FUNCTIONS OF THREE VARIABLES WHICH SATISFY BOTH THE HEAT EQUATION AND LAPLACE'S EQUATION IN TWO VARIABLES

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1. Introduction

In a recent paper on statistical fluid mechanics Professor J. Kampé de Fériet [1] employed several integrals of which the following is a typical example

$$(1.1) \qquad u(x, y, t) = \int_0^\infty e^{-xr^2} \cos yr^2 \cos tr\, \phi(r) dr.$$

The function $u(x, y, t)$, which it defines, formally satisfies the following three classical differential equations,

$$(1.2) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x}, \qquad \frac{\partial^4 u}{\partial t^4} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The last is a consequence of the first two and is known as the equation for the transverse vibrations of a bar [2]. The integral is both a Laplace integral and a Fourier integral. Such integrals seem to invite theoretical study independent of their statistical applications. We are able to give several necessary and sufficient conditions on a function $u(x, y, t)$ in order that it should have a representation (1.1).

If the variables x and t in the heat equation, the second of those in (1.2), are interchanged we obtain

$$(1.3) \qquad \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}.$$

A function which formally satisfies these three equations is

$$(1.4) \qquad u(x, y, t) = \int_{-\infty}^\infty e^{ixr - y|r| - tr^2} \phi(r) dr,$$

and we find that this integral representation is typical. In fact we show that every solution of (1.3) which is non-negative and of class L (as a function of x) on $-\infty < x < \infty$ for each $y > 0$, $t > 0$ is equal to an integral

(1.4) with $\phi(r)$ a positive definite function. It is to be understood that all functions $u(x, y, t)$ are assumed to have continuous second order derivatives, at least.

2. Two preliminary results

We here establish two lemmas which are not without intrinsic interest. One involves positive harmonic functions, the other positive temperature functions. The proof of the latter will appear in [3] but a sketch thereof is given here for completeness.

LEMMA 2.1. *Necessary and sufficient conditions that $u(x, y)$ (of class C^2) should have the representation*

$$u(x, y) = \int_{-\infty}^{\infty} e^{ixr-y|r|}\phi(r)dr \qquad y > 0, \tag{2.1}$$

where $\phi(r)$ is positive definite, are:

A. $$u_{xx}+u_{yy} = 0, \qquad u(x, y) \geqq 0 \qquad y > 0$$

B. $$\int_{-\infty}^{\infty} u(x, y_0)dx < \infty \qquad \textit{for some } y_0 > 0.$$

Under these conditions the integral B is independent of y_0 and

$$|\phi(r)| \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, y)dx \qquad -\infty < r < \infty, \qquad y > 0. \tag{2.2}$$

First assume (2.1) with $\phi(r)$ positive definite [4]. That is, for some real non-decreasing bounded functions $\alpha(s)$

$$\phi(r) = \int_{-\infty}^{\infty} e^{-irs}d\alpha(s). \tag{2.3}$$

Then by Fubini's theorem and the known Fourier transform of $\exp(-y|r|)$ we have for $y > 0$ that

$$\begin{aligned} u(x, y) &= \int_{-\infty}^{\infty} d\alpha(s) \int_{-\infty}^{\infty} e^{i(x-s)r-y|r|}dr \\ &= 2y\int_{-\infty}^{\infty} \frac{d\alpha(s)}{(x-s)^2+y^2} \geqq 0 \end{aligned}$$

$$\int_{-\infty}^{\infty} u(x, y)dx = 2\pi\int_{-\infty}^{\infty} d\alpha(s) \geqq 2\pi|\phi(r)|.$$

We have thus proved the necessity of the conditions as well as the conclusion (2.2).

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Conversely, condition A is known [5] to imply that

$$u(x, y) = Py + 2y\int_{-\infty}^{\infty} \frac{d\alpha(r)}{(x-r)^2+y^2} \qquad y > 0, \tag{2.4}$$

where $\alpha(r)$ is non-decreasing and P is a non-negative constant. Hence

$$2y\int_{-\infty}^{\infty} \frac{d\alpha(r)}{(x-r)^2+y^2} \leqq u(x, y) \qquad y > 0,$$

and by Fubini's theorem

$$\int_{-\infty}^{\infty} d\alpha(r)\int_{-\infty}^{\infty} \frac{2y\,dx}{(x-r)^2+y^2} = 2\pi\int_{-\infty}^{\infty} d\alpha(r) \leqq \int_{-\infty}^{\infty} u(x, y)dx.$$

Since by condition B the integral on the right is finite for $y = y_0$, the function $\alpha(r)$ is bounded. Hence $\phi(r)$, as defined by (2.3), is positive definite. The constant P is necessarily zero, as we see by integrating both sides of equation (2.4) over the whole x-axis for $y = y_0$. But we have already seen that the integral (2.4) is equal to the integral (2.1) when $\phi(r)$ is defined by (2.3). This completes the proof.

