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ISOMORPHISMS OF PARTIAL GROUP RINGS

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Abstract. We consider the isomorphism problem for partial group rings $R_{par}G$ and show that, in the modular case, if $char(R) = p$ and $R_{par}G_1 \cong R_{par}G_2$ then the corresponding group rings of the Sylow p-subgroups are isomorphic. We use this to prove that finite abelian groups having isomorphic modular partial group algebras are isomorphic. Moreover, in the integral case, we show that the isomorphism of partial group rings of finite groups G_1 and G_2 implies $\mathbb{Z}G_1 \cong \mathbb{Z}G_2$.

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1. Introduction. Partial representations of groups were introduced independently by R. Exel [**4**] and J. C. Quigg and I. Raeburn [**8**] in the context of C^*-algebras, motivated by the desire to study algebras generated by partial isometries on a Hilbert space. The *partial group ring* of a group G over a ring R was defined in [**3**] and plays a role in the theory of partial representations similar to that of the group ring in representation theory.

DEFINITION 1.1. Given a group G and a ring R with unity we consider the semigroup S_G generated by the set of symbols $\{[g] \mid g \in G\}$ with relations:
(1) $[e] = 1$;
(2) $[s^{-1}][s][t] = [s^{-1}][st]$;
(3) $[s][t][t^{-1}] = [st][t^{-1}]$;
for all $s, t \in G$.

The *partial group ring* $R_{par}G$ of G over R is the semigroup ring of S_G over R.

If R is a commutative ring, then an alternative definition can be given by the universal property which puts the representations of $R_{par}G$ into one-to-one correspondence with the partial representations of G (see [**3**, p. 512]). Given two R-algebras A and B, an isomorphism $A \cong B$ will mean an R-isomorphism of algebras.

It is well known that for group rings, if K is a field, in general KG does *not* determine G up to isomorphisms. Indeed, E. C. Dade [**1**] gave an example of two non-isomorphic groups G and H such that $KG \cong KH$ for all fields K. Even in the stronger hypothesis that $\mathbb{Z}G \cong \mathbb{Z}H$, a counterexample has been given recently by M. Hertweck [**5**]. It is then natural to consider the isomorphism problem of partial group rings to see if these carry more information about the group than group rings in the usual sense. (For more

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information on the isomorphism problem of group rings, the reader may see [**9**], [**10**] or [**11**].)

The structure of partial group rings was described in [**3**] where it was also shown that if G and H are finite abelian groups and R is an integral domain whose characteristic does not divide the order of G then $R_{par}G \cong R_{par}H$ if and only if $G \cong H$. An example was given to show that there do exist non-isomorphic finite groups such that their partial group rings over an algebraically closed field of characteristic 0 are actually isomorphic.

In this paper we fill a gap in formula (14) of [**3**] and prove a similar result on isomorphisms in the modular case; i.e., when the characteristic of R divides the order of G. We also show that if R is of characteristic $p > 0$ and G and H are arbitrary finite groups such that their partial group rings over R are isomorphic, then the group rings of the corresponding Sylow p-subgroups are also isomorphic. Furthermore, we prove that if $\mathbb{Z}_{par}G \cong \mathbb{Z}_{par}H$ for finite groups G and H, then also $\mathbb{Z}G \cong \mathbb{Z}H$ showing that the hypothesis of having isomorphic partial group rings over the integers is even stronger than having isomorphic integral group rings.

2. The structure of partial group rings. Throughout this paper, G will always denote a finite group. In this section, we shall describe the structure of $R_{par}G$, correcting a gap in the proof of the recursive formula (14) of [**3**, Theorem 3.2].

We shall consider the Brandt groupoid associated with G, denoted $\Gamma = \Gamma(G)$, whose elements are pairs (A, g), where g is an element of G and A is a subset of G containing its identity e and the element g^{-1}. Note that $e, g \in gA$. The multiplication of pairs $(A, g)(B, h)$ in Γ is defined only in the case when $A = hB$ where we set:

$$(hB, g) \cdot (B, h) = (B, gh).$$

Notice that the set of *units* of Γ, denoted by $\Gamma^{(0)}$, is the set of all elements of the form (A, e).

