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Real Hypersurfaces in Complex Two-plane Grassmannians with Reeb Parallel Ricci Tensor in the GTW Connection

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Abstract. There are several kinds of classification problems for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Among them, Suh classified Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel Ricci tensor in Levi–Civita connection. In this paper, we introduce the notion of generalized Tanaka–Webster (GTW) Reeb parallel Ricci tensor for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Next, we give a complete classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with GTW Reeb parallel Ricci tensor.

Introduction

The classification of real hypersurfaces in Hermitian symmetric spaces is one of interesting parts in the field of differential geometry. Among them, we introduce a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is a kind of Hermitian symmetric space of compact irreducible type with rank 2. Remarkably, the manifolds are equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ satisfying $JJ_\nu = J_\nu J$ $(\nu = 1,2,3)$, where $\{J_\nu\}_{\nu=1,2,3}$ is an orthonormal basis of $\mathfrak{J}$. When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann Manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper we assume that m is not less than 3.

Let N be a local unit normal vector field of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. Since $G_2(\mathbb{C}^{m+2})$ has the Kähler structure J, we can define a *Reeb vector field* $\xi = -JN$ and a 1-dimensional distribution $[\xi] = \mathcal{C}^\perp$, where $\mathcal{C}$ denotes the orthogonal complement in T_xM, $x \in M$, of the Reeb vector field ξ. The Reeb vector field ξ is said to be *Hopf* if $\mathcal{C}$ (or $\mathcal{C}^\perp$) is invariant under the shape operator A of M. The 1-dimensional foliation of M by the integral curves of ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* if and only if the Hopf foliation of M is totally geodesic.

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By the formulas in [11, Section 2], it can easily be seen that ξ is Hopf if and only if M is Hopf.

From the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$, there naturally exist *almost contact* 3*-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Let $\mathcal{Q}^\perp = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent space T_xM of M at $x \in M$. In addition, $\mathcal{Q}$ stands for the orthogonal complement of $\mathcal{Q}^\perp$ in T_xM. It is the quaternionic maximal subbundle of T_xM. Thus the tangent space of M consists of the direct sum of $\mathcal{Q}$ and $\mathcal{Q}^\perp$ as follows: $T_xM = \mathcal{Q} \oplus \mathcal{Q}^\perp$.

For two distributions $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ defined above, we consider two natural invariant geometric properties under the shape operator A of M, that is, $A\mathcal{C}^\perp \subset \mathcal{C}^\perp$ and $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$. In a paper due to Suh [17, Theorem 1.1] we introduce the following theorem.

Theorem A *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{Q}^\perp$ are invariant under the shape operator of M if and only if either*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In case (A), we say M is of Type (A). Similarly, in case (B) we say M is of Type (B). Until now, by using Theorem A, many geometers have investigated some characterizations for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with geometric quantities, shape operator, normal (or structure) Jacobi operator, Ricci tensor, and so on. Actually, Lee and Suh [11] gave a characterization for a real hypersurface of Type (B) as follows.

Theorem B *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathcal{Q}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$. In other words, M is locally congruent to a real hypersurface of Type (B).*

In particular, there are various well-known results with respect to the Ricci tensor S on Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$. From such a point of view, Suh [16] gave a characterization of a model space of Type (A) in $G_2(\mathbb{C}^{m+2})$ under the condition $S\phi = \phi S$, where ϕ denotes the structure tensor field of M. In [17] and [18], he also considered the parallelism of Ricci tensor with respect to the Levi–Civita connection and gave the following theorem.

Theorem C ([18]) *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with non-vanishing geodesic Reeb flow. If the Ricci tensor is Reeb parallel, $\nabla_\xi S = 0$. Then M is locally congruent to one of the following:*

(i) *a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \neq \frac{\pi}{4\sqrt{2}}$,*

(ii) *a tube over a totally geodesic $\mathbb{H}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$ with radius r such that*

$$\cot^2(2r) = \frac{1}{2m-1}$$

and ξ-parallel eigenspaces $T_{\cot r}$ and $T_{\tan r}$.

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Now we introduce another connection different from the Levi–Civita one, called the *generalized Tanaka–Webster* (*GTW*) *connection* $\widehat{\nabla}^{(k)}$ on M given by

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + \widehat{F}^{(k)}_X Y,$$

where k is a non-zero real number (see [1, 2, 5]) and

$$\widehat{F}^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y.$$

The operator $\widehat{F}^{(k)}$ is a skew-symmetric (1,1) type tensor, that is,

$$g(\widehat{F}^{(k)}_X Y, Z) = -g(Y, \widehat{F}^{(k)}_X Z)$$

for all tangent vector fields X, Y, and Z on M and is said to be a *Tanaka–Webster* (or *k-th-Cho*) *operator* with respect to X. Recently, in [5] Jeong, Kimura, Lee, and Suh defined the notion of Reeb parallel shape operator with GTW connection, $\widehat{\nabla}^{(k)}_\xi A = 0$, for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ and gave the following theorem.