LEMMA 2.2. *Necessary and sufficient conditions that*

$$u(x, t) = \int_{-\infty}^{\infty} e^{ixr-tr^2}\phi(r)dr \qquad t > 0, \tag{2.5}$$

where $\phi(r)$ is positive definite, are:

A. $$u_{xx} = u_t, \qquad u(x, t) \geqq 0 \qquad t > 0$$

B. $$\int_{-\infty}^{\infty} u(x, t_0)dx < \infty \qquad \textit{for some } t_0 > 0.$$

Under these conditions the integral B *is independent of t_0 and*

$$|\phi(r)| \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, t)dx \qquad -\infty < r < \infty, t > 0. \tag{2.6}$$

If we define $\phi(r)$ by (2.3) and recall that

$$k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixr-tr^2}dr \qquad t > 0,$$

where $k(x, t)$ is the non-negative fundamental solution of the heat equation,

$$k(x, t) = (4\pi t)^{-\frac{1}{2}}e^{-x^2/(4t)} \qquad t > 0,$$

then the integral (2.5) becomes

$$u(x, t) = 2\pi\int_{-\infty}^{\infty} k(x-s, t)d\alpha(s). \tag{2.7}$$

Thus the necessity of the conditions is apparent. The inequality (2.6) follows from the familiar equation

$$\int_{-\infty}^{\infty} k(x, t)dx = 1 \qquad t > 0.$$

Conversely, an earlier result of the author [6] shows that condition A implies the representation (2.7) for some non-decreasing function $\alpha(s)$. We now use condition B to show that $\alpha(s)$ is bounded. Then $\phi(r)$ as defined by (2.3) is positive definite. The boundedness of $\alpha(s)$ allows us to use Fubini's theorem to establish the equality of integrals (2.7) and (2.5). This completes the proof.

3. Solutions of equations (1.2)

We show first that certain functions $u(x, y, t)$ which satisfy equations (1.2) have a Gauss-Weierstrass representation.

THEOREM 3.1. *Necessary and sufficient conditions that*

$$u(x, y, t) = \mathrm{Re} \int_{-\infty}^{\infty} k(t-r, x+iy)d\alpha(r) \qquad x > 0, \tag{3.1}$$

where $\alpha(r)$ is non-decreasing, are:
A. *For each real t $u(x, y, t)$ is the real part of a function of $(x+iy)$ which is analytic for $x > 0$ and whose restriction to the real axis is real.*

B. $$u(x, 0, t) \geqq 0, \qquad u_x(x, 0, t) = u_{tt}(x, 0, t) \qquad \text{for all } x > 0.$$

Assume first equation (3.1). When $y = 0$ we have the familiar Gauss-Weierstrass integral

$$u(x, 0, t) = \int_{-\infty}^{\infty} k(t-r, x)d\alpha(r) \qquad x > 0, \tag{3.2}$$

so that condition B is immediate. The integral (3.2) may be rewritten as follows,

$$u(x, 0, t) = (4\pi x)^{-\frac{1}{2}} \int_0^{\infty} e^{-s/4x} d[\alpha(t-\sqrt{s})-\alpha(t+\sqrt{s})].$$

This integral is a Laplace transform in the variable $1/4x$, and as such is known to be the (real) restriction to the x-axis of a function of $(x+iy)$ which is analytic for $x > 0$. Since the same is true of the factor $(4\pi x)^{-\frac{1}{2}}$ the necessity of condition A is also established.

Conversely, by a theorem of the author [6] about positive temperature functions, hypothesis B implies equation (3.2) with $\alpha(r)$ some non-decreasing function. Since the analytic function described in hypothesis A is real on the real axis it must equal $u(x, 0, t)$ there and hence must equal the analytic function defined by the integral (3.1). This completes the proof.

