Canad. Math. Bull. Vol. 17 (4), 1974

ON THE MINIMUM ORDER OF GRAPHS WITH GIVEN GROUP

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For G a finite group let $\alpha(G)$ denote the minimum number of vertices of the graphs X the automorphism group $A(X)$ of which is isomorphic to G.

G. Sabidussi proved [1], that $\alpha(G)=O(n \log d)$ where $n=|G|$ and d is the minimum number of generators of G. As $O(\log n)$ is the best possible upper bound for d, the result established in [1] implies that $\alpha(G)=O(n \log \log n)$.

We prove that

$$\alpha(G) = O(n).$$

More exactly, apart from the cyclic groups Z_3, Z_4 and Z_5 we prove

$$\alpha(G) \leq 2n.$$

It is obvious that for $n=p$ prime, this result is sharp, i.e. $\alpha(Z_p)=2p$. As pointed out in [2], $\alpha(Z_3)=9$, $\alpha(Z_5)=15$. $\alpha(Z_4)=10$ was proved by R. L. Meriwether in 1963 (unpublished, see [3]).

THEOREM. *If G is different from the cyclic groups of order* 3, 4, 5 *then* $\alpha(G)\leq 2\,|G|$.

Proof.

(i) For $|G|\leq 2$ $\alpha(G)=|G|$. If G is the Klein group ($|G|=4$), then the figure shows a graph Y with $A(Y)\cong G$ and $|V(Y)|=4$.

(ii) Let $|G|=n\geq 6$. If G is cyclic, the theorem was proved in [1].

(iii) Assume that G is not cyclic and let $H=\{h_1, \ldots, d_d\}$ be a minimal generating system of G $(d\geq 2)$. In the following, $V(X)$ denotes the set of vertices, $E(X)$ the set of edges of the graph X.

Let us define the graphs X_1 and X_2 by

$$V(X_s) = G \qquad (s = 1, 2);$$
$$E(X_1) = \{[h_i g, h_{i+1} g] : g \in G,\ i = 1, \ldots, d-1\};$$
$$E(X_2) = \{[h_1 g, g] : g \in G\}.$$

The right regular representation of G is a transitive subgroup of both $A(X_1)$ and $A(X_2)$. Hence the graphs X_s $(s=1,2)$ are regular. Let ρ_s be the valency of the vertices of X_s. Clearly $\rho_2\leq 2$. Hence if $\rho_1=\rho_2$ then

$$n-1-\rho_2 \geq n-3 \geq 3 > \rho_2 = \rho_1. \tag{1}$$

Received by the editors September 13, 1972.

https://doi.org/10.4153/CMB-1974-082-9 Published online by Cambridge University Press

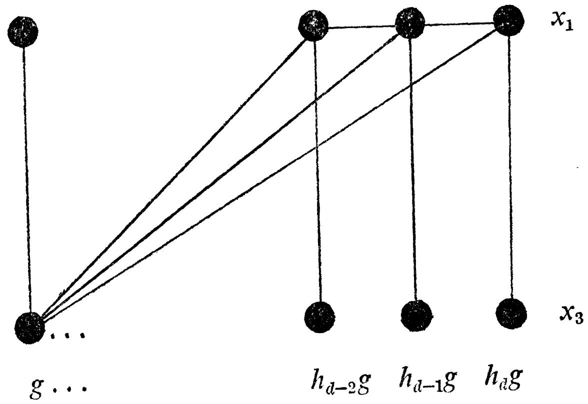

Figure 1

Let $X_3=X_2$ if $\rho_1\neq\rho_2$, and $X_3=\bar{X}_2$ (the complement of X_2, i.e. $V(X_3)=V(X_2)$; $e\in E(X_3)\leftrightarrow e\notin E(X_2)$) if $\rho_1=\rho_2$. Let ρ_3 be the valency of the vertices of X_3. From (1) it follows that

$$\rho_3\neq\rho_1.$$

Let us define the graph X (see Fig. 2) by

$$V(X)=G\times\{1,3\};$$

$$E(X)=\{[(a,s),(b,s)]:s=1,3,\ [a,b]\in E(X_s)\}$$

$$\cup\ \{[(g,3),(h_ig,1)]:g\in G,\ i=1,\ldots,d\}$$

$$\cup\ \{[(g,3),(g,1)]:g\in G\}.$$

Evidently $|V(X)|=2n$.

Let $A=\{\pi_g:g\in G\}$ denote the permutation group consisting of the permutations

$$\pi_g:(a,s)\to(ag,s)\qquad(a\in G,\ s=1,3).$$

A acts on $V(X)$, and clearly $A\cong G$, $A\subseteq A(X)$. We prove that $A=A(X)$.

The valency of the vertices in $G\times\{s\}$ is ρ_s+d+1. Hence, as $\rho_3\neq\rho_1$, both $G\times\{1\}$ and $G\times\{3\}$ are invariant under the action of $A(X)$.