Let R be a commutative ring. We recall that if Γ is any groupoid, then the *groupoid algebra* $R\Gamma$ is the free R-module freely generated over R by the elements of Γ, with multiplication given by:

$$\gamma_1 \cdot \gamma_2 = \begin{cases} \gamma_1\gamma_2, & \text{if the product is defined} \\ 0, & \text{otherwise,} \end{cases}$$

and extended linearly on $R\Gamma$.

Let $R\Gamma(G)$ denote the R-algebra of the groupoid $\Gamma(G)$. The dimension of this algebra is equal to the cardinality of $\Gamma(G)$, which is the number of pairs (A, g) as described above. We remind that if $|G| = n$, it was shown in [**3**] that:

$$\dim(R\Gamma(G)) = 2^{n-2}(n+1). \tag{1}$$

Observe that, since the right-hand side of (1) is a strictly increasing function on n, it follows that if G and H are finite groups such that $R\Gamma(G)$ is isomorphic to $R\Gamma(H)$, then $|G| = |H|$.

Notice that the units of Γ, of the form (A, e), are *idempotents* in $R\Gamma(G)$, they are mutually orthogonal and their sum is the identity of $R\Gamma(G)$.

It was shown in [**3**, Corollary 2.7] that *the groupoid algebra* $R\Gamma(G)$ *is isomorphic to the partial group algebra* $R_{par}G$.

It is sometimes useful to represent a groupoid Γ as an oriented graph E_Γ, whose vertices are the units of the groupoid. To each element $(A, g) \in \Gamma$ we assign an oriented edge of E_Γ from (A, e) to (gA, e) corresponding to the map $A \ni a \mapsto ga \in gA$. Each connected component of E_Γ represents a subgroupoid of Γ.

Let A be any subset of G containing the identity. In what follows, we will identify A with the vertex (A, e) of the graph $E_{\Gamma(G)}$. We denote by H the *stabilizer* of A in G; i.e.:

$$H = \{g \in G \mid gA = A\}.$$

In the graph $E_{\Gamma(G)}$, H corresponds to the set of edges starting and ending at the vertex (A, e). Notice that, since $e \in A$, then $H \subseteq A$.

Since H acts on the left on A, then the orbits of this action are the right cosets of H, and A is a union of them, say:

$$A = \bigcup_{i=1}^{m} Ht_i, \qquad t_1 = e,$$

where

$$m = \frac{|A|}{|H|}.$$

Let Γ_A denote the subgroupoid of $\Gamma(G)$ corresponding to the connected component of the vertex A of the graph $E_{\Gamma(G)}$. It was shown in [**3**, Proposition 3.1] that the groupoid algebra $K\Gamma_A$ is isomorphic to $M_m(KH)$. Clearly, $K\Gamma(G)$ is the direct sum of the algebras arising from all the connected components. We wish to compute the number of these direct summands.

Let $\mathcal{C}$ denote a full set of representatives of the conjugacy classes of subgroups of G. Given a subgroup $H \in \mathcal{C}$ and a positive integer m, we need to count the number of subsets A of G, of order $|A| = m \times |H|$ whose stabilizer is precisely H. We denote by $b_m(H)$ the number of all such subsets and by $c_m(H)$ the number of distinct direct summands of the form $M_m(KH')$ where H' is any subgroup of G conjugate to H.