Theorem D *Let M be a connected orientable Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the shape operator A is generalized Tanaka–Webster Reeb parallel, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Motivated by these works, in this paper we consider the notion of Reeb parallelism for the Ricci tensor S with respect to the GTW connection on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. The Ricci tensor S is said to be *generalized Tanaka–Webster Reeb parallel* (*GTW Reeb parallel*) if the covariant derivative in GTW connection $\widehat{\nabla}^{(k)}$ of S along the Reeb direction vanishes, that is, $(\widehat{\nabla}^{(k)}_\xi S)Y = 0$. In terms of this condition, we assert the following theorem.

Theorem 1 *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\alpha \neq 2k$. The Ricci tensor S on M is GTW Reeb parallel if and only if M is locally congruent to one of the following:*

(i) *a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius r such that*

$$r \neq \frac{1}{2\sqrt{2}}\cot^{-1}\Big(\frac{k}{\sqrt{2}}\Big),$$

or

(ii) *a tube over a totally geodesic $\mathbb{H}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$ with radius r such that*

$$r = \frac{1}{2}\cot^{-1}\Big(\frac{-k}{4(2n-1)}\Big).$$

When we consider the notion of GTW parallel Ricci tensor, that is, $(\widehat{\nabla}^{(k)}_X S) = 0$ for arbitrary tangent vector field X on M, by Theorem 1 we can assert the following theorem.

Theorem 2 *There does not exist any Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $\alpha \neq 2k$, satisfying $(\widehat{\nabla}^{(k)}_X S)Y = 0$ for any tangent vector fields X and Y on M.*

On the other hand, in [6] Jeong, Lee, and Suh gave a characterization of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\nabla A = \widehat{\nabla}^{(k)}A$. So naturally we consider that $\nabla S = \widehat{\nabla}^{(k)}S$, that is, the covariant derivative of the Ricci tensor S coincides with the derivative of S in the GTW connection. This is equivalent to the fact that the Ricci tensor S commutes with the Tanaka–Webster operator $\widehat{F}_X^{(k)}$, that is, $S \cdot \widehat{F}_X^{(k)} = \widehat{F}_X^{(k)} \cdot S$. It means that any eigenspace of the *Ricci tensor* S *is invariant* under the Tanaka–Webster operator $\widehat{F}_X^{(k)}$. In terms of this condition, we assert the following theorem.

Theorem 3 *There does not exist any Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying $(\widehat{\nabla}_X^{(k)}S)Y = (\nabla_X S)Y$ for any tangent vector fields X and Y on M.*

In order to get our results, in Section 1 we will give the fundamental formulas related to the Reeb parallel Ricci tensor. In Section 2, we want to give a complete proof of Theorem 1 for $\alpha = g(A\xi, \xi) \neq 2k$. In Sections 3 and 4 we give complete proofs of Theorem 2 and 3, respectively.

1 Basic Formulas for Ricci Tensor in $G_2(\mathbb{C}^{m+2})$

In this paper, we refer the reader to [3, 4, 7–9, 12, 13, 15, 16, 19–22] for Riemannian geometric structures of Hermitian symmetric spaces and its geometric quantities. Hereafter, let us denote by M a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, and let S denote the Ricci tensor of M. From [14], the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, is given by

$$
\begin{aligned}
SX &= (4m+7)X - 3\eta(X)\xi + hAX - A^2X \\
&\quad + \sum_{\nu=1}^{3}\big\{-3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X - \eta_\nu(\phi X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\big\},
\end{aligned}
\tag{1.1}
$$

where h denotes the trace of the shape operator A, that is, $h = \mathrm{Tr}A$. Then the derivative of the Ricci tensor S becomes

$$
\begin{aligned}
(\nabla_X S)Y = &-3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX \\
&+ (Xh)AY + h(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y \\
&- 3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\Big\} \\
&+ \sum_{\nu=1}^{3}\Big\{2\eta_\nu(\phi AX)\phi_\nu\phi Y + g(AX, \phi_\nu\phi Y)\phi_\nu\xi - \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\
&\qquad + \eta_\nu(\phi Y)\eta(AX)\xi_\nu - \eta_\nu(\phi Y)\phi_\nu\phi AX - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu\Big\},
\end{aligned}
\tag{1.2}
$$

for any tangent vector fields X and Y on M.

