u, $v + u^2$ and $w \pm u^2 + \alpha v^2$, where α is a smooth modulus distinct from $-1, 0$.

The germ u corresponds to the case when the Monge–Taylor map is transverse to the $A_3 \cap \Delta$-stratum.

The configuration here is a pair of tangential curves (see Proposition 3.4 and Figure 1*a*, top right). The germ $v + u^2$ corresponds to the case when the Monge–Taylor map is transverse to Δ and the A_3 strata but not to the $A_3 \cap \Delta$-stratum. Here, on the surface M, both the parabolic and the A_3-sets are smooth but have higher contact. In a generic one-parameter family we have a birth/annihilation of two A_3-points on a smooth parabolic curve (see Figure 1*a*, bottom right).

The germ $w \pm u^2 + \alpha v^2$ yields Morse transitions on both surfaces $w = 0$ and $w - v^2 = 0$. The restriction of this function to $w = 0$ is given by $\pm u^2 + \alpha v^2$ and its restriction to $w - v^2 = 0$ is given by $\pm u^2 + (\alpha + 1)v^2$. We obtain six different cases (see Figure 2). On the surface M, these transitions occur when the Monge–Taylor map fails to be transverse to both Δ and the A_3-strata. In a generic one-parameter family, both the parabolic and A_3-sets undergo Morse transitions. The six cases depend on whether the parabolic/A_3-sets are born in a hyperbolic or elliptic region. These transitions are drawn in Figure 2. We can show that all the transitions occur, and now give some examples realizing them.

Hyperbolic case: $(f^1, f^2) = (xy + y^3, x^2 + x^2y^2 + xy^3 + \lambda y^4)$

We have $j^2\delta = x^2 + 3xy + 3(2\lambda + 3)y^2$, and the 2-jet of the function giving the A_3-set is $-x^2 + \frac{8}{3}\lambda xy + \frac{16}{3}\lambda(\lambda + \frac{3}{2})y^2$. (When Δ is an isolated point the surface patch is hyperbolic.) The exceptional values for λ are $-\frac{9}{8}, 0$, when one of the above sets is not Morse, together with $-\frac{3}{2}$. The latter value is best understood by looking at the equation determining the asymptotic directions in § 4. When $\lambda = -\frac{3}{2}$ we have a change from a foci type equation to one involving saddles. We obtain the transitions in Figure 2*f* if $\lambda < -\frac{3}{2}$, Figure 2*e* if $-\frac{3}{2} < \lambda < -\frac{9}{8}$, Figure 2*b* if $-\frac{9}{8} < \lambda < 0$, and in Figure 2*c* if $0 < \lambda$ (birth of an elliptic island in a hyperbolic sea).

Elliptic case: $(f^1, f^2) = (xy + y^3, x^2 + x^2y - 3x^2y^2 + \lambda y^4)$

We have $j^2\delta = -x^2 - 2xy + (2\lambda + 3)y^2$ and the 2-jet of the function giving the A_3-set is $x^2 - \lambda xy + \lambda(\lambda + \frac{3}{2})y^2$. (When Δ is an isolated point the surface patch is elliptic.) The missing transitions in the hyperbolic case (Figure 2*a*) is obtained for the values $\lambda < -2$

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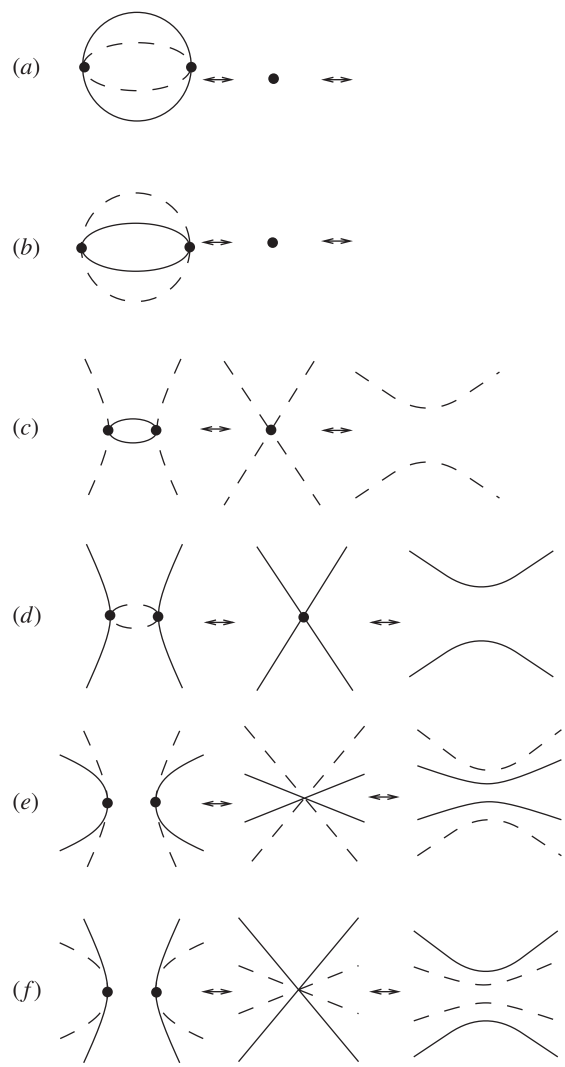

Figure 2. Morse transitions on the parabolic (continuous) and A_3-sets (dashed).

(birth of a hyperbolic island in an elliptic sea), and Figure 2d is obtained for the values $0 < \lambda$.

