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ON THE ISOMETRY OF SURFACES PRESERVING THE LINES OF CURVATURE

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In this paper we study the isometric deformations of surfaces in E^3, which preserve the lines of curvature. We call a surface M and LC-surface if it admits a non-trivial deformation of this type. We distinguish three types of LC-surfaces, and obtain some new results about these three types of surfaces.

0. INTRODUCTION

Bryant, Chern and Griffiths [1] studied the isometric deformations of surfaces in 3-dimensional Euclidean space E^3, which preserve the lines of curvature. They obtained a main result as follows:

THEOREM A. *In the three-dimensional Euclidean space E^3 consider two pieces of surfaces M, M^*, such that (a) their Gaussian curvature is not zero and they have no umbilics; and (b) they are connected by an isometry $f: M \to M^*$ preserving the lines of curvature. Then M and M^* are in general congruent or symmetric. There are surfaces M, for which the corresponding M^* is distinct relative to rigid motions. The Molding surfaces, and only these, are such surfaces belonging to a continuous family of distinct surfaces, which are connected by isometries preserving the lines of curvature (see* [1], *p.283).*

In the present paper we shall study deeply the deformations. First of all, we have the following definition.

DEFINITION. Let M and M^* be two surfaces in E^3 and I and I^* be the first fundamental forms of M and M^*. Suppose a map $F: M \to M^*$ is a diffeormorphism, and

(a) F is an I-isometry

$$F^*(I^*) = I,$$

where F^* represents F's cotangent map.

(b) F preserves the lines of curvature.

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In this case, we call F an isometry preserving the lines of curvature, or an 'LC-isometry'. If a surface M admits a non-trivial LC-isometry, we call M an LC-surface.

Relative to the integrability conditions of a differential equation (see (1.26) and (1.27)), we can distinguish three types of LC-surfaces. We obtain some new results about the three types of LC-surfaces.

(1) Theorem A supposes that the Gaussian curvatures of the surfaces M and M^* are not zero. We shall prove that any developable surface (with Gaussian curvature zero) is an LC-surface of the first type (see Section 2, Theorem 1).

(2) An LC-surface of the second type is just a Molding surface which was studied in Theorem A. We shall get some new properties of the surface (see Section 3, Theorem 2, and Theorem 3 and its Corollary).

(3) We shall get a necessary and sufficient condition for a surface to be an LC-surface of the third type (see Section 4, Theorem 4).

1. Isometry preserves the lines of curvature

We study a piece of an oriented surface M in 3-dimensional Euclidean space E^3, and suppose it to be sufficiently differentiable and with no umbilic points. On M there is a field of orthonormal frames $me_1e_2e_3$, such that $m \in M$, where e_1 and e_2 are unit vectors along the principal directions of M at m, and e_3 is the unit normal vector to M at m. We have

$$dm = \omega_1 e_1 + \omega_2 e_2, \tag{1.1}$$

$$de_i = \omega_{ij} e_j, \tag{1.2}$$

$$\omega_{ij} + \omega_{ji} = 0, \qquad i, j = 1, 2, 3, \tag{1.3}$$

$$\omega_{13} = a\omega_1, \ \omega_{23} = c\omega_2. \tag{1.4}$$

We assume $a > c$, where a and c are the two principal curvatures of M at m. The mean curvature and the Gaussian curvature of M are

$$2H = a + c, \qquad K = ac. \tag{1.5}$$

The structural equations of M are

$$d\omega_1 = -\omega_2 \wedge \omega_{12}, \qquad d\omega_2 = \omega_1 \wedge \omega_{12}, \tag{1.6}$$

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 = -\omega_{13} \wedge \omega_{23}, \tag{1.7}$$

$$d\omega_{13} = -\omega_{23} \wedge \omega_{12}, \qquad d\omega_{23} = \omega_{13} \wedge \omega_{12}. \tag{1.8}$$

The connection form of M is

$$\omega_{12} = h\omega_1 + k\omega_2. \tag{1.9}$$

The differentials of the two principal curvatures a and c are

$$\begin{aligned} da &= a_1\omega_1 + a_2\omega_2, \\ dc &= c_1\omega_1 + c_2\omega_2. \end{aligned} \tag{1.10}$$

Define two functions

$$f = a - c > 0, \qquad g = a + c. \tag{1.11}$$

Using (1.4), (1.9), (1.10) and (1.11), the structural equations (1.7) imply

$$a_2 = fh, \qquad c_1 = fk. \tag{1.12}$$

Suppose M^* is another surface in the space E^3. We shall denote the quantities pertaining to M^* by the same symbols with star "$*$", for example ω^*_{ij}, a^*, c^*, ...