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In particular, substituting $X = \xi$ into (1.2) and using the condition that M is Hopf, that is, $A\xi = \alpha\xi$, we get

$$(\nabla_\xi S)Y = 4\alpha \sum_{\nu=1}^{3} \Big\{ \eta_\nu(\phi Y)\xi_\nu - \eta_\nu(Y)\phi_\nu\xi \Big\} + (\xi h)AY \\ + h(\nabla_\xi A)Y - (\nabla_\xi A)AY - A(\nabla_\xi A)Y.$$

Moreover, by the definition of GTW connection $\widehat{\nabla}^{(k)}$ the covariant derivative of S with respect to the GTW connection becomes

$$(1.3)\qquad (\widehat{\nabla}^{(k)}_X S)Y = \widehat{\nabla}^{(k)}_X (SY) - S(\widehat{\nabla}^{(k)}_X Y) \\ = \nabla_X(SY) + g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY \\ - S\nabla_X Y - g(\phi AX, Y)S\xi + \eta(Y)S\phi AX + k\eta(X)S\phi Y.$$

It yields

$$(1.4)\quad (\widehat{\nabla}^{(k)}_X S)Y = -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX \\ + (Xh)AY + h(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y \\ + g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY \\ - g(\phi AX, Y)S\xi + \eta(Y)S\phi AX + k\eta(X)S\phi Y \\ - 3\sum_{\nu=1}^{3} \Big\{ g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \Big\} \\ + \sum_{\nu=1}^{3} \Big\{ 2\eta_\nu(\phi AX)\phi_\nu\phi Y + g(AX, \phi_\nu\phi Y)\phi_\nu\xi - \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\ + \eta_\nu(\phi Y)\eta(AX)\xi_\nu - \eta_\nu(\phi Y)\phi_\nu\phi AX - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu \Big\}.$$

From now on, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with GTW Reeb parallel Ricci tensor, that is, S satisfies:

$$\text{(C-1)}\qquad (\widehat{\nabla}^{(k)}_\xi S)X = 0.$$

By (1.3), it becomes

$$(1.5)\qquad (\widehat{\nabla}^{(k)}_\xi S)X = \widehat{\nabla}^{(k)}_\xi (SX) - S(\widehat{\nabla}^{(k)}_\xi X) \\ = \nabla_\xi(SX) + g(\phi A\xi, SX)\xi - \eta(SX)\phi A\xi - k\eta(\xi)\phi SX \\ - S(\nabla_\xi X) - g(\phi A\xi, X)S\xi + \eta(X)S\phi A\xi + k\eta(\xi)S\phi X \\ = (\nabla_\xi S)X - k\phi SX + kS\phi X.$$

Thus, condition (C-1) is equivalent to $(\nabla_\xi S)X = k\phi SX - kS\phi X$, which yields

$$(1.6)\qquad 4(k-\alpha)\sum_{\nu=1}^{3} \Big\{ \eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi \Big\} \\ = (\xi h)AX + h(\nabla_\xi A)X - (\nabla_\xi A)AX - A(\nabla_\xi A)X - kh\phi AX \\ + k\phi A^2 X + khA\phi X - kA^2\phi X$$

from (1.1), (1.2), and [10, Section 2].

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Using these equations, we prove that ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$, as follows.

Lemma 1.1 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M has a GTW Reeb parallel Ricci tensor, then ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.*

Proof In order to prove this lemma, we put

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{1.7}$$

for some unit vectors $X_0 \in \mathcal{Q}$ and $\xi_1 \in \mathcal{Q}^\perp$. Putting $X = \xi$ in (1.6), by (1.7) and basic formulas in [10, Section 2], it follows that

$$4(\alpha - k)\eta_1(\xi)\phi_1\xi = \alpha(\xi h)\xi - h(\xi\alpha)\xi - 2\alpha(\xi\alpha)\xi, \tag{1.8}$$

where we have used $(\nabla_\xi A)\xi = (\xi\alpha)\xi$ and $(\nabla_\xi A)A\xi = \alpha(\xi\alpha)\xi$.

Taking the inner product of (1.8) with $\phi_1\xi$, we have

$$4(\alpha - k)\eta_1(\xi)\eta^2(X_0) = 0,$$

because of $\eta^2(X_0) + \eta^2(\xi_1) = 1$. From this, we have the following three cases.

Case 1: $\alpha = k$. From the definition of GTW connection we see that α must be a non-zero real number. By virtue of $Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$ in [10, Lemma A], the Reeb vector field ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$.