Notice that, since conjugation is an automorphism of G, if H and H' are conjugate in G, it follows by symmetry that $b_m(H) = b_m(H')$, for all m. Thus, the total number of sets of order $m \times |H|$ whose stabilizer is either H or one of its conjugates is $b_m(H)(G : N_G(H))$, where $N_G(H)$ denotes the normalizer of H in G. In each connected component we have m of these sets (whose stabilizers are pairwise conjugate), whence:

$$c_m(H) = \frac{b_m(H)(G : N_G(H))}{m}. \tag{1}$$

A recursive formula for the coefficients $b_m(H)$ can be obtained as in [**3**]. The number of subsets of G which is a union of m cosets of H, one of which is always H itself, is clearly $\binom{(G:H)-1}{m-1}$. Some of these may have a stabilizer B which is actually bigger that H, and their number is given by:

$$\sum_{\substack{H < B \leq G \\ (B:H) \mid m}} b_{m/(B:H)}(B).$$

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So we have:

$$b_m(H) = \binom{(G:H)-1}{m-1} - \sum_{\substack{H<B\leq G\\(B:H)|m}} b_{m/(B:H)}(B).$$

From formula (1), we have that

$$b_m(H) = \frac{mc_m(H)}{(G:N_G(H))},$$

hence

$$c_m(H) = \frac{1}{m}(G:N_G(H))\left(\binom{(G:H)-1}{m-1} - \sum_{\substack{H<B\leq G\\(B:H)|m}} \frac{m/(B:H)c_{m/(B:H)}(B)}{(G:N_G(B))}\right). \tag{2}$$

Thus, we come to the following reformulation of Theorem 3.2 of [**3**].

THEOREM 2.1. *Let R be a commutative ring, G a finite group and let $\mathcal{C}$ denote a full set of representatives of the conjugacy classes of subgroups of G. Then the partial group ring of G over R is of the form*

$$R_{par}G \cong \bigoplus_{\substack{H\in\mathcal{C}\\1\leq m\leq(G:H)}} c_m(H)\,M_m(RH),$$

where $c_m(H)\,M_m(RH)$ means the direct sum of $c_m(H)$ copies of $M_m(RH)$ and the coefficients $c_m(H)$ are given by the recursive formula (2) *above.*

In the light of this fact, Corollary 3.3 of [**3**] should now be stated as follows.

COROLLARY 2.2. *Let G_1 and G_2 be two finite groups. Assume that there exists an isomorphism between the lattices of subgroups of G_1 and of G_2 that preserves conjugacy and such that corresponding subgroups have isomorphic group rings over R. Then $R_{par}G_1 \cong R_{par}G_2$.*

One should notice that the counterexample given in [**3**, Remark 4.6] to show that there exist noncommutative groups G_1 and G_2 which are not isomorphic and such that $K_{par}G_1 \cong K_{par}G_2$, where K denotes an algebraically closed field of characteristic 0, remains valid since the lattices of subgroups of these groups also fulfill the conditions of Corollary 2.2.

3. Isomorphisms of modular partial group algebras. In this section we shall consider partial group algebras over a field K of characteristic $p > 0$ which divides the order of the given groups.

The following easy fact will be needed in the sequel.

LEMMA 3.1 ([**6**, Proposition 22.1]). *Let R be a ring with unity and let $1 = e_1 + \cdots + e_r$ and $1 = f_1 + \cdots + f_s$ be two decompositions of the identity into a sum of minimal central idempotents. Then $r = s$ and there exists a permutation $\sigma \in S_r$ such that $e_i = f_{\sigma(i)}$, $1 \leq i \leq r$.*

It follows that if a ring R possesses a decomposition into a finite product of indecomposable rings then such decomposition is unique up to a permutation of direct factors.

REMARK 3.2. Let R be a commutative ring and let G be a group. Write

$$R_{par}G \cong \bigoplus_{\substack{H \in \mathcal{C} \\ 1 \leq m \leq (G:H)}} c_m(H)\, M_m(KH),$$

the decomposition of $R_{par}G$ given by Theorem 2.1. If we write each group ring KH as a direct sum of indecomposable two-sided ideals, $KH = \oplus_i I_i(H)$ then the ideals of the form $M_m(I_i(H))$ are the indecomposable direct summands of $R_{par}G$.