We return now to the case $a_0 r_3 - m b_2 = 0$. Then the A_3-stratum is no longer a smooth hypersurface, and we can change coordinates in the transversal so that it is given by

$$\bar{a}_2 = \lambda^2,$$
$$\bar{b}_3 = \lambda \bar{r}_3.$$

This is a generalized cross-cap tangent to the parabolic stratum along $\bar{a}_2 = \bar{b}_3 = 0$. We get a product stratification and a three-dimensional model is given by $w(v^2 - u^2 w) = 0$, removing the handle of the Whitney umbrella (Figure 1b, left). In other words this is

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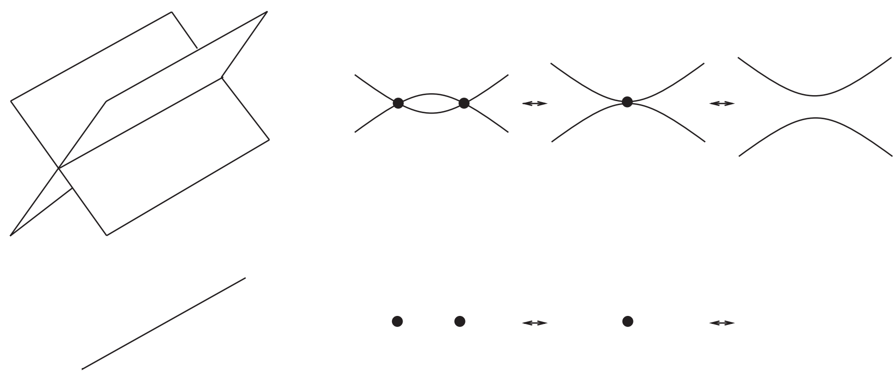

Figure 3. Changes on the parabolic set at a non-versal D_4: real type top right and imaginary type bottom right.

the form of the pull-back in the $M \times I$ space. The Monge–Taylor map is generically transverse to the $A_3 \cap \Delta$-stratum, so we are seeking germs of submersions transverse to the u-axis. These are all equivalent to germs of the form $u + w$ (this follows from [**8**] and an extra calculation) and the transition on the A_3 and Δ-sets are shown in Figure 1*b*, right. The A_3-set undergoes the cusp transitions here.

Proposition 3.5. *Changes on the parabolic set away from umbilic points occur at A_3-points. In a generic one-parameter family of surfaces the A_3 and parabolic sets undergo the Morse transitions described in Figure 2. The A_3-set also undergoes the transitions in Figure 1 on a smooth parabolic curve.*

3.3. Changes on Δ at an umbilic

In the setting of § 2 the origin is an umbilic point when $j^2 f^2 = 0$. If the curvature $\kappa = m^2 - ln \neq 0$ (see [**20**]) and $m \neq 0$, a transversal to the $\mathcal{G}$-orbit of (f^1, f^2) in $V_3 \times V_3$ is given by

$$(lx^2 + 2mxy + ny^2 + f_3^1 + \overline{f_3^1},\ \bar{a}_0 x^2 + \bar{a}_2 y^2 + f_3^2 + \overline{f_3^2}),$$

where $\overline{f_3^1}$ and $\overline{f_3^2}$ are general cubics and $\bar{a}_0, \bar{a}_2 \in \mathbb{R}$. (If $m = 0$ we can consider another transversal and the results follow in the same way.) In this transversal the D_4-stratum is given by $\bar{a}_0 = \bar{a}_2 = 0$ and the parabolic stratum by

$$n^2 \bar{a}_0^2 + 2(2m^2 - nl)\bar{a}_0 \bar{a}_2 + l^2 \bar{a}_2^2 = 0.$$

This is a non-degenerate quadratic if $m^2 - ln \neq 0$ and $m \neq 0$. In this case a three-dimensional transversal consists of two intersecting planes at an inflection point of real type, with the intersecting set the D_4-stratum, and of a line (the D_4-stratum) in the case of an inflection point of imaginary type. Models for these are provided by the varieties $V = \{(u, v, w) \in \mathbb{R},\ 0 : w^2 \pm v^2 = 0\}$ (Figure 3, left).

It follows that when the Monge–Taylor map is transverse to the D_4-stratum, the parabolic set on the surface consists of a pair of transverse-intersecting curves at an

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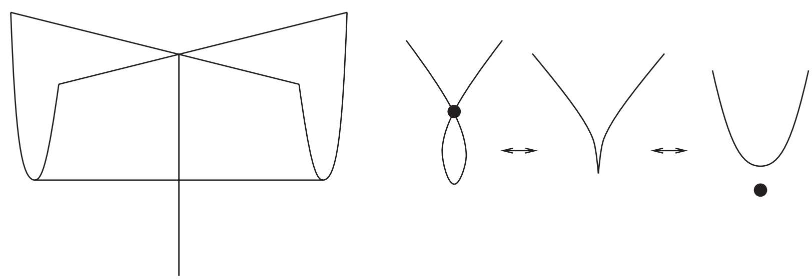

Figure 4. Changes on the parabolic set at a D_4 point with $\kappa(p) = 0$.

inflection point of real type, and an isolated point at an inflection point of imaginary type. (This is proved in [**21**] by direct calculations.)

The Monge–Taylor map fails to be transverse to the D_4-stratum if and only if the family of height functions is not a versal unfolding of the D_4-singularity at the origin. Then in a generic one-parameter family of surfaces, we obtain transitions on Δ equivalent to those given by Morse sections on V. These are represented by the germ $v + u^2$ and are as in Figure 3.

When the curvature is zero, a transversal has the form

$$(lx^2 + 2(m + \overline{m})xy + ny^2 + f_3^1 + \overline{f_3^1},\ \overline{a}_0x^2 + \overline{a}_2y^2 + f_3^2 + \overline{f_3^2}).$$

Now the quadratic part of the equation giving the parabolic stratum is degenerate. (Note that when $\kappa = 0$, the family of height functions remains a versal unfolding of the D_4-singularity.) One can show that, in an appropriate system of coordinates in the transversal, this stratum is given by $v^2 - uw^2 = 0$ (Figure 4, left). So we have a product stratification and a three-dimensional transversal yields a Whitney umbrella. Generic sections on a Whitney umbrella are as in Figure 4, right (see [**8**]).