Case 2: $\eta(\xi_1) = 0$. By the notation (1.7), we see that ξ belongs to $\mathcal{Q}$.

Case 3: $\eta(X_0) = 0$. This case implies that ξ belongs to $\mathcal{Q}^\perp$ from (1.7).

Accordingly, summing up these cases, the proof of our Lemma is completed. ∎

2 Proof of Theorem 1

Hereafter, let M be a Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$ with GTW Reeb parallel Ricci tensor. Then by Lemma 1.1 we divide our consideration in two cases depending on whether ξ belongs to $\mathcal{Q}^\perp$ or $\mathcal{Q}$.

First of all, if we assume $\xi \in \mathcal{Q}$, then a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with GTW Reeb parallel Ricci tensor, and $\alpha = g(A\xi, \xi) \neq 2k$ is locally congruent to a real hypersurface of Type (B) by virtue of Theorem B given in the introduction.

Next let us consider the case, $\xi \in \mathcal{Q}^\perp$. Accordingly, we can put $\xi = \xi_1$. Since M is a Hopf hypersurface with GTW Reeb parallel Ricci tensor, equation (1.6) becomes

$$\begin{aligned}(\xi h)AX + h(\nabla_\xi A)X - (\nabla_\xi A)AX - A(\nabla_\xi A)X = \\ k(h\phi AX - \phi A^2X - hA\phi X + A^2\phi X).\end{aligned} \tag{2.1}$$

From the Codazzi equation [10, Section 2] and differentiating $A\xi = \alpha\xi$, we obtain

$$\begin{aligned}(\nabla_\xi A)X &= (\nabla_X A)\xi + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 \\ &= (X\alpha)\xi + \alpha\phi AX - A\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3.\end{aligned}$$

Using the equation from [10, Lemma 2.1] and the previous one, we get

$$(\nabla_\xi A)X = \frac{\alpha}{2}\phi AX - \frac{\alpha}{2}A\phi X + (\xi\alpha)\eta(X)\xi.$$

Therefore, (2.1) can be written as

$$(\xi h)AX + \widetilde{\kappa} h\phi AX - \widetilde{\kappa} hA\phi X + (h-2\alpha)(\xi\alpha)\eta(X)\xi - \widetilde{\kappa}\phi A^2 X + \widetilde{\kappa} A^2\phi X = 0, \tag{2.2}$$

where $\widetilde{\kappa} = (\frac{\alpha}{2} - k)$.

Since $\widetilde{\kappa} \neq 0$ is equivalent to the given condition $\alpha \neq 2k$, (2.2) yields

$$\frac{(\xi h)}{\widetilde{\kappa}} AX + h\phi AX - hA\phi X + \frac{(h-2\alpha)}{\widetilde{\kappa}}(\xi\alpha)\eta(X)\xi - \phi A^2 X + A^2\phi X = 0. \tag{2.3}$$

Now we consider the case $\xi h = 0$. Then (2.3) can be reduced to

$$h\phi AX - hA\phi X + \frac{(h-2\alpha)}{\widetilde{\kappa}}(\xi\alpha)\eta(X)\xi - \phi A^2 X + A^2\phi X = 0. \tag{2.4}$$

Taking the inner product of (2.4) with ξ, we have $\frac{(h-2\alpha)}{\widetilde{\kappa}}(\xi\alpha)\eta(X) = 0$. Thus, (2.4) becomes

$$h\phi AX - \phi A^2 X - hA\phi X + A^2\phi X = 0. \tag{2.5}$$

On the other hand, from equation (1.1) we calculate

$$S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2 X.$$

Then by (2.5) it follows that $S\phi X = \phi SX$ for any tangent vector field X on M. Hence, by Suh [16] we assert that M satisfying our assumptions must be a model space of Type (A).

We now assume $\xi h \neq 0$. Putting $\sigma = \frac{(\xi h)}{\widetilde{\kappa}}(\neq 0)$ and $\tau = \frac{(h-2\alpha)}{\widetilde{\kappa}}(\xi\alpha)$, equation (2.3) becomes

$$\sigma AX + h\phi AX - hA\phi X + \tau\eta(X)\xi - \phi A^2 X + A^2\phi X = 0. \tag{2.6}$$

Applying ϕ to (2.6) and replacing X by ϕX in (2.6), respectively, we get the following two equations:

$$\sigma\phi AX - hAX + h\alpha\eta(X)\xi - h\phi A\phi X + A^2 X - \alpha^2\eta(X)\xi + \phi A^2\phi X = 0$$
$$\sigma A\phi X + h\phi A\phi X + hAX - h\alpha\eta(X)\xi - \phi A^2\phi X - A^2 X + \alpha^2\eta(X)\xi = 0.$$