Let the mapping $F\colon M \to M^*$ be an isometry from M to M^*, and $me_1e_2e_3$ and $m^*e^*_1e^*_2e^*_3$ be the fields of principal frames over M and M^*, respectively.

Since F is an isometry, we have

$$\begin{aligned} \omega^*_1 &= \omega_1 \cos\tau - \omega_2 \sin\tau \\ \omega^*_2 &= \omega_1 \sin\tau + \omega_2 \cos\tau \end{aligned} \tag{1.13}$$

where τ is an angle of rotation of the principal directions during the isometric deformation. Now we suppose that F preserves the lines of curvature, so it preserves the principal directions; so the angle $\tau = 0$, and from (1.13), we get

$$\omega^*_1 = \omega_1, \quad \omega^*_2 = \omega_2, \quad \omega^*_{12} = \omega_{12}. \tag{1.14}$$

Since Gaussian curvature is preserved under isometry, $ac = a^*c^*$, and we can let

$$a^* = ta, \quad c^* = t^{-1}c, \quad t \neq 0. \tag{1.15}$$

The geometric interpretations of t and t^{-1} are as change-coefficients of the two principal curvatures a and c, respectively. From (1.9), (1.11) and (1.14), we have

$$\begin{aligned} &h^* = h, \qquad k^* = k, \\ &f^* = a^* - c^* = ta - t^{-1}c, \quad g^* = a^* + c^* = ta + t^{-1}c. \end{aligned} \tag{1.16}$$

From equation (1.12) when applied to M^*, we get

$$(ta)_2 = f^*h, \qquad \left(t^{-1}c\right)_1 = f^*k. \tag{1.17}$$

Using (1.12) and (1.17) gives

$$\begin{aligned} ct_1 &= t\left(1 - t^2\right)ak, \\ at_2 &= -t^{-1}\left(1 - t^2\right)ch. \end{aligned} \tag{1.18}$$

We consider two cases depending on whether M is a developable surface or not.

I. M is a developable surface. In this case, $K = ac = 0$. We suppose $a \neq 0$, $c = 0$. From (1.18) we get

$$k = 0, \qquad t_2 = 0 \tag{1.19}$$

and

$$\omega_{13} = a\omega_1, \qquad \omega_{23} = 0. \tag{1.20}$$

Using (1.8) and (1.20), we have $d\omega_{13} = 0$, so set ω_{13} as the total differential of the function u_1, a first parameter of M,

$$\omega_{13} = du_1, \qquad \omega_1 = a^{-1}du_1. \tag{1.21}$$

On the surface M the set of u_1-curves (the curves along which $\omega_2 = 0$) is a family of lines of curvature which are not straight lines (because $a \neq 0$). But the other family of lines of curvature (the curves along $\omega_1 = 0$) are straight lines. We can choose the arclength of the straight lines as a second parameter u_2, so

$$\omega_2 = du_2. \tag{1.22}$$

Hence the differential equation of t (1.18) or (1.19) becomes

$$\frac{\partial t}{\partial u_2} = 0;$$

its solution is

$$t = t(u_1). \tag{1.23}$$

Consequently, any developable surface M is an LC-surface.

II. M is not a developable surface. In this case, $K = ac \neq 0$, $a \neq 0$, $c \neq 0$. We rewrite (1.18) as

$$\begin{aligned} t_1 &= t(1 - t^2)ac^{-1}k, \\ t_2 &= -t^{-1}(1 - t^2)ca^{-1}h. \end{aligned} \tag{1.18$'$}$$

From now on we assume $t_2 \neq 1$, discarding the trivial case that M^* is congruent or symmetric to M. Now we let

$$\begin{aligned} \theta_1 &= ac^{-1}k\omega_1, \\ \theta_2 &= ca^{-1}h\omega_2, \end{aligned} \tag{1.24}$$

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and

$$T = t^2 \neq 0, 1 \tag{1.25}$$

so that (1.18′) can be written

$$dT = 2(1-T)(T\theta_1 - \theta_2). \tag{1.26}$$

Taking exterior derivatives of (1.26) gives

$$T(d\theta_1 - 2\theta_1 \wedge \theta_2) = d\theta_2 - 2\theta_1 \wedge \theta_2. \tag{1.27}$$

Equation (1.26) is the total differential equation satisfied by the square T of the change-coefficient t of the principal curvature a. Equation (1.27) is an integrability condition of (1.26). When solving (1.27) for T, on substituting into (1.26), we get the condition on the surface M, to be an LC-surface but not a developable surface.