The curvature does not vanish in general at a D_5-singularity and the Monge–Taylor map is transverse to the D_4-stratum, so no transitions occur here on the parabolic set. However, we shall see below that changes occur on the A_3-set in this situation.

Proposition 3.6. *The parabolic set undergoes the following transitions at an umbilic.*

(1) *Birth of two inflection points of real or imaginary types at a non-versal D_4 (Figure 3).*

(2) *Change from a real-type to imaginary-type inflection via sections of a Whitney umbrella at a D_4 singularity when the curvature is zero (Figure 4).*

(3) *No changes occur on the parabolic set at a D_5-singularity.*

3.4. Changes on the A_3-set at an umbilic

To simplify the calculations, we shall take here, without loss of generality, $j^2f = (x^2 \pm y^2, 0)$ at an umbilic point. Then $j^3f = (x^2 \pm y^2 + f_3^1, f_3^2)$, where f_3^1, f_3^2 are general cubics. We can rotate coordinates so that $f_3^2 = x(x + \alpha y)(x + \beta y)$ at an elliptic umbilic

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(D_4^+) with $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \neq \beta$, and to $f_3^2 = x(x^2 + 2sxy + ty^2)$ with $s^2 - t < 0$ at a hyperbolic umbilic (D_4^-). We shall treat the D_4^+ case in detail, the D_4^- follows in the same way and is easier. A transversal to the $\mathcal{G}$-orbit of (f^1, f^2) in $V_3 \times V_3$ is given by

$$(x^2 \pm y^2 + f_3^1 + \overline{f_3^1},\ \overline{a}_0x^2 + \overline{a}_1xy + x(x + (\alpha + \overline{\alpha})y)(x + (\beta + \overline{\beta})y)),$$

where $\overline{f_3^1}$ is a general cubic and $\overline{a}_0, \overline{a}_1, \overline{\alpha}, \overline{\beta} \in \mathbb{R}, 0$.

In this transversal the D_4-stratum is given by $\overline{a}_0 = \overline{a}_1 = 0$. We are seeking the A_3-stratum. Recall that the height function along a normal direction $(0, 0, \lambda, 1)$, with λ close to zero, is given by $f_2 + \lambda f_1 = Q + C$, where

$$Q = (\overline{a}_0 + \lambda)x^2 + \overline{a}_1xy \pm \lambda y^2,$$
$$C = x(x + (\alpha + \overline{\alpha})y)(x + (\beta + \overline{\beta})y) + \lambda(f_3^1 + \overline{f_3^1}).$$

As λ is close to zero we can write the cubic C of the form

$$C = (x + \lambda A_1y)(x + (\alpha + \overline{\alpha} + \lambda A_2)y)(x + (\beta + \overline{\beta} + \lambda A_3)y).$$

The height function along $(0, 0, \lambda, 1)$ has an A_3-singularity if and only if $Q = L^2$ and L divides the cubic C. For the quadratic to be degenerate we need

$$\overline{a}_1^2 \mp 4\lambda(\overline{a}_0 + \lambda) = 0. \tag{3.1}$$

Then it is not hard to show that L is a multiple of the first factor of C if and only if $\overline{a}_1 = 0$.

For the remaining two factors in C, we can assume that the coefficient of x^2 in Q is not zero, that is, $(\overline{a}_0 + \lambda) \neq 0$. In this case L is a multiple of $x + (\alpha + \overline{\alpha} + \lambda A_2)y$ if and only if

$$\overline{a}_1 - 2(\overline{a}_0 + \lambda)(\alpha + \overline{\alpha} + \lambda A_2) = 0. \tag{3.2}$$

Substituting this in (3.1) we get

$$(\overline{a}_0 + \lambda)(\alpha + \overline{\alpha} + \lambda A_2)^2 \mp \lambda = 0.$$

We can parametrize $\overline{a}_0$ and $\overline{a}_1$ in this equation and equation (3.2) by λ and the remaining variables in the transversal. This results in a smooth hypersurface in $V_3 \times V_3$. In the same way, when L is a multiple of the last factor in C, we obtain a smooth hypersurface in $V_3 \times V_3$. So the A_3-stratum in the transversal consists of three hypersurfaces meeting transversally along the D_4-stratum when $\alpha \neq \beta$ and $\alpha\beta \neq \mp 1$, and a three-dimensional model is given by $V = \{(u, v, w) : vw(v - w) = 0\}$. The transverse sections are equivalent to u. These represent the case when the Monge–Taylor map is transverse to the D_4-stratum, so the A_3-set consists here of three smooth curves meeting transversally at the elliptic umbilic. One can show that the Monge–Taylor map fails to be transversal to the D_4-stratum if and only if the height function is not a versal unfolding of the D_4-singularity. In this case the Morse functions on V are equivalent to $v + aw + u^2$ with $a \neq -1, 0$, and the transitions of the A_3-curves are drawn in Figure 5*a* (see also Figure 5*b* for the hyperbolic umbilic case).