Summing up the above two equations, we obtain $\phi A + A\phi = 0$. Thus, equation (2.6) implies

$$\sigma AX + 2h\phi AX + \tau\eta(X)\xi = 0.$$

Let the orthogonal projection of X onto the distribution $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$ be denoted $X_{\mathcal{C}}$. Inserting this into the previous equation yields $\sigma AX_{\mathcal{C}} + 2h\phi AX_{\mathcal{C}} = 0$. In addition, applying ϕ to this equation, it follows that $\sigma\phi AX_{\mathcal{C}} - 2hAX_{\mathcal{C}} = 0$. Thus, we obtain

$$\begin{pmatrix} \sigma & 2h \\ -2h & \sigma \end{pmatrix} \begin{pmatrix} AX_{\mathcal{C}} \\ \phi AX_{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the square matrix of order 2, that is, $\sigma^2 + 4h^2 \geq \sigma^2 \neq 0$, so we get $AX_{\mathcal{C}} = 0$ for any $X_{\mathcal{C}} \in \mathcal{C}$. Substituting $X_{\mathcal{C}}$ as ξ_2 and ξ_3 implies that $A\xi_2 = 0$ and $A\xi_3 = 0$, respectively. Hence, we can assert that the distribution $\mathcal{Q}^\perp$ is invariant under the shape operator, that is, M is a $\mathcal{Q}^\perp$-invariant real hypersurface. Thus, by virtue of Theorem A, we conclude that with our assumptions M must be a model space of Type (A).

Summing up these discussions, we conclude that if a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfies (C-1), and $\alpha \neq 2k$, then M is of Type (A) or (B).

Now, let us check whether the Ricci tensor S of a model space of Type (A) (or of Type (B)) satisfies the Reeb parallelism with respect to $\widehat{\nabla}^{(k)}$ by using the principal curvature vectors and their corresponding principal curvature values for each eigenspace with respect to the shape operator A given in [10, Proposition A] (resp. [10, Proposition B]).

Let us denote by M_A a model space of Type (A). From now on, using the equations (1.1), (1.2), and [10, Proposition A], let us check whether or not the Ricci tensor S satisfies (1.6), which is equivalent to our condition (C-1) for each eigenspace T_α, T_β, T_λ, and T_μ on $T_x M_A$, $x \in M_A$. In order to do this, we find one equation related to S from (1.6) using the property of M_A, $\xi = \xi_1$ as follows:

$$
\begin{aligned}
(2.7) \qquad (\widehat{\nabla}^{(k)}_\xi S)X = &-h(\nabla_\xi A)X + (\nabla_\xi A)AX + A(\nabla_\xi A)X + kh\phi AX \\
&- k\phi A^2 X - khA\phi X + kA^2\phi X,
\end{aligned}
$$

since $h = \alpha + 2\beta + 2(m-2)\lambda$ is a constant.

Case A-1: $X = \xi(= \xi_1) \in T_\alpha$. Since $(\nabla_\xi A)\xi = 0$, we see that $(\widehat{\nabla}^{(k)}_\xi S)\xi = 0$ from the equation (2.7). It means that the Ricci tensor S becomes GTW Reeb parallel on T_α.

Case A-2: $X \in T_\beta = \operatorname{Span}\{\xi_2, \xi_3\}$. For $\xi_\mu \in T_\beta$, $\mu = 2, 3$ we have

$$
\begin{aligned}
(\nabla_\xi A)\xi_\mu &= \beta(\nabla_\xi \xi_\mu) - A(\nabla_\xi \xi_\mu) \\
&= \beta q_{\mu+2}(\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\xi)\xi_{\mu+2} + \alpha\beta\phi_\mu\xi \\
&\quad - q_{\mu+2}(\xi)A\xi_{\mu+1} + q_{\mu+1}(\xi)A\xi_{\mu+2} - \alpha A\phi_\mu\xi,
\end{aligned}
$$

which yields that $(\nabla_\xi A)\xi_2 = 0$ and $(\nabla_\xi A)\xi_3 = 0$. Therefore, from equation (2.7) we obtain, respectively,

$$
\begin{aligned}
(\widehat{\nabla}^{(k)}_\xi S)\xi_2 &= kh\phi A\xi_2 - k\phi A^2\xi_2 - khA\phi\xi_2 + kA^2\phi\xi_2 \\
&= (-kh\beta + k\beta^2 + kh\beta - k\beta^2)\xi_3 = 0,
\end{aligned}
$$

and $(\widehat{\nabla}^{(k)}_\xi S)\xi_3 = 0$ by similar methods. So, we assert that the Ricci tensor S of M_A is Reeb parallel on T_β.