THEOREM 3.3. *Let R be an integral domain of characteristic $p > 0$ and let G_1, G_2 be two finite groups such that $R_{par}G_1 \cong R_{par}G_2$. Let S_i denote a Sylow p-subgroup of G_i, $i = 1, 2$. Then $RS_1 \cong RS_2$.*

Proof. Since S_1 is a p-group and $char(R) = p$, it follows directly from Theorem 2.1 that KS_1 is an indecomposable direct summand of $R_{par}G_1$. By the remark above, there exists a subgroup H of G_2 and an indecomposable direct summand I of KH such that $RS_1 \cong M_m(I)$, for some positive integer m. Notice that RS_1 contains no idempotent elements, so neither does $M_m(I)$ and hence we must have $m = 1$.

Since I is a direct summand of RH, there exists an idempotent e, which is central in RH, such that $I = RHe$.

Claim 1. *The p'-elements of H act as scalars on e (i.e., if $h \in H$ is a p'-element, then there exists an element $\beta \in R$ such that $he = \beta e$).*

Indeed, it is well known that $\Delta(S_1) = \langle x - 1 \mid x \in S_1 \rangle$ is a nilpotent ideal of RS_1 and $RS_1 = R \oplus \Delta(S_1)$ as R-modules. Since $I \cong RS_1$, for an element $h \in H$ we have that he can be written in the form $he = \alpha e + \eta$ where $\alpha \in R$ and η is nilpotent. So, there exists a positive integer n such that $\eta^{p^n} = 0$ and we have $(he)^{p^n} = (\alpha e)^{p^n}$. As h is a p'-element, there exists a positive integer s such that $h^{sp^n} = h$, thus $he = \beta e$, where $\beta = \alpha^{sp^n}$.

Claim 2. *$I = KSe$, where S is a Sylow p-subgroup of H.*

In fact, since e is central, He is a group and if S is a Sylow p-subgroup of H, then Se is a Sylow p-subgroup of He. By the claim above, the set N of p'-elements of He is central, so $He = Se \times N$. Every element $x \in RHe$ can be written in the form $x = \sum_{i,j} r_{ij} y_i h_j e$ where each $y_i e \in Se$ and each $h_j e \in N$ so, using again the previous claim, we obtain

$$x = \sum_{i,j} r_{ij} y_i \beta_j e = \sum_{i,j} r_{ij} \beta_j y_i e \in RSe.$$

Conclusion.

As Se is a set of generators of the R-module RSe and the image of S_1 under the isomorphism is a linearly independent set in RSe we have that $|S_1| \leq |S| \leq |S_2|$. As $|G_1| = |G_2|$ it follows immediately that S is a Sylow p-subgroup of G_2 and clearly $RS_1 \cong I \cong RS_2$. □

COROLLARY 3.4. *Let R be an integral domain of characteristic $p > 0$ and let G_1, G_2 be two finite p-groups such that $R_{par}G_1 \cong R_{par}G_2$. Then $RG_1 \cong RG_2$. Moreover, if G_1 is abelian then $G_1 \cong G_2$.*

Proof. The first part of our statement follows directly from the theorem above. If G_1 is abelian then, by a result of Deskins [**2**], it follows that $G_1 \cong G_2$. □

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In order to prove our next theorem we shall need some technical results. For a finite abelian group G, we shall denote by $\gamma_m(G)$ the number of subgroups of G of order m. We shall say that a divisor $k > 0$ of $|G|$ is *small* if every prime that divides $|G|$ divides also $\frac{|G|}{k}$.

PROPOSITION 3.5. *Let K be an algebraically closed field of characteristic $p > 0$ and G a finite abelian group. Let k be a small divisor of $|G|$ with $|G| \neq 2k$. Then, the multiplicity of $M_{\frac{|G|}{k}-1}(K)$ in the decomposition of $K_{par}G$ as a direct sum of indecomposable two-sided ideals is*

$$\frac{k}{|G|-k} \sum_{\substack{m \mid |G|, p \nmid m \\ 1 \leq m < \frac{k|G|}{|G|-k}}} \binom{\frac{|G|}{m}-1}{\frac{|G|}{k}-2} m\gamma_m(G).$$