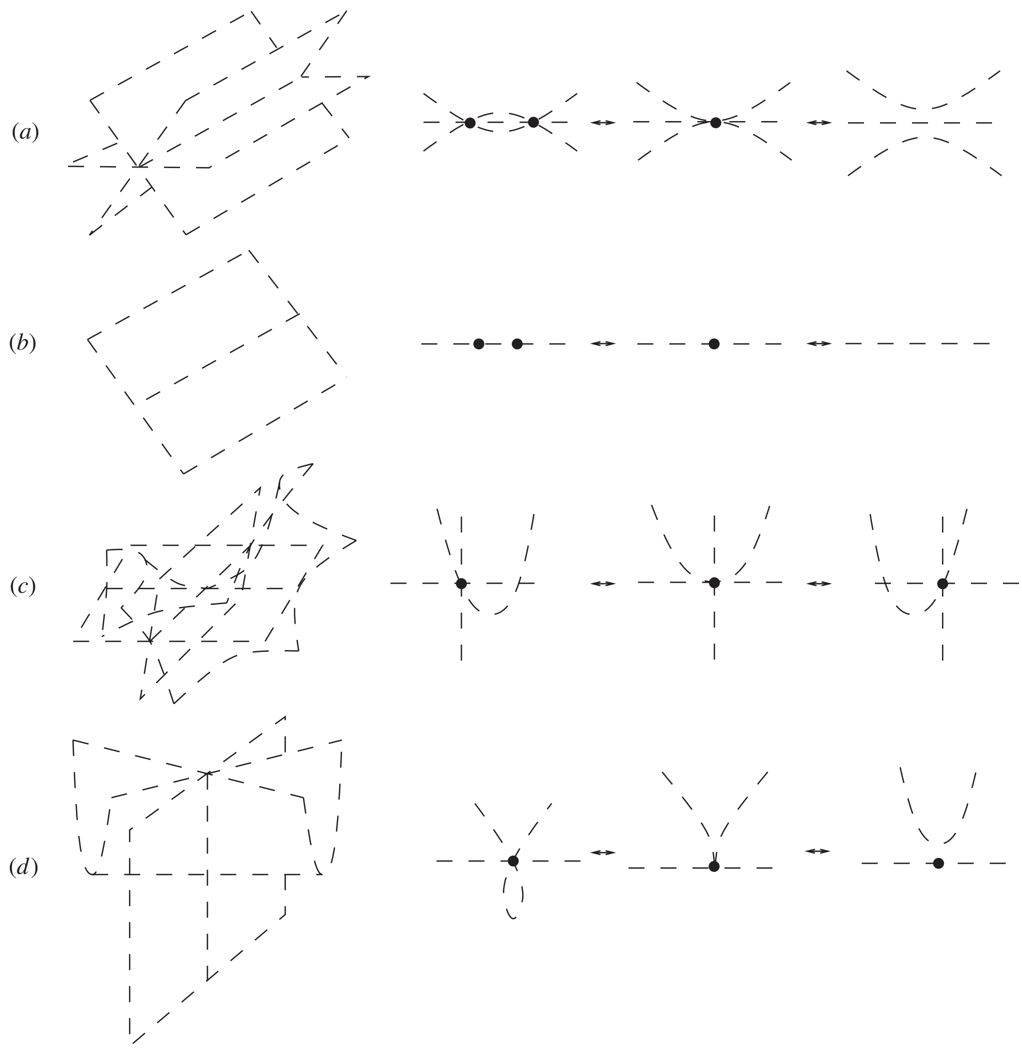

Figure 5. Changes on the A_3-set at an umbilic: (a), (b) a non-versal D_4; (c) $\alpha\beta = \mp 1$; and (d) a D_5-umbilic.

We can also obtain transitions on the A_3-set when the roots of the cubic f_3^2 satisfy $\alpha\beta = \mp 1$ or when f_3^2 has a double root (at a D_5 umbilic).

Following the above calculations further, we can show that when $\alpha\beta = \mp 1$, the A_3-stratum is given, after changes of coordinates, by

$$\overline{a}_1\overline{a}_0(\overline{a}_0 - \overline{a}_1\overline{\beta}) = 0.$$

Generic sections of this set have been determined in [**11**] and are as shown Figure 5c.

At a D_5-singularity, similar calculations show that the A_3-stratum is given, after changes of coordinates, by $\overline{a}_1(\overline{a}_1^2 + \overline{\alpha}\,\overline{a}_0^2) = 0$ (Figure 5d, left), and the generic sections on this set are also given in [**11**] and are as in Figure 5d, right.

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Proposition 3.7. *At an umbilic point, the A_3-set consists generically of three transverse smooth curves at a D_4^+ and a single smooth curve at a D_4^-.*

The A_3-set undergoes the following transitions at an umbilic.

(1) *Birth of two elliptic or hyperbolic umbilics at a non-versal D_4 (Figure 5a, b).*

(2) *Removing the tangency of two A_3 curves at an elliptic umbilic (Figure 5c).*

(3) *Change from an elliptic to a hyperbolic umbilic at a D_5 singularity of the height function (Figure 5d).*

We can use the fact that the A_3-singularities occur only in the non-elliptic region to sketch all possible joint transitions of the parabolic and A_3-set at an umbilic point, by superimposing the configurations in Figures 3 and 4 with those in Figure 5. (Note that we can have an inflection point of real or imaginary type at both elliptic and hyperbolic umbilics.)

4. Asymptotic directions on M

The contact of a surface $M \subset \mathbb{R}^4$ with lines is determined by the family of orthogonal projections

$$P : M \times S^3 \to TS^3$$
$$(p, v) \mapsto (v, p- < p, v > v),$$

where TS^3 is the tangent bundle to the unit sphere. For most (respectively, all) tangent directions v at a hyperbolic (respectively, elliptic) point of M the corresponding germ $P_v : M \to \mathbb{R}^3$ is stable. At a hyperbolic point there are two directions of projection, called the *asymptotic directions*, where the singularity of P_v is degenerate (worse than a cross-cap).