By the structure of a tangent vector space $T_x M_A$ at $x \in M_A$, we see that the distribution $\mathcal{Q}$ is composed of two eigenspaces T_λ and T_μ. On the distribution $\mathcal{Q} = T_\lambda \oplus T_\mu$ we obtain

$$
(2.8) \qquad (\nabla_\xi A)X = \alpha\phi AX - A\phi AX + \phi X + \phi_1 X
$$

by virtue of the Codazzi equation [10, Section 2]. Using this equation we consider the following two cases.

Case A-3: $X \in T_\lambda = \{X \mid X \in \mathcal{Q},\ JX = J_1X\}$. We naturally see that if $X \in T_\lambda$, then $\phi X = \phi_1 X$. Moreover, the vector ϕX also belongs to the eigenspace T_λ for any $X \in T_\lambda$, that is, $\phi T_\lambda \subset T_\lambda$. From these and (2.8), we obtain

$$
(\nabla_\xi A)X = (\alpha\lambda - \lambda^2 + 2)\phi X, \quad \text{for } X \in T_\lambda.
$$

From these facts and (2.7), we obtain

$$(\widehat{\nabla}_{\xi}^{(k)} S)X = (\alpha\lambda - \lambda^2 + 2)(2\alpha - h)\phi X,$$

which implies that the Ricci tensor S must be Reeb parallel for $\widehat{\nabla}^{(k)}$ on T_λ, since $\alpha\lambda - \lambda^2 + 2 = 0$.

Case A-4 : $X \in T_\mu = \{ X \mid X \in \mathfrak{Q},\ JX = -J_1 X \}$. If $X \in T_\mu$, then $\phi X = -\phi_1 X$, $\phi T_\mu \subset T_\mu$ and $\mu = 0$. So, from (2.8), we obtain $(\nabla_\xi A)X = 0$, moreover $(\widehat{\nabla}_{\xi}^{(k)} S)X = 0$ for any $X \in T_\mu$.

Summing up all of the cases mentioned above, we can assert that the Ricci tensor S of a real hypersurface M_A in $G_2(\mathbb{C}^{m+2})$ is GTW Reeb parallel.

Now let us consider our problem for a model space of Type (B), which will be denoted by M_B. In order to do this, let us calculate the fundamental equation related to the covariant derivative of the Ricci tensor S of M_B along the direction of ξ in GTW connection. On $T_x M_B$, $x \in M_B$, since $\xi \in \mathfrak{Q}$ and $h = \mathrm{Tr}(A) = \alpha + (4n-1)\beta$ is a constant, equation (1.6) is reduced to

$$\begin{aligned}(\widehat{\nabla}_{\xi}^{(k)} S)X = 4(k - \alpha)\sum_{\nu=1}^{3}\Big\{ \eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi \Big\} \\ - h(\nabla_\xi A)X + (\nabla_\xi A)AX + A(\nabla_\xi A)X \\ + kh\phi AX - k\phi A^2 X - khA\phi X + kA^2\phi X.\end{aligned}$$

Moreover, by the equation of Codazzi and [10, Proposition B] we obtain that for any $X \in T_x M_B$,

$$\begin{aligned}(\nabla_\xi A)X &= \alpha\phi AX - A\phi AX + \phi X - \sum_{\nu=1}^{3}\big\{ \eta_\nu(X)\phi_\nu\xi + 3g(\phi_\nu\xi, X)\xi_\nu \big\} \\ &= \begin{cases} 0 & \text{if } X \in T_\alpha, \\ \alpha\beta\phi\xi_\ell & \text{if } X \in T_\beta = \mathrm{Span}\{\xi_\ell \mid \ell = 1,2,3\}, \\ -4\xi_\ell & \text{if } X \in T_\gamma = \mathrm{Span}\{\phi\xi_\ell \mid \ell = 1,2,3\}, \\ (\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda, \\ (\alpha\mu + 2)\phi X & \text{if } X \in T_\mu. \end{cases}\end{aligned}$$

From these two equations, it follows that

$$(2.9)\qquad (\widehat{\nabla}_{\xi}^{(k)} S)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha, \\ (\alpha - k)(4 - h\beta + \beta^2)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta, \\ (4(\alpha - k) + (h - \beta)(4 + k\beta))\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma, \\ (h - \beta)(k\lambda - k\mu - \alpha\lambda - 2)\phi X & \text{if } X \in T_\lambda, \\ (h - \beta)(k\mu - k\lambda - \alpha\mu - 2)\phi X & \text{if } X \in T_\mu. \end{cases}$$

So, we see that M_B has Reeb parallel GTW Ricci tensor, when α and h satisfy the conditions $\alpha = k$ and $h - \beta = 0$, which means $r = 1/2\cot^{-1}(-k/4(2n-1))$. Moreover, this radius r satisfies our condition $\alpha \neq 2k$.