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Since $k(t, x)$ satisfies the heat equation it is clear from the representation (3.1) that $u(x, y, t)$ satisfies all three equations (1.2). We point out that the theorem would be false without the requirement, in hypothesis A, of reality on the real axis. Thus the function

$$u(x, y, t) = e^{x+t} \cos y - y$$

would satisfy the altered hypotheses. For, it is the real part of the entire function of $z = x+iy$,

$$u(x, y, t) = \operatorname{Re} [e^{z+t} + iz],$$

and $u(x, 0, t)$ satisfies hypothesis B. But the equation

$$e^{z+t} + iz = \int_{-\infty}^{\infty} k(t-r, z) d\alpha(r)$$

can have no real solution $\alpha(r)$. (Consider real z).

We turn next to the integrals (1.1) and prove the following result.

THEOREM 3.2. *Necessary and sufficient conditions that*

$$u(x, y, t) = \int_0^{\infty} e^{-xr^2} \cos tr \cos yr^2 \phi(r) dr \qquad x > 0, \tag{3.3}$$

where $\phi(r)$ is even and positive definite, are:

A. *For each real t $u(x, y, t)$ is the real part of a function of $(x+iy)$ which is analytic for $x > 0$ and whose restriction to the x-axis is real.*

B.
$$u(x, 0, t) \geqq 0,\ u_x(x, 0, t) = u_{tt}(x, 0, t) \qquad \textit{for all } x > 0,$$
$$\int_{-\infty}^{\infty} u(x_0, 0, t) dt < \infty \qquad \textit{for some } x_0 > 0.$$

C.
$$u(x, y, -t) = u(x, y, t).$$

Assume first the representation (3.3). Condition C is evident. Since $\phi(r)$ is even and positive definite it is real and equal to

$$\phi(r) = \int_{-\infty}^{\infty} \cos rs \, d\alpha(s)$$

for some bounded non-decreasing function $\alpha(s)$. Hence

$$u(x, y, t) = \operatorname{Re} \int_0^{\infty} e^{-(x+iy)r^2} \cos tr \, \phi(r) dr. \tag{3.4}$$

Since the integral (3.4) is a Laplace transform, convergent for $x > 0$, it clearly has the properties of the analytic function described in hypothesis A. Since $\phi(r)$ is even

$$u(x, 0, t) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-xr^2+itr} \phi(r) dr \qquad x > 0, \tag{3.5}$$

and condition B follows from Lemma 2.2.

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Conversely, condition B also implies equation (3.5), where $\phi(r)$ is positive definite. By C

$$(3.6) \qquad u(x, 0, t) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-xr^2} \cos tr\, \phi(r) dr.$$

By B $u(x, 0, t)$ is real so that if $\phi = \phi_1 + i\phi_2$

$$(3.7) \qquad u(x, 0, t) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-xr^2} \cos tr\, \phi_1(r) dr = \int_0^{\infty} e^{-xr^2} \cos tr\, \phi_1(r) dr$$

$$(3.8) \qquad \phi_1(r) = \int_{-\infty}^{\infty} \cos rs\, d\alpha(s),$$

where $\alpha(s)$ is non-decreasing and bounded. We assert that $\phi_1(r)$ is itself positive definite. This becomes evident if we rewrite the integral (3.8) as follows

$$(3.9) \qquad \phi_1(r) = \int_{-\infty}^{\infty} \cos rs\, d\left(\frac{\alpha(s) - \alpha(-s)}{2}\right) = \int_{-\infty}^{\infty} e^{-irs} d\left(\frac{\alpha(s) - \alpha(-s)}{2}\right).$$

Since $\alpha(s)$ is non-decreasing and bounded, the same is true of $[\alpha(s) - \alpha(-s)]/2$. If x is replaced by $z = x + iy$ in (3.7), the integral then defines a function of z analytic for $x > 0$, reducing to the real function $u(x, 0, t)$ when $y = 0$. Since there can be only one such analytic function, its real part must be $u(x, y, t)$ by hypothesis A. On the other hand its real part is the integral (3.3) with ϕ replaced by ϕ_1. This completes the proof.

An example of the theorem is

$$u(x, y, t) = \operatorname{Re} k(t, x+iy) = \frac{1}{\pi}\int_0^{\infty} e^{-xs^2} \cos ts \cos ys^2\, ds \qquad x > 0.$$

Here the even positive definite function ϕ is the constant $1/\pi$.

Corollary 3.2. *Under the conditions of Theorem 3.2 $u(x, y, t)$ has the representation (3.1), where $\alpha(r)$ is odd, non-decreasing and bounded.*

For, by equation (3.9) the function $\phi(r)$ of (3.3) may be represented as

$$\phi(r) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-irs} d\alpha(s),$$

where $\alpha(s)$ is odd, non-decreasing and bounded. But

$$k(t-r, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t-r)s - zs^2} ds \qquad x > 0.$$

If this is substituted in (3.1) we find after simple calculations that $u(x, y, t)$ is equal to the integral (3.3), so that the equivalence of the two representations is evident.