It is shown in [**5**] that the contact of a surface $M \subset \mathbb{R}^4$ with lines is also determined by a pencil of binary forms. Again suppose that the surface is given locally by $(x, y, f^1(x, y), f^2(x, y))$ with the 1-jets of f^1 and f^2 identically zero, and denote by (Q_1, Q_2) the 2-jet of (f^1, f^2). Then projecting along a tangent direction $(\alpha, \beta, 0, 0)$ yields a map-germ with 2-jet $(-\beta x + \alpha y, Q_1(x, y), Q_2(x, y))$. As $[A : B]$ varies in the projective line $\mathbb{R}P^1$, the 2-jet of the projection defines a pencil of binary forms with $-\beta x + \alpha y$ as a factor. This gives a line in $\mathbb{R}P^2$, one for each direction, parametrized by $[s : t] \mapsto [-\beta s : (\alpha s - \beta t)/2 : \alpha t]$, or with equation $\alpha^2 A + 2\alpha\beta B + \beta^2 C = 0$, and it is the tangent line to $B^2 - AC = 0$ at $[\beta^2 : -\alpha\beta : \alpha^2]$ that corresponds to $(-\beta x + \alpha y)^2$. Therefore, the pencil of quadratic forms correspond to the tangent lines to the conic of degenerate forms.

Proposition 4.1 (see [5]). *The direction of projection yields a cross-cap unless the line it determines passes through one of the points of intersection of the conic of degenerate forms with the pencil (Q_1, Q_2) and is consequently tangent to the conic there. So there are two asymptotic directions at a hyperbolic point, one at a parabolic point and none at an elliptic point.*

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Corollary 4.2. *The asymptotic directions are solutions of the BDE*

$$(am - bl)\,\mathrm{d}x^2 + (an - cl)\,\mathrm{d}x\,\mathrm{d}y + (bn - cm)\,\mathrm{d}y^2 = 0,$$

where a, b, c and l, m, n are the coefficients of the quadratic forms Q_1 and Q_2 at (x, y). This equation is affine invariant and can also be written in the following form:

$$\begin{vmatrix} \mathrm{d}y^2 & -\mathrm{d}x\,\mathrm{d}y & \mathrm{d}x^2 \\ a & b & c \\ l & m & n \end{vmatrix} = 0.$$

The discriminant of the BDE is given by the zero set of the function

$$\delta = (an - cl)^2 - 4(am - bl)(bn - cm)$$

and coincides with the parabolic set of the surface.

Proof. Using Proposition 4.1, one only has to determine the condition for the projection along the direction $[\mathrm{d}x : \mathrm{d}y]$ to be worse than a cross-cap. The equation follows from a straightforward calculation. □

To study the configurations of the asymptotic curves, and the way they bifurcate in generic one-parameter families of surfaces, we need to recall some results on BDEs. These equations are a particular type of implicit differential equation of the form

$$A(x, y)\,\mathrm{d}x^2 + 2B(x, y)\,\mathrm{d}x\,\mathrm{d}y + C(x, y)\,\mathrm{d}y^2 = 0,$$

where A, B, C are smooth germs of functions $\mathbb{R}^2, 0 \to \mathbb{R}$. A BDE defines pairs of directions at points (x, y) in the plane where $\delta = B^2 - AC > 0$. These directions coincide on the discriminant Δ given by $\delta = 0$; the BDE has no solutions at points where $\delta < 0$.

One way to proceed in the study of BDEs is to consider in $\mathbb{R}^2 \times \mathbb{R}P^1$ the set S of points $(x, y, [\alpha : \beta])$, where $\delta(x, y) \geqslant 0$ and the direction $[\alpha : \beta]$ is a solution of the BDE at (x, y). One can lift the bivalued field defined by the BDE to a single-valued field ξ on S. Generically M is smooth, and there is a natural involution on S that interchanges points with the same image under the projection to $\mathbb{R}^2$. The set of fixed points of this involution is the lift of the discriminant. By studying this single field together with the involution, a number of useful classifications have been carried out for BDEs with the simplest discriminants (those which are smooth or have only Morse singularities).

One can separate BDEs into two categories distinguished by whether the coefficients all vanish at the origin or not. In the first category, the stable configurations of the integral curves at a point on the discriminant are obtained in [**13**–**15**]. If the discriminant is smooth and the unique direction defined by the equation is transverse to it, then a smooth model is given by $\mathrm{d}x^2 + y\,\mathrm{d}y^2 = 0$ [**13**, **14**]. If the unique direction is tangent to the discriminant, then the stable BDEs are smoothly equivalent to $\mathrm{d}x^2 + (-x + \lambda y^2)\,\mathrm{d}y^2 = 0$ with $\lambda \neq 0, \frac{1}{16}$ [**15**]. The singularity is a *well-folded saddle* if $\lambda < 0$, a *well-folded node* if $0 < \lambda < \frac{1}{16}$, and a *well-folded focus* if $\frac{1}{16} < \lambda$. The modulus λ can be eliminated topologically [**12**, **15**].

The bifurcations in generic one-parameter families of BDEs with non-vanishing coefficients have also been established. These consists of the *well-folded saddle/node bifurcations* ($\lambda = 0$ above) and occur when the discriminant is smooth and the lifted field ξ has a saddle-node singularity [**16**]. When $\lambda = \frac{1}{16}$ we have a change from node to focus and the normal form is given in [**17**]. The other case occurs when the discriminant has a Morse singularity. These equations are labelled *Morse type 1* and are studied in [**12**].

When the coefficients of the BDE vanish at the origin the surface S is smooth if and only if δ has a Morse singularity [**6**]. These equations are of codimension 1 and are labelled *Morse type 2* [**6**, **7**]. In this case the whole projective line (the exceptional fibre) $0 \times \mathbb{R}P^1$ is in S, and the field ξ has generically three or one zeros on the exceptional fibre of type saddles (S) or nodes (N). When the discriminant is an isolated point (A_1^+ singularity) the coefficients of the topological models are:

(i) ($\mathcal{L}$) lemon (1S), $(y, x, -y)$,

(ii) ($\mathcal{S}$) star (3S), $(y, -x, -y)$, and

(iii) ($\mathcal{M}$) monstar (2S + 1N), $(y, \frac{1}{4}x, -y)$,

where the numbers 1S, 3S, 2S + 1N indicate the number and type of the singularities of the field ξ on S.