Hence summing up these considerations, we give a complete proof of Theorem 1 in the introduction.

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For the case $\alpha = 2k$, the Reeb vector field ξ of a Hopf hypersurface M with GTW Reeb parallel Ricci tensor belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$. So, for the case $\xi \in \mathcal{Q}^\perp$, equation (2.2) becomes $\xi h = 0$; that is, the trace h of the shape operator A is constant along ξ.

For the case $\xi \in \mathcal{Q}$, it is a well-known fact that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ must be a model space M_B of Type (B) (see [11]). On the other hand, from (2.9) and $\alpha = 2k$, the GTW covariant derivative of the Ricci tensor S of M_B along the direction of ξ is given by

$$(\widehat{\nabla}^{(k)}_\xi S)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha, \\ k(4 - h\beta + \beta^2)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta, \\ (4k + (h-\beta)(4 + k\beta))\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma, \\ -(h-\beta)(k\beta + 2)\phi X & \text{if } X \in T_\lambda, \\ -(h-\beta)(k\beta + 2)\phi X & \text{if } X \in T_\mu. \end{cases}$$

Actually, since $\alpha = 2k$, we naturally have $k\beta + 2 = 0$. It follows that the Ricci tensor S is GTW Reeb parallel on T_λ and T_μ. In order to be the GTW Reeb parallel Ricci tensor on the other eigenspaces T_β and T_γ, we should have the following two equations:

$$4 - h\beta + \beta^2 = 0 \quad \text{and} \quad 4k + (h-\beta)(4 + k\beta) = 0.$$

Combining these two equations, we have $2k + h - \beta = 0$. Since

$$h = \alpha + 3\beta + (4n-4)(\lambda + \mu) = \alpha + (4n-1)\beta \quad \text{and} \quad \alpha = 2k,$$

it follows that $\alpha = -(2n-1)\beta$. By virtue of [10, Proposition B], $\alpha = -2\tan(2r)$ and $\beta = 2\cot(2r)$, where $r \in (0, \pi/4)$, we obtain $\tan(2r) = \sqrt{2n-1}$. From such assertions, we conclude that a model space of Type (B) has GTW Reeb parallel Ricci tensor for special radius r such that $r = \frac{1}{2}\tan^{-1}(\sqrt{2n-1})$.

From the above, we have the following. Let M be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with GTW Reeb parallel Ricci tensor for $\alpha = 2k$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}$, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$ with radius r such that $r = \frac{1}{2}\tan^{-1}\sqrt{2n-1}$.

3 Proof of Theorem 2

Bear in mind that the notion of GTW parallel Ricci tensor is stronger than GTW Reeb parallel Ricci tensor, and in the previous section, we got that a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, $(\alpha \not\equiv 2k)$ satisfying GTW Reeb parallel Ricci tensor, then M is locally congruent to of Type M_A or Type M_B.

Hereafter, let us check whether the Ricci tensor S of a model space M_A (or M_B) satisfies the parallelism with respect to $\widehat{\nabla}^{(k)}$ by using the principal curvature vectors and their corresponding principal curvature values for each eigenspace with respect to the shape operator A given in [10, Proposition A] (or [10, Proposition B], respectively).

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Suppose that M is of Type (A). Remember that $A\xi = \alpha\xi$, $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, with $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$. Take $Y = \xi$, $X = \xi_2$ in (1.4). We have

$$4\beta\xi_3 + h\nabla_{\xi_2}\alpha\xi - hA\phi A\xi_2 - \nabla_{\xi_2}\alpha^2\xi + A^2\phi A\xi_2 = \\ \beta\{g(\xi_3, S\xi)\xi - \eta(S\xi)\xi_3 + S\xi_3\}.$$

Since the Reeb function α is constant, $S\xi = (4m + h\alpha - \alpha^2)\xi$, and

$$S\xi_3 = (4m + 6 + h\beta - \beta^2)\xi_3,$$

from (1.4) we arrive at $\beta\xi_3 = 0$, which is impossible. Thus, M_A does not have GTW parallel Ricci tensor.