Now let us distinguish three cases of LC-surfaces.

FIRST TYPE. M is a developable surface, $K = 0$, or $\theta_1 = \theta_2 = 0$.

SECOND TYPE. M is not a developable surface, $K \neq 0$, and

$$d\theta_1 - 2\theta_1 \wedge \theta_2 = d\theta_2 - 2\theta_1 \wedge \theta_2 = 0. \tag{1.28}$$

Then (1.27) holds identically for all T, and (1.26) has a continuum of solutions, each depending on an arbitrary constant. Thus we have a one-parameter family of surfaces LC-isometric to M. This is just the case in Theorem A (see [**1**]).

THIRD TYPE. M is not a developable surface, $K \neq 0$, and (1.28) does not hold.

$$d\theta_1 - 2\theta_1 \wedge \theta_2 \neq 0, \qquad d\theta_2 - 2\theta_1 \wedge \theta_2 \neq 0. \tag{1.29}$$

Then from (1.27) we get T, inserting T into (1.26). Thus we can obtain a single surface which is LC-isometric to M.

2. LC-SURFACES OF THE FIRST TYPE — DEVELOPABLE SURFACES

Suppose M is a developable surface, and choose the frame field and the parameters u_1, u_2, so that (1.21) and (1.22) hold.

$$\omega_1 = a^{-1}du_1, \qquad \omega_2 = du_2, \qquad a \neq 0, \tag{2.1}$$

$$\omega_{13} = du_1, \qquad \omega_{23} = 0, \qquad c = 0, \tag{2.2}$$

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where the differential of the first parameter u_1 is the normal connection form ω_{13}, and the second parameter u_2 is the arclength of the straight lines on the surface M. The change-coefficient t of the principal curvatures as (1.23)

$$t = t(u_1) \tag{2.3}$$

is any function of u_1. From (1.15) and (1.14) we have some quantities of the surface M^*:

$$\begin{gathered} a^* = t(u_1)a, \quad c^* = 0 \\ \omega_1^* = \omega_1 = a^{-1}du_1, \quad \omega_2^* = \omega_2 = du_2 \\ \omega_{12}^* = \omega_{12}, \quad \omega_{13}^* = t(u_1)du_1, \quad \omega_{23}^* = 0. \end{gathered} \tag{2.4}$$

From the above discussion we obtain the following theorem.

THEOREM 1. *Any developable surface is an LC-surface of the first type. In other words, any developable surface can deform continuously to another developable surface by any isometry preserving the lines of curvature.*

For the developable surface there exist three cases as follows:

(1). Cylinder M:

$$m(s, z) = m(s) + zk,$$
$$m(s) = x(s)i + y(s)j,$$

where $Oijk$ is a frame in E^3, $m(s)$ a plane curve parametrised by its arclength s, and

$$m' = \alpha, \qquad \alpha' = \kappa\beta.$$

Choosing a frame field over M, by

$$e_1 = \alpha, \qquad e_2 = k, \qquad e_3 = -\beta,$$

we have

$$\begin{gathered} dm = \omega_1 e_1 + \omega_2 e_2, \\ \omega_1 = ds, \quad \omega_2 = dz, \quad \omega_{12} = 0, \\ \omega_{13} = a\omega_1, \quad a = -\kappa \neq 0, \quad \omega_{23} = 0. \end{gathered}$$

Define the two parameters

$$u_1 = \int a(s)ds = -\int \kappa(s)ds, \qquad u_2 = z.$$

Suppose M^* is another cylinder which is LC-isometric to M. We have

$$\omega_{13}^* = -\kappa^*(s^*)ds^* = t(s)\omega_{13} = -t(s)\kappa(s)ds.$$

Hence

$$\kappa^*(s^*)ds^* = t(s)\kappa(s)ds. \tag{2.5}$$

This is the equation of an LC-isometry F.

(2). Cone M:

$$m(s, v) = vm(s),$$
$$m^2(s) = 1, \qquad v > 0,$$

where $m(s)$ is a curve on the unit sphere centered at the origin and parametrised by its arclength s. We have

$$m' = \alpha, \qquad \alpha' = \kappa\beta.$$

Choose a frame field over M, by

$$e_1 = \alpha, \quad e_2 = m, \quad e_3 = \alpha \times m,$$

and we have

$$dm = \omega_1 e_1 + \omega_2 e_2,$$
$$\omega_1 = v ds, \quad \omega_2 = dv,$$
$$\omega_{12} = v^{-1}\omega_1, \quad \omega_{13} = a\omega_1, \quad a = \kappa v^{-1}, \quad \omega_{23} = 0.$$

The two parameters of M are

$$u_1 = \int a\omega_1 = \int \kappa(s) ds, \quad u_2 = v.$$

Suppose M^* is another cone which is LC-isometric to M. We have the equation (2.5), too.