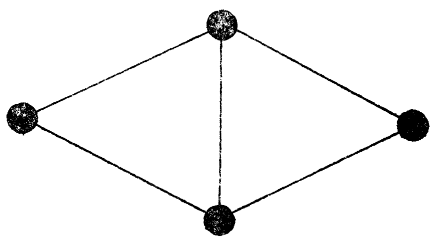

Figure 2

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Assume that $[h_j g_0, g_0] \in E(X_1)$ for some $g_0 \in G$, $1 \leq j \leq d$. Then

$$[h_j g_0, g_0] = [h_i g, h_{i+1} g] \quad \text{for some } g \in G, \qquad 1 \leq i \leq d-1.$$

That is, either

$$h_j g_0 = h_i g, \qquad g_0 = h_{i+1} g;$$

or

$$h_j g_0 = h_{i+1} g, \qquad g_0 = h_i g.$$

In the first case $h_j h_{i+1} = h_i$, in the second case $h_j h_i = h_{i+1}$; both contradict the minimality of H.

Thus

(2) $$[h_j g, g] \notin E(X_1) \qquad (j = 1, \ldots, d;\ g \in G).$$

Let us consider the section graph S_g of X on the vertices in $G \times \{1\}$ which are adjacent to $(g, 3)$,

$$V(S_g) = \{h_1 g, \ldots, h_d g, g\} \times \{1\}.$$

From (2) it follows that $(g, 1)$ is an isolated vertex in S_g.

Since S_g contains a connected section graph on its vertices which are different from $(g, 1)$ and since

$$|V(S_g)| = d+1 \geq 3,$$

$(g, 1)$ is the only isolated vertex in S_g. Hence if $\phi \in A(X)$, $(g, 3)\phi = (g', 3)$ then $(g, 1)\phi = (g', 1)$.

That is, if $\phi \in A(X)$, then there is a permutation $\bar{\phi}$ of G such that

$$(g, s)\phi = (g\bar{\phi}, s) \qquad (s = 1, 2;\ g \in G).$$

Let $\phi \in A(X)$, $g_0 \in G$, $g_0 \bar{\phi} = x$. Put $\psi = \phi \pi_x^{-1}$. Then

(3) $$g_0 \bar{\psi} = g_0.$$

It suffices to prove that $\bar{\psi}$ is the identity. (3) implies that S_{g_0} is invariant under $\bar{\psi}$. The only edges of S_{g_0} are

(4) $$[h_i g_0, h_{i+1} g_0] \qquad (i = 1, \ldots, d-1).$$

For if $[h_k g_0, h_l g_0] \in E(X_1)$ $(i \leq k, l \leq d, k \neq l)$, then $[h_k g_0, h_l g_0] = [h_i g, h_{i+1} g]$ for suitable g and i $(g \in G, 1 \leq i \leq d-1)$. That is, either

$$h_k g_0 = h_i g, \qquad h_l g_0 = h_{i+1} g;$$

or

$$h_l g_0 = h_i g, \qquad h_k g_0 = h_{i+1} g.$$

It is sufficient to deal with the first case:

$$g g_0^{-1} = h_i^{-1} h_k = h_{i+1}^{-1} h_l.$$

As a consequence of this and the minimality of H we obtain $\{i, i+1\}=\{k, l\}$, which proves (4).

(4) implies that $A(S_{g_0})$ contains only two elements: the identity and the reflection

$$\tau: h_i g_0 \to h_{d+1-i} g_0 \qquad (i = 1, \ldots, d). \tag{5}$$

We now prove

$$[h_d g_0, g_0] \notin E(X_2). \tag{6}$$

For if $[h_d g_0, g_0] \in E(X_2)$, then $[h_d g_0, g_0]=[h_1 g, g]$ for a suitable $g \in G$. That is, either

$$g_0 = g, \qquad h_d = h_1;$$

or

$$h_d g_0 = g, \qquad g_0 = h_1 g,$$

i.e. $h_d=h_1^{-1}$; both are impossible by $d \geq 2$ and the minimality of H.

From (6) and $[h_1 g_0, g_0] \in E(X_2)$ it follows that

$$[h_1 g_0, g_0] \in E(X_3) \leftrightarrow [h_d g_0, g_0] \notin E(X_3).$$

Thus the reflection τ defined in (5) cannot be the restriction of $\bar{\psi}$ to $V(S_{g_0})$ (since $g_0\bar{\psi}=g_0$). Hence, as a consequence of (5), $\bar{\psi} \mid V(S_{g_0})$ is the identity. That is, the elements of Hg_0 are fixed under $\bar{\psi}$. By induction we obtain that the elements of $H^2 g_0, H^3 g_0, \ldots$ are also fixed under $\bar{\psi}$. As H generates the finite group G, $H \cup H^2 \cup \cdots \cup H^r = G$ for some r. Thus $\bar{\psi}$ is the identity permutation of G which completes the proof.

References

1. G. Sabidussi, *On the minimum order of graphs with given automorphism group,* Monatsh. Math. **63,** (1959) pp. 124–127.

2. F. Harary and E. Palmer, *The smallest graph whose group is cyclic*, Czechoslovak Math. J. **16,** (91) (1966), pp. 70–71.

3. G. Sabidussi, Review 2563, Math. Rev. **33,** No. 3, March 1967.

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