When the discriminant is a crossing (A_1^+ singularity) the topological models are:

(i) ($\mathcal{U}_1$) 1S, (y, x, y),

(ii) ($\mathcal{U}_2$) 1N, $(y, -\frac{1}{4}x, y)$,

(iii) ($\mathcal{U}_3$) 3S, $(y, -2x, y)$,

(iv) ($\mathcal{U}_4$) 2S + 1N, $(y, y - x, y)$, and

(v) ($\mathcal{U}_5$) 1S + 2N, $(y, -\frac{2}{3}x, y)$.

We shall now determine the relevant local models for the BDE of the asymptotic curves, i.e. the conditions for the BDE to be equivalent to one of the above models, and where possible the geometrical interpretation of each case. There are two situations to consider here depending on whether the origin is an umbilic (inflection) point or not.

4.1. The origin is not an umbilic point

In this case the pair of quadratics (Q_1, Q_2) could be taken, after affine changes of coordinates, in the form (xy, x^2), so that the height function along the normal direction $(0, 0, 0, 1)$ has a degenerate singularity. We can then write the functions f^1 and f^2 in § 2 in the form

$$f^1 = xy + r_0x^3 + r_1x^2y + r_2xy^2 + r_3y^3 + \sum_{i=0}^{i=4} s_i x^{4-i}y^i + \cdots,$$
$$f^2 = x^2 + b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + \sum_{i=0}^{i=4} c_i x^{4-i}y^i + \cdots.$$

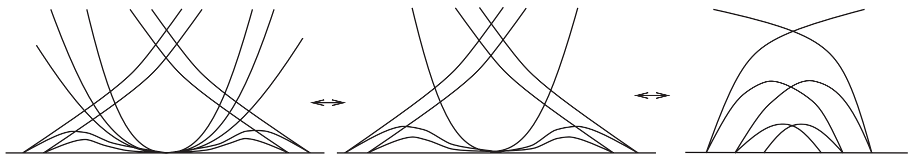

Figure 6. Changes from well-folded node to well-folded focus.

The 2-jets of the coefficients of the second fundamental forms are given by

$$\begin{aligned} a &= \tfrac{1}{2}f^1_{xx} = \tfrac{1}{2}(6r_0x + 2r_1y + 12s_0x^3 + 6s_1xy + 2s_2y^2),\\ b &= \tfrac{1}{2}f^1_{xy} = \tfrac{1}{2}(1 + 2r_1x + 2r_2y + 3s_1x^2 + 4s_2xy + 3s_3y^2),\\ c &= \tfrac{1}{2}f^1_{yy} = \tfrac{1}{2}(2r_2x + 6r_3y + 2s_2x^2 + 6s_3xy + 12s_4y^2),\\ l &= \tfrac{1}{2}f^2_{xx} = \tfrac{1}{2}(2 + 6b_0x + 2b_1y + 12c_0x^3 + 6c_1xy + 2c_2y^2),\\ m &= \tfrac{1}{2}f^2_{xy} = \tfrac{1}{2}(2b_1x + 2b_2y + 3c_1x^2 + 4c_2xy + 3c_3y^2),\\ n &= \tfrac{1}{2}f^2_{yy} = \tfrac{1}{2}(2b_2x + 6b_3y + 2c_2x^2 + 6c_3xy + 12c_4y^2), \end{aligned}$$

and the resulting 1-jets of the coefficients of the BDE of the asymptotic curves are given, after scaling, by

$$\begin{aligned} A &= 1 + (3b_0 + 2r_1)x + (b_1 + 2r_2)y,\\ B &= r_2x + 3r_3y,\\ C &= -b_2x - 3b_3y. \end{aligned}$$

The BDE is equivalent to $\mathrm{d}x^2 + y\,\mathrm{d}y^2$ if and only if $b_3 \neq 0$ [**13, 14**], that is, when the origin is an A_2 singularity of the height function. In this case the asymptotic curves (integral curves of the BDE) form a family of cusps.

When $b_3 = 0$ but $b_2 \neq 0$, that is, a height function has an A_3 singularity and the parabolic set is smooth, we can reduce the 2-jet, and hence the BDE, to

$$\mathrm{d}x^2 + (-x - (3/b_2^2)(3r_3^2 - \tfrac{5}{2}b_2r_3 + 2c_4)y^2)\,\mathrm{d}y^2 = 0$$

if $\lambda \neq 0, \frac{1}{16}$, where $\lambda = -(3/b_2^2)(3r_3^2 - \frac{5}{2}b_2r_3 + 2c_4)$. The singularity of the asymptotic field has a *well-folded singularity* [**15**] of type saddle if $\lambda < 0$, node if $0 < \lambda < \frac{1}{16}$ and focus if $\frac{1}{16} < \lambda$.

When $\lambda = 0$ we get, in a generic one-parameter family of surfaces, a *well-folded saddle/node bifurcation* [**16**]. This condition is precisely that for the Monge–Taylor map to fail to be transverse to the A_3-stratum at a parabolic point (see § 3.2, where $a_0 = 1$ and $m = \frac{1}{2}$ here). In particular, one of the newly born A_3-points is of type well-folded saddle and the other well-folded node.

When $\lambda = \frac{1}{16}$ we get, in a generic one-parameter family of surfaces, a change from a well-folded node to a well-folded focus [**17**]. A normal form for this family is given by $\mathrm{d}y^2 + (-y + (\frac{1}{16} + t)x^2)\,\mathrm{d}x^2 = 0$. We draw in Figure 6 the changes of integral curves of this family, as they have not been drawn anywhere else. We observe that the condition

$\lambda = \frac{1}{16}$ for the BDE of the asymptotic curves is not related to any of those arising from the family of height functions or from the Monge–Taylor map.