Proof. Notice that $K_{par}G$ is a direct sum of two-sided ideals of the form $M_m(KH)$, where H is a subgroup of G. Writing $H = P \times N$ where P is a p-group and $p \nmid |N|$, if $P \neq 1$, we have that $M_m(KH) = M_m((KN)P) = M_m((K \oplus \cdots \oplus K)P) \cong M_m(KP) \oplus \cdots \oplus M_m(KP)$. Thus, the indecomposable two sided direct summands of $K_{par}G$ are either of the form $M_m(K)$ or $M_m(KP)$ where P is a p-subgroup of G, $m \geq 1$. Notice that summands of the form $M_m(K)$ come only from the decompositions of summands $M_m(KH)$ where H is a p'-subgroup.

Let H be a p'-subgroup of G with $(\frac{|G|}{k}-1)|H| \leq |G|$ and let A be a subset of G such that $1 \in A$ and $|A| = (\frac{|G|}{k}-1)|H|$. We claim that if the stabilizer $S(A)$ of A contains H then $S(A) = H$. In fact, if q is a prime dividing $|S(A)|/|H|$ then q divides $(\frac{|G|}{k}-1)$. As $q \mid |G|$, it follows from the hypothesis that $q \mid |G|/k$, a contradiction. Thus $S(A) = H$ and it follows that any set which is a union of $(\frac{|G|}{k}-1)$ distinct cosets of H has stabilizer equal to H so the multiplicity of $M_{\frac{|G|}{k}-1}(KH)$ in the decomposition given in Theorem 2.1 is

$$\frac{k}{|G|-k}\binom{(G:H)-1}{\frac{|G|}{k}-2}.$$

Since $M_{\frac{|G|}{k}-1}(KH)$ is a direct sum of $|H|$ copies of $M_{\frac{|G|}{k}-1}(K)$ we have that in the decomposition of $K_{par}G$ into a direct sum of indecomposable two-sided ideals the total number of summands isomorphic to $M_{\frac{|G|}{k}-1}(K)$ coming from subgroups of a fixed order m with $p \nmid m$ is

$$\frac{k}{|G|-k}\binom{(G:H)-1}{\frac{|G|}{k}-2} m\gamma_m(G).$$

The result follows. □

Next we adapt Corollary 4.2 of [**3**] to the modular case.

COROLLARY 3.6. *Let K be an algebraically closed field of characteristic $p > 0$ and G_1 and G_2 finite abelian groups with $|G_1| = |G_2| = q^n a$ where $q \neq p$ is a prime and $q \nmid a$. If $K_{par}G_1 \cong K_{par}G_2$ then $\gamma_{q^j}(G_1) = \gamma_{q^j}(G_2)$, for all positive integers j such that $2j \leq n$.*

Proof. If $a = 1$ the the result follows from [**3**, Corollary 4.2], so one may assume that $q^n a$ is not a power of a prime. We prove more in general that if k is a small divisor of $|G_1|$ with $p \nmid k$, and $|G_1| > k^2$, then $\gamma_m(G_1) = \gamma_m(G_2)$ for all $m \leq k$ with $p \nmid m$. Under our hypotheses, the number $k = q^j$ satisfies these conditions.

We use induction on k, the case $k = 1$ being trivial. Let $k > 1$ be fixed. One easily observes that the inequality $m < \frac{|G_1|k}{|G_1|-k}$ holds if and only if $m \leq k$. Moreover, $|G_1| > 2k$ as $|G_1| > k^2$. Thus, using Proposition 3.5, we have that the multiplicity of $M_{\frac{|G_1|}{k}-1}(K)$ in the decomposition of $K_{par}G_1$ into a direct sum of indecomposable two-sided ideals is:

$$\frac{k}{|G_1|-k} \sum_{\substack{m \mid |G_1|,\, p \nmid m \\ 1 \leq m \leq k}} \binom{\frac{|G_1|}{m} - 1}{\frac{|G_1|}{k} - 2} m\gamma_m(G_1).$$