If the factor $\cos tr$ in the integral (3.3) is replaced by $\sin tr$ the resulting integral defines a function $u(x, y, t)$ which also satisfies the equations (1.2). We can characterize a class of such functions as follows.

THEOREM 3.3. *Necessary and sufficient conditions that*

$$u(x, y, t) = \int_0^\infty e^{-xr^2} \sin tr \cos yr^2 \, \phi(r) dr \qquad x > 0, \tag{3.10}$$

where $|r|\phi(r)$ is an even positive definite function, are:

A. *For each t $u(x, y, t)$ is the real part of a function of $(x+iy)$ which is real for $y = 0$ and is analytic for $x > 0$.*

B. $$u_t(x, 0, t) \geqq 0, \qquad u_{tx}(x, 0, t) = u_{ttt}(x, 0, t) \qquad \textit{for all } x > 0,$$
$$\int_{-\infty}^\infty u_t(x_0, 0, t) dt < \infty \qquad \textit{for some } x_0 > 0.$$

C. $$u(x, y, -t) = -u(x, y, t), \quad u(x, y, 0) = 0.$$

Assume first equation (3.10). Condition C is evident. Set

$$|r|\phi(r) = \int_{-\infty}^\infty \cos rs \, d\alpha(s),$$

where $\alpha(s)$ is non-decreasing and bounded. It is permissible to assume that $\alpha(s)$ is odd, as (3.9) shows. Then

$$u(x, 0, t) = \int_0^\infty e^{-xr^2} \sin tr \, \phi(r) dr$$

$$u_t(x, 0, t) = \int_0^\infty e^{-xr^2} \cos tr \, \phi(r) r \, dr$$

$$= \frac{1}{2} \int_{-\infty}^\infty e^{itr-xr^2} |r| \phi(r) dr. \tag{3.11}$$

By Lemma 2.2, condition B follows. By (3.10)

$$u(x, y, t) = \operatorname{Re} \int_0^\infty e^{-(x+iy)r^2} \sin tr \, \phi(r) dr,$$

so that condition A follows by the same argument as that used in the proof of Theorem 3.2.

Conversely, Condition B implies (3.11) where $|r|\phi(r)$ is positive definite. By C and B $u_t(x, 0, t)$ is even in t and real, so that

$$u_t(x, 0, t) = \frac{1}{2} \int_{-\infty}^\infty e^{-xr^2} \cos tr \, \phi(r) |r| dr$$

$$= \frac{1}{2} \int_{-\infty}^\infty e^{-xr^2} \cos tr \, dr \int_{-\infty}^\infty \cos rs \, d\alpha(s)$$

$$= \int_0^\infty e^{-xr^2} \cos tr \, \phi_1(r) r \, dr$$

$$u(x, 0, t) = \int_0^\infty e^{-xr^2} \sin tr \, \phi_1(r) dr.$$

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Here we have integrated with respect to t, a valid step by uniform convergence, and used the fact that $u(x, 0, 0) = 0$. Finally, Condition A now gives (3.10) as in the previous proof.

An example of the theorem is

$$u(x, y, t) = \operatorname{Re} \int_0^t k(r, x+iy)dr \qquad x > 0$$

$$= \frac{1}{\pi} \int_0^\infty e^{-xr^2} \sin tr \cos yr^2 \frac{dr}{r}.$$

Here $|r|\phi(r) = 1/\pi$, an even positive definite function.

Integrals of type (3.3) and (3.10) in which the factor $\cos yr^2$ is replaced by $\sin yr^2$ could be characterized in an analogous way. Condition A would deal with the imaginary part of an analytic function. Condition B would need to be altered suitably in both cases since $u(x, 0, t)$ and $u_t(x, 0, t)$ would vanish identically. We omit details.

4. Solutions of equations (1.3)

Equally interesting are the functions $u(x, y, t)$ which satisfy the three equations (1.3). We have now reversed the rôles of the time and space variables of the heat equation, and as a consequence the third equation has become the "backward" heat equation. We first prove a result analogous to that of Theorem 3.1.