When $b_3 = b_2 = 0$, the discriminant is singular, and as the coefficients of the BDE are not all zero at the origin, we generally get a *Morse type 1* singularity [**12**]. Changes of coordinates show that the 2-jet, and hence the BDE, is equivalent to

$$\mathrm{d}x^2 + [(-r_2^2 + 2b_1r_2 - c_2)x^2 + 3(-2r_2r_3 + 2b_1r_3 - c_3)xy - 3(3r_3^2 + 2c_4)y^2]\,\mathrm{d}y^2 = 0,$$

when $3(-2r_2r_3 + 2b_1r_3 - c_3)^2 + 4(3r_3^2 + 2c_4)(-r_2^2 + 2b_1r_2 - c_2) \neq 0$ (Morse condition on the discriminant) and $-3r_3^2 - 2c_4 \neq 0$. The last condition ensures that the reduction is valid [**12**]. It also means that the BDE has multiplicity 2; alternatively, the unique direction determined by the equation is not tangent to the discriminant. We get two well-folded saddles on one side of the transition when $-3r_3^2 - 2c_4 < 0$ and two well-folded foci otherwise. We can use the fact that the A_3-set and the asymptotic curves lie in the hyperbolic region to draw the configurations of the asymptotic directions on Figure 2 using the models in [**12**].

Proposition 4.3. *Away from umbilic points the asymptotic lines have the following stable topological configurations at a parabolic point on a surface $M \in \mathbb{R}^4$.*

(1) *A family of cusps at an ordinary parabolic point.*

(2) *A well-folded singularity at an A_3-point of the height function.*

In a generic one-parameter family of surfaces these curves undergo the following bifurcations.

(3) *Well-folded saddle/node bifurcations at a non-transverse A_3-point on a smooth parabolic set.*

(4) *Changes from well-folded node to well-folded focus at an A_3-point on a smooth parabolic set when the lifted field has one eigenvalue with multiplicity* 2.

(5) *Morse type 1 bifurcations at a Morse singularity of the parabolic set.*

4.2. The origin is an umbilic point: the stable structures

When the origin is an umbilic, the coefficients of the BDE of the asymptotic curves vanish, and in general the discriminant has a Morse singularity so we have a *Morse type 2* BDE. Although this BDE is of codimension 1 in the set of all BDEs, it is stable in this context as generic umbilic points are stable. We shall determine here only the type of this BDE at generic umbilics. In order to determine its bifurcations we need to understand codimension 2 phenomena in BDEs with zero coefficients. This is part of a future investigation.

There are two types of umbilics: imaginary type, where the pair of quadratics (Q_1, Q_2) is equivalent to $(x^2 + y^2, 0)$ (Δ is an isolated point); and the real type, when (Q_1, Q_2) is equivalent to $(xy, 0)$ (Δ is a crossing). We shall treat the cases separately.

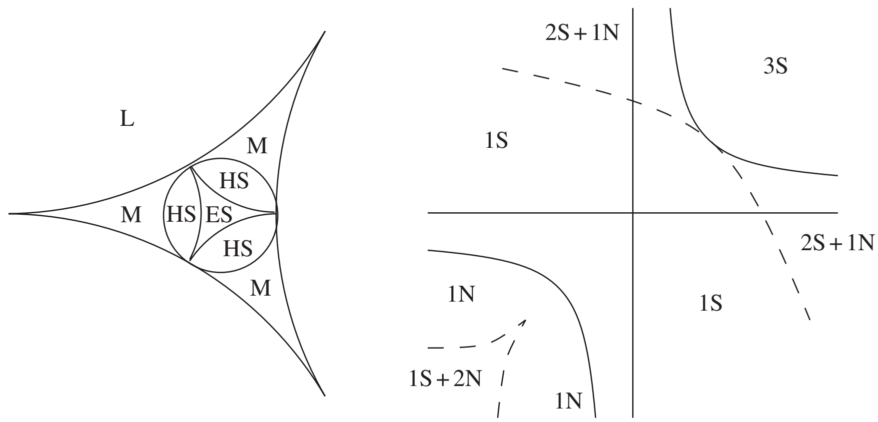

Figure 7. Partition of the set of cubics left, and of the (a,b)-plane right ('L' denotes lemon, 'M' denotes monstar, 'ES' denotes elliptic star, and 'HS' denotes hyperbolic star).

The imaginary type: $(x^2+y^2,0)$

(This case has also been dealt with in [**18**] using an alternative method.) The 3-jet of the parametrization of the surface is of the form $(x,y,x^2+y^2+O(3),v(x,y)+O(3))$, where $v(x,y)=b_0x^3+b_1x^2y+b_2xy^3+b_3y^3$. Then the 1-jet of the BDE is given by

$$v_{xy}\,\mathrm{d}x^2+(v_{yy}-v_{xx})\,\mathrm{d}x\,\mathrm{d}y-v_{xy}\,\mathrm{d}y^2.$$

This is precisely the 1-jet of the BDE giving the principal directions of the surface $(x,y,a_0(x^2+y^2)+v(x,y)+O(3))$ in $\mathbb{R}^3$ at the origin, an umbilic point (see [**4**]). So its configuration follows the same pattern, and, in particular, if the cubic v is written in the complex form $\mathrm{Re}(z^3+\beta z^2\overline{z})$, we obtain the well-known partition of the β-plane into the various regions where the *lemon, star and monstar* umbilic occur (Figure 7, left). As inflection points are stable on surfaces in $\mathbb{R}^4$, the BDEs giving the asymptotic curves are stable in this context.