By Lemma 3.1, this number is invariant under K-algebra isomorphisms. Since the coefficient of $\gamma_k(G_1)$ in the above formula is non-zero, to conclude the proof it suffices to show that $\gamma_m(G_1) = \gamma_m(G_2)$ for all $m < k$ with m dividing $|G_1| = |G_2|$ and $p \nmid m$. One checks this similarly as in the proof of [**3**, Corollary 4.2]. □

THEOREM 3.7. *Let R be an integral domain of characteristic $p > 0$ and let G_1, G_2 be two finite abelian groups such that $R_{par}G_1 \cong R_{par}G_2$. Then $G_1 \cong G_2$.*

Proof. Since, for any field $E \supset R$ we have that

$$E_{par}G_1 \cong E \otimes_R R_{par}G_1 \cong E \otimes_R R_{par}G_2 \cong E_{par}G_2,$$

we may assume, without loss of generality, that $K_{par}G_1 \cong K_{par}G_2$ for an algebraically closed field K.

Let $G_1 = P_1 \times \cdots \times P_t$ and $G_2 = Q_1 \times \cdots \times Q_t$ be the decompositions of G_1 and G_2 as a direct product of Sylow subgroups respectively and assume that P_1 and Q_1 are the subgroups corresponding to the prime p. Then, by Theorem 3.3 we have that $KP_1 \cong KQ_1$ and hence, by Deskins' result [**2**], $P_1 \cong Q_1$.

Let $q_i \neq p$ be the prime divisor of $|G_1|$ corresponding to the Sylow subgroups P_i and Q_i respectively. It follows from Corollary 3.6 that $\gamma_{q_i^j}(G_1) = \gamma_{q_i^j}(G_2)$ for all positive integers j such that q_i^{2j} divides $|G_1|$. Now, [**3**, Lemma 4.3] shows directly that $P_i \cong Q_i$, $2 \leq i \leq t$. □

4. Integral partial group rings. Let G_1 and G_2 be finite groups. We will show that $\mathbb{Z}_{par}G_1 \cong \mathbb{Z}_{par}G_2$ is a stronger restriction than having isomorphic integral group rings. Actually, we have the following.

THEOREM 4.1. *Let G_1 and G_2 be finite groups such that $\mathbb{Z}_{par}G_1 \cong \mathbb{Z}_{par}G_2$. Then, for every subgroup H of G_1 there exists a subgroup N of G_2 such that $\mathbb{Z}H \cong \mathbb{Z}N$. In particular, $\mathbb{Z}G_1 \cong \mathbb{Z}G_2$.*

Proof. Write

$$\mathbb{Z}_{par}G_1 \cong \bigoplus_{\substack{H \in \mathcal{C} \\ 1 \leq m \leq (G_1:H)}} c_m(H)\, M_m(\mathbb{Z}H),$$

$$\mathbb{Z}_{par}G_2 \cong \bigoplus_{\substack{N \in \mathcal{C}' \\ 1 \leq m \leq (G_2:N)}} c_m(N)\, M_m(\mathbb{Z}N),$$

the decompositions of $\mathbb{Z}_{par}G_1$ and $\mathbb{Z}_{par}G_2$ given by Theorem 2.1.

Notice that, since it follows immediately from a theorem of Kaplansky [**7**, Theorem 7.2.3] that integral group rings of finite groups contain no nontrivial idempotent elements, the summands in the decompositions above are all indecomposable.

Given a subgroup H of G_1, we have that $\mathbb{Z}H$ appears as a summand of $\mathbb{Z}_{par}G_1$ so we obtain from Lemma 3.1 that there is a subgroup N of G_2 and a positive integer m such that $\mathbb{Z}H \cong M_m(\mathbb{Z}N)$. As $\mathbb{Z}H$ contains no idempotent elements, it follows immediately that $m = 1$ and $\mathbb{Z}H \cong \mathbb{Z}N$. As G_1 and G_2 are of maximal order, the last part of our statement is obvious. □

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