THEOREM 4.1. *Necessary and sufficient conditions that*

$$u(x, y, t) = e^{y^2/4t} \int_{-\infty}^{\infty} k(x-r, t) \cos \frac{(x-r)y}{2t} \, d\alpha(r) \qquad t > 0, \tag{4.1}$$

where $\alpha(r)$ *is non-decreasing, are*:

A. *For each* $t > 0$, $u(x, y, t)$ *is the real part of an entire function of* $(x+iy)$ *which is real for* $y = 0$.

B. $$u(x, 0, t) \geqq 0, \qquad u_{xx}(x, 0, t) = u_t(x, 0, t) \qquad \textit{for } t > 0.$$

Assume the representation (4.1), which is equivalent to

$$u(x, y, t) = \operatorname{Re} \int_{-\infty}^{\infty} k(x+iy-r, t)d\alpha(r) \qquad t > 0. \tag{4.2}$$

Then

$$u(x, 0, t) = \int_{-\infty}^{\infty} k(x-r, t)d\alpha(r). \tag{4.3}$$

It is a familiar fact that the Gauss-Weierstrass integral (4.2) is an entire function of the complex variable $z = x+iy$ and that the integral (4.3) defines a real non-negative solution of the heat equation. Hence the necessity of the conditions is immediate.

Conversely, reference [6], used above, implies (4.3) with $\alpha(r)$ non-decreasing. Since there is one analytic function reducing to (4.3) when $y = 0$ it must be the integral (4.2). Hypothesis A insures the representation (4.2) or (4.1), and the proof is complete.

A case in point is provided by the example

$$u(x, y, t) = e^{x+t} \cos y.$$

It is the real part of the entire function e^{z+t}, and $\alpha(r) = e^r$.

A subclass of that just considered is characterized in the following theorem.

THEOREM 4.2. *Necessary and sufficient conditions that*

$$u(x, y, t) = \operatorname{Re} \int_{-\infty}^{\infty} e^{ixr-yr-tr^2} \phi(r) dr \qquad t > 0, \tag{4.4}$$

where $\phi(r)$ is positive definite, are conditions A and B of Theorem 4.1 and in addition condition

C. $$\int_{-\infty}^{\infty} u(x, 0, t_0) dx < \infty \qquad \textit{for some } t_0 > 0.$$

An obvious application of the equation

$$k(x+iy, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x+iy)r-tr^2} dr \qquad t > 0$$

shows the equivalence of the integrals (4.2) and (4.4) when $\phi(r)$ is positive definite. The proof is completed by use of Lemma 2.2.

COROLLARY 4.2. *Conditions A, B, C of Theorems 4.1 and 4.2 are necessary and sufficient for the representation*

$$u(x, y, t) = \int_{-\infty}^{\infty} e^{-yr-tr^2} dr \int_{-\infty}^{\infty} \cos r(x-s) d\alpha(s), \tag{4.5}$$

where $\alpha(s)$ is non-decreasing and bounded.

Equation (4.5) is equivalent to (4.4) when $\phi(r)$ is defined by (2.3). An example is provided by the function

$$\begin{aligned} u(x, y, t) &= \operatorname{Re} k(x+iy, t) \\ &= e^{y^2/4t} k(x, t) \cos \frac{xy}{2t} \qquad t > 0. \end{aligned}$$

Here $\phi(r)$ is the constant $(2\pi)^{-1}$ and $\alpha(r)$ is constant except for a jump at $r = 0$.

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A further subclass of those thus far considered in this section is of special interest.

THEOREM 4.3. *Necessary and sufficient conditions that*

$$u(x, y, t) = \int_{-\infty}^{\infty} e^{ixr-y|r|-tr^2}\phi(r)dr, \tag{4.6}$$

where $\phi(r)$ is positive definite, are:

A. $u(x, y, t) \geqq 0,\ u_{xx}(x, y, t) = -u_{yy}(x, y, t) = u_t(x, y, t)$ *for* $y > 0, t > 0$.

B. $\int_{-\infty}^{\infty} u(x, y_0, t_0)dx < \infty$ *for some* $y_0 > 0, t_0 > 0$.

First assume (4.6) with $\phi(r)$ defined by (2.3). Then

$$\begin{aligned} u(x, y, t) &= \int_{-\infty}^{\infty} d\alpha(s) \int_{-\infty}^{\infty} e^{ixr-irs-y|r|-tr^2}dr \\ &= \int_{-\infty}^{\infty} g(x-s, y, t)d\alpha(s), \end{aligned} \tag{4.7}$$

where

$$g(x, y, t) = \int_{-\infty}^{\infty} e^{ixr-y|r|-tr^2}dr.$$

But

$$e^{-tr^2} = \int_{-\infty}^{\infty} e^{-irs}k(s, t)ds \qquad t > 0, \tag{4.8}$$

so that

$$\begin{aligned} g(x, y, t) &= \int_{-\infty}^{\infty} k(s, t)ds \int_{-\infty}^{\infty} e^{i(x-s)r-y|r|}dr \\ &= 2\int_{-\infty}^{\infty} \frac{k(s, t)y}{(x-s)^2+y^2}\,ds. \end{aligned} \tag{4.9}$$