In the case of M_B, if we take $X = \xi_1$, $Y = \xi$ in (1.4) and bear in mind that $S\xi = (4m+4+h\alpha-\alpha^2)\xi$ and $S\phi_1\xi = (4m+8)\phi_1\xi$, we obtain $\alpha h = 0$, where $\alpha = -2\tan(2r)$. As $\alpha \neq 0$ we must have $h = 0$. With similar computations, we obtain $6\beta\xi_3 = 0$, for $\beta = 2\cot(2r)$, when $X = \xi_1$, $Y = \xi_2$ in (1.4). As this is impossible, M_B does not have a GTW parallel Ricci tensor, and this completes the proof of Theorem 2. ■

4 Proof of Theorem 3

Recently, in [6] Jeong, Lee, and Suh gave a characterization of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\widehat{\nabla}^{(k)}A = \nabla A$. So naturally we consider that $(\widehat{\nabla}^{(k)}_X S)Y = (\nabla_X S)Y$; that is, the parallel Ricci tensor in GTW connection coincides with the parallel Ricci tensor in Levi–Civita connection. As a special case, we restrict $X = \xi$ as follows:

(C-2) $$(\widehat{\nabla}^{(k)}_\xi S)X = (\nabla_\xi S)X$$

for any tangent vector field X on M.

By virtue of equation (1.5) and being Hopf, condition (C-2) is equivalent to $S\phi = \phi S$; thus, we have the following remark [16].

Remark 4.1 Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then $\widehat{\nabla}^{(k)}_\xi S = \nabla_\xi S$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

By Remark 4.1, if a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies $\widehat{\nabla}^{(k)}S = \nabla S$, then naturally (C-2) holds on M. So M is of Type (A), that is, M_A. Now let us check whether a model space M_A satisfies our condition

(C-3) $$(\widehat{\nabla}^{(k)}_X S)Y = (\nabla_X S)Y$$

for any tangent vector fields X, $Y \in T_xM_A$, $x \in M_A$. In order to do this, we assume that the Ricci tensor S of M_A satisfies (C-3). That is, we have

(4.1) $$\begin{aligned} 0 &= (\widehat{\nabla}^{(k)}_X S)Y - (\nabla_X S)Y \\ &= g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY \\ &\quad - g(\phi AX, Y)S\xi + \eta(Y)S\phi AX + k\eta(X)S\phi Y \end{aligned}$$

for any $X, Y \in T_xM_A$.

Since $T_xM_A = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\gamma$, equation (4.1) holds for $X \in T_\beta$ and $Y \in T_\alpha$. For the sake of convenience we put $X = \xi_2 \in T_\beta$ and $Y = \xi \in T_\alpha$. Since $S\xi = \delta\xi$ and

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$S\xi_3 = \sigma\xi_3$ where $\delta = (4m + h\alpha - \alpha^2)$ and $\sigma = (4m + 6 + h\beta - \beta^2)$, equation (4.1) reduces to $\beta(\delta - \sigma)\xi_3 = 0$. By [10, Proposition A], since the principal curvature $\beta = \sqrt{2}\cot(\sqrt{2}r)$ for $r \in (0, \pi/\sqrt{8})$ is non-zero, it follows that $\delta - \sigma = 0$. In other words, by [10, Proposition B] we obtain

$$\begin{aligned} -(\delta - \sigma) &= 6 - \alpha\beta + \beta^2 + (2m-2)\beta\lambda - (2m-2)\alpha\lambda \\ &= 8 - 4(m-1)\tan^2(\sqrt{2}r), \end{aligned}$$

which gives us

$$\tan^2(\sqrt{2}r) = \frac{2}{m-1}. \tag{4.2}$$

In addition, since (4.1) holds for $X \in T_\lambda$ and $Y = \xi$, we obtain

$$0 = (\widehat{\nabla}^{(k)}_X S)\xi - (\nabla_X S)\xi = \lambda(\tau - \delta)\phi X,$$

where in the second equality we have used $\phi X \in T_\lambda$ and $SX = (4m+6+h\lambda-\lambda^2)X = \tau X$ for any $X \in T_\lambda$. As $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ where $r \in (0, \pi/\sqrt{8})$ is non-zero, we have also $\tau - \delta = 0$. By a straightforward calculation, it is

$$\tau - \delta = 6 + h\lambda - \lambda^2 - h\alpha + \alpha^2 = 4m - 4\cot^2(\sqrt{2}r) = 0.$$

From (4.2), it becomes $2m + 2 = 0$, which gives us a contradiction. Accordingly, it completes the proof of Theorem 3 given in the introduction. ■

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