(3). Tangent surface M: $m(s, \bar{v}) = m(s) + \bar{v}\alpha(s), \quad \bar{v} > 0$,
where $m(s)$ is a nonplanar curve parametrised by its arclength s. We have

$$m' = \alpha, \quad \alpha' = \kappa\beta, \quad \kappa \neq 0,$$
$$\beta' = -\kappa\alpha + \tau\gamma, \quad \gamma' = -\tau\beta, \quad \tau \neq 0.$$

Choose a frame field of M, by

$$e_1 = \beta, \quad e_2 = \alpha, \quad e_3 = -\gamma$$

and we get
$$dm = (\kappa\bar{v} ds)\beta + (ds + d\bar{v})\alpha$$

or
$$dm = \omega_1 e_1 + \omega_2 e_2,$$
$$\omega_1 = \kappa(v - s) ds, \quad \omega_2 = dv,$$
$$\bar{v} = v - s, \quad v > s.$$
$$\omega_{12} = -\kappa ds, \quad \omega_{13} = a\omega_1, \quad \omega_{23} = 0,$$
$$a = -\tau\kappa^{-1}(v - s)^{-1}.$$

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The two parameters of M are

$$u_1 = -\int \kappa(s)ds, \quad u_2 = v.$$

Let M^* be another tangent surface which is LC-isometric to M. We have the equation of the LC-isometry F as

$$\tau^*(s^*)\left[\kappa^*(s^*)(v^* - s^*)\right]^{-1} ds^* = t(s)\tau(s)\left[\kappa(s)(v - s)\right]^{-1} ds.$$

3. LC-SURFACES OF THE SECOND TYPE – MOLDING SURFACES

We now consider LC-surfaces of the second type. As is well known, [**1**] shows that these surfaces are Molding surfaces (see Theorem A), but in this section we obtain some new properties of these surfaces.

Suppose Z is a cylinder and C is a plane curve on some tangent plane π of Z. The surface M is the locus described by C in space as the tangent plane π rolls about Z. Such a surface M is called a Molding surface (or M-surface).

THEOREM 2. *An LC-surface of the second type is an M-surface.*

This theorem is just Theorem A, and its proof is essentially contained in [**1**].

PROOF I: We first prove $\sigma \cong hk = 0$. According to the definition of an LC-surface of the second type, we have $K \neq 0$, $a \neq 0$, $c \neq 0$, so the two forms ω_{13} and ω_{23} are independent forms. We can choose them as fundamental forms for computation. Rewriting (1.24), in view of (1.4),

$$\begin{aligned} \theta_1 &= c^{-1}k(a\omega_1) = k'\omega_{13} \\ \theta_2 &= a^{-1}h(c\omega_2) = h'\omega_{23} \end{aligned} \tag{3.1}$$

where

$$h' = a^{-1}h, \quad k' = c^{-1}k \tag{3.2}$$

so that

$$\omega_{12} = h'\omega_{13} + k'\omega_{23}. \tag{3.3}$$

Denote

$$\sigma = hk = (ac)(h'k') = K\sigma', \tag{3.4}$$

where

$$\sigma' = h'k'. \tag{3.5}$$

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The structural equations (1.7) and (1.8) of the surface M are

$$d\omega_{12} = -*1, \quad *1 \cong \omega_{13} \wedge \omega_{23}, \tag{3.6}$$

$$d\omega_{13} = h'*1, \quad d\omega_{23} = k'*1. \tag{3.7}$$

Inserting (3.1) into (1.27), in view of (3.2) and (3.5), we have

$$\begin{aligned}(dh' - \sigma'\omega_{13}) \wedge \omega_{23} &= 0,\\ (dk' + \sigma'\omega_{23}) \wedge \omega_{13} &= 0.\end{aligned} \tag{3.8}$$

Using Cartan's Lemma from (3.8) implies

$$\begin{aligned}dh' &= \sigma'\omega_{13} + q'\omega_{23},\\ dk' &= p'\omega_{13} - \sigma'\omega_{23},\end{aligned} \tag{3.9}$$

where p' and q' are two functions.

Taking exterior derivatives of (3.3), using (3.9) and (3.7) from (3.6) we obtain

$$p' - q' = -\left(h'^2 + k'^2 + 1\right). \tag{3.10}$$

We define a function u by

$$p' + q' = 2u. \tag{3.11}$$

Solving (3.10) and (3.11) for p', q'

$$\begin{aligned}p' &= u - \left(h'^2 + k'^2 + 1\right)/2,\\ q' &= u + \left(h'^2 + k'^2 + 1\right)/2.\end{aligned} \tag{3.12}$$

Taking derivatives of (3.5), in view of (3.9), we get

$$d\sigma' = (h'p' + k'\sigma')\omega_{13} + (k'q' - h'\sigma')\omega_{23}. \tag{3.13}$$

Taking exterior derivatives of (3.9), using (3.13), we have

$$\begin{aligned}(dq' + 2h'\sigma'\omega_{13}) \wedge \omega_{23} &= 0,\\ (dp' + 2k'\sigma'\omega_{23}) \wedge \omega_{13} &= 0,\end{aligned} \tag{3.14}$$

so

$$\begin{aligned}dp' &= p''\omega_{13} - 2k'\sigma'\omega_{23},\\ dq' &= -2h'\sigma'\omega_{13} + q''\omega_{23}.\end{aligned} \tag{3.15}$$

Taking derivatives of (3.10), using (3.9) and (3.15), we get

$$p'' = -2(k'p' + 2h'\sigma'),$$
$$q'' = 2(h'q' - 2k'\sigma'). \tag{3.16}$$

Taking derivatives of (3.11), from (3.15) and (3.16), we get

$$du = \left[k'\left(-u + \frac{1}{2}\right) - \frac{1}{2}k'^3 - \frac{5}{2}h'\sigma'\right]\omega_{13} + \left[h'\left(u + \frac{1}{2}\right) + \frac{1}{2}h'^3 - \frac{5}{2}k'\sigma'\right]\omega_{23}. \tag{3.17}$$

Taking exterior derivatives of (3.17), using (3.9), (3.13) and (3.17), we get

$$\sigma' = 0 \quad \text{or} \quad \sigma = 0.$$

We suppose

$$h \neq 0, \quad k = 0. \tag{3.18}$$

From (3.9) and (3.10), we have

$$p' = 0, \quad q' = 1 + h'^2. \tag{3.19}$$

For this surface, we also have

$$\omega_{12} = h\omega_1, \quad \omega_{13} = a\omega_1, \quad \omega_{23} = c\omega_2, \tag{3.20}$$

$$dh' = \left(1 + h'^2\right)\omega_{23}, \tag{3.21}$$

$$da = h\omega_2 + \lambda\omega_1, \quad dc = \mu\omega_2 \tag{3.22}$$

where λ and μ are two functions.

PROOF II: Secondly we prove that the surface is an M-surface. For this purpose we examine its family of lines of curvature, which are defined by the equation

$$\omega_1 = 0,$$

and are denoted by $\{\Gamma_2\}$. Along any curve Γ_2, we have $\omega_{12} = h\omega_1 = 0$, so that Γ_2 is a geodesic. Letting $\omega_2 = ds$, we obtain equations for Γ_2,

$$\frac{dx}{ds} = e_2, \quad \frac{de_2}{ds} = ce_3,$$
$$\frac{de_3}{ds} = -ce_2, \quad \frac{de_1}{ds} = 0. \tag{3.23}$$

Hence Γ_2 is a plane curve with curvature c, and the plane has normal vector e_1. From (3.23),

$$dc = \mu\omega_2 \cong 0 (\text{mod } \omega_2)$$

meaning that all the curves $\{\Gamma_2\}$ are congruent to each other, because their curvatures are the same.

Since

$$\begin{aligned} de_1 &= \omega_{12}e_2 + \omega_{13}e_3 = (he_2 + ae_3)\omega_1 \\ &= a(h'e_2 + e_3)\omega_1, \end{aligned}$$

the intersection of two neighbouring planes of the curve Γ_2 is a line in the direction

$$e_1 \times de_1 = a(-e_2 + h'e_3)\omega_1.$$

By (3.21), we have

$$d(-e_2 + h'e_3) = ch'(-e_2 + h'e_3)\omega_2.$$

Hence this direction is fixed. It follows that all planes of the lines Γ_2 of curvature are the tangent planes of a cylinder Z. The generator of Z is parallel to the fixed direction. So the surface M is a Molding surface. □

We wish to explain geometrical properties of the other family of lines of curvature $\{\Gamma_1\}$, which are defined by the equation

$$\omega_2 = 0.$$

It is known that all curves Γ_