The real type: $(xy,0)$

The surface is given by $(x,y,xy+O(3),v(x,y)+O(3))$, where $v(x,y)=x^3+b_1x^2y+b_2xy^3+y^3$ after scaling provided $b_0b_3\neq 0$ (see [**5**]). The 1-jet of the BDE is then given by

$$v_{xy}\,\mathrm{d}x^2-v_{yy}\,\mathrm{d}y^2,$$

and its discriminant by $v_{xy}v_{yy}=(3x+b_1y)(b_2x+3y)=0$. This quadratic is non-degenerate (singularity of Morse type) if $b_1b_2-9\neq 0$. The curve $b_1b_2-9=0$ in the (b_1,b_2)-plane also represents the set of cubics where the family of height functions fails to be a versal unfolding of the D_4 singularity at the origin (see Proposition 2.2 and Figure 7, right, the continuous hyperbola).

There are two curves of interest in the (b_1,b_2)-plane (see [**6**]). The first represents the cubics where the singularity of the lifted field are not stable (i.e, its linearization

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has a zero eigenvalue at a singular point). This set coincides with the non-Morse curve $b_1b_2 - 9 = 0$.

The second curve consists of points (b_1, b_2), where the cubic giving the zeros of the lifted field has a double root. This cubic is $\phi(p) = 3p^3 + b_2p^2 - b_1p - 3$ and the double root curve is given by

$$b_1(t) = -3(t^2 - (2/t)),$$
$$b_2(t) = 3(2t - (1/t^2)),$$

or alternatively given by the equation

$$(b_1b_2 - 81)^2 - 4(b_2^2 + 9b_1)(b_1^2 + 9b_2) = 0$$

(Figure 7, right, the dashed curve). This curve coincides with the set of points where the direction of the orthogonal projection has a double S_2 singularity [**5**]. In Figure 7, right, we indicate the different topological types in each region bounded by the above curves. We observe that all the cases of *Morse type 2* BDEs occur in this context. As hyperbolic umbilics are stable points on a surface in $\mathbb{R}^4$, the BDE of asymptotic lines provides a geometric context where *Morse type 2* BDEs with a discriminant having an A_1^- singularity are stable.

Proposition 4.4. *The asymptotic directions have the configurations of the lemon, star or monstar BDEs at an umbilic point of imaginary type. At an umbilic point of real type it is topologically equivalent to one of the five models of BDEs of Morse type 2 with a discriminant of type* A_1^-. *In both cases the configurations are stable.*

We shall now deduce some global properties of a compact surface in $\mathbb{R}^4$ from the singularities of the BDE of the asymptotic lines. We shall denote by HC (hyperbolic cusp) (respectively, EC (elliptic cusp)) the well-folded saddle singularity (respectively, well-folded node/focus singularity) of the BDE of the asymptotic curves. The lemon, star and monstar singularities of this BDE are denoted by $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{M}$, respectively. The different types of singularities of *Morse type 2* BDEs where the discriminant is a node are denoted by $\mathcal{U}_1, \dots, \mathcal{U}_5$ (see the beginning of this section). The symbol $n(\text{sing})$ indicates the number of singularities on a surface M of one of the given types above, M_{h} denotes the hyperbolic region of M, $\overline{M_{\text{h}}}$ its closure, and $\mathcal{X}(M)$ denotes the Euler characteristic of M.

Theorem 4.5. *Let M be a generic compact embedded surface in $\mathbb{R}^4$. Then*

$$\mathcal{X}(\overline{M_{\text{h}}}) = \tfrac{1}{2}[n(\mathcal{L}) - n(\mathcal{S}) + n(\mathcal{M})] + \tfrac{1}{2}[n(EC) - n(HC)] + \tfrac{1}{2}[n(\mathcal{U}_1) + 3n(\mathcal{U}_2) - n(\mathcal{U}_3) + n(\mathcal{U}_4) + 3n(\mathcal{U}_5)].$$

In particular, if N is convex, then

$$\mathcal{X}(M) = \tfrac{1}{2}[n(\mathcal{L}) - n(\mathcal{S}) + n(\mathcal{M})],$$

and if M has no inflection points of real type, then

$$\mathcal{X}(\overline{M_{\text{h}}}) = \tfrac{1}{2}[n(\mathcal{L}) - n(\mathcal{S}) + n(\mathcal{M})] + \tfrac{1}{2}[n(EC) - n(HC)].$$

Proof. As in the proof of Theorem 2.1 in [**4**], we consider in the projectivized tangent bundle to M, $\mathrm{P}(TM)$, the set $\tilde{M}$ which consists of the asymptotic directions at non-umbilics and all directions at umbilic points. Since the surface M is generic, the parabolic set is either smooth or has Morse singularities at umbilic points, and hence the surface $\tilde{M}$ is smooth [**6**], and the boundary of $\overline{M}_h$, the closure of the hyperbolic region of M, is a union of circles. The projection $\pi : \tilde{M} \to \overline{M}_h$ is a smooth 2-fold covering away from the parabolic set and umbilics. The exceptional fibres over umbilic points are real projective lines, that is circles, with cylindrical neighbourhoods. The set $\tilde{M}$ is obtained by deleting disk neighbourhoods of the umbilic points of $\overline{M}_h$ taking two copies of the result, gluing along the boundary of $\overline{M}_h$ and sewing in the cylinders. It is not hard to see that $\mathcal{X}(\tilde{M}) = 2(\mathcal{X}(\overline{M}_h) - m)$, where m is the number of umbilics. The bivalued asymptotic field on M_{h} lifts to a smooth line field ξ on $\tilde{M}$. For a generic surface M this field has one or three zeros on each exceptional fibre [**6**], and well-folded saddles/nodes/foci on the lift of the parabolic set. The result now follows using Poincaré's Theorem. □

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