Here we have used the known Fourier transform of $e^{-y|r|}$. From (4.9) it is seen that $g(x, y, t) \geqq 0$ and that $\int_{-\infty}^{\infty} g(x, y, t)dx = 2\pi$ for $t > 0$, $y > 0$. From (4.7) the same is true of $u(x, y, t)$. That $u(x, y, t)$ satisfies equations (1.3) follows by differentiation of the integral (4.6). The differentiated integral converges uniformly for all x, all $y \geqq 0$, and for $a \leqq t \leqq b (a > 0)$. Thus the necessity of conditions A and B is proved.

Conversely, let us apply Lemma 2.1 for each $t > 0$. We obtain

$$u(x, y, t) = \int_{-\infty}^{\infty} e^{ixr-y|r|}\phi(r, t)dr \qquad y > 0,$$

where $\phi(r, t)$ is a positive definite, and hence a bounded continuous, function of r for each $t > 0$. Similarly Lemma 2.2 gives for each $y > 0$

$$u(x, y, t) = \int_{-\infty}^{\infty} e^{ixr-tr^2}\psi(r, y)dr \qquad t > 0,$$

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where $\psi(r, y)$ is a positive definite function of r for each $y > 0$. By the uniqueness of Fourier transforms in L we have

$$e^{-y|r|}\phi(r, t) = e^{-tr^2}\psi(r, y), \qquad -\infty < r < \infty, \qquad y > 0, \qquad t > 0.$$

But this equation implies that each side of the equation

$$\phi(r, t)e^{tr^2} = \psi(r, y)e^{y|r|}$$

is a function of r alone which we shall call $\phi(r)$. By Lemmas 2.1 and 2.2 the integral

$$\int_{-\infty}^{\infty} u(x, y, t)dx = M$$

is independent of y and t and is hence a constant M. Thus by (2.2)

$$|\phi(r)| = e^{tr^2}|\phi(r, t)| \leqq \frac{e^{tr^2}}{2\pi}\int_{-\infty}^{\infty} u(x, y, t)dx$$

$$|\phi(r)| \leqq \frac{Me^{tr^2}}{2\pi} \qquad -\infty < r < \infty, \qquad t > 0.$$

But this implies that $\phi(r)$ is bounded. Equation (4.6) is now established, and it remains only to show that $\phi(r)$ is positive definite.

From equations (4.6) and (4.8) we obtain

$$u(x, y, t) = \int_{-\infty}^{\infty} k(s, t)ds \int_{-\infty}^{\infty} e^{i(x-s)r-y|r|}\phi(r)dr$$

when $y > 0$, $t > 0$. Here we have used Fubini's theorem, applicable in view of the boundedness of $\phi(r)$. That is,

$$u(x, y, t) = \int_{-\infty}^{\infty} k(x-r, t)h(r, y)dr \qquad y > 0, \qquad t > 0,$$

where

$$h(r, y) = \int_{-\infty}^{\infty} e^{irs-y|s|}\phi(s)ds. \tag{4.10}$$

Since $u(x, t, 0+) = h(x, y)$ by a basic property of the Gauss-Weierstrass integral, we conclude from the positive nature of $u(x, y, t)$ that $h(x, y) \geqq 0$ for $y > 0$. Moreover,

$$\int_{-\infty}^{\infty} u(x, y, t)dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k(x-r, t)h(r, y)dr$$

$$= \int_{-\infty}^{\infty} h(r, y)dr = M \qquad y > 0, \qquad t > 0.$$

Hence we may apply Lemma 2.1 to $h(x, y)$ and conclude from (4.10) that $\phi(s)$ is positive definite. This completes the proof.

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Observe the similarity and the distinction between Theorems 4.2 and 4.3. The integrals (4.4) and (4.6) differ only in the presence of an absolute value sign in the latter. Its presence guarantees that the latter is real. Note also that the integral $u(x, y, t)$ of Theorem 4.2 need not be positive for $y > 0$, $t > 0$, as the above illustrative example shows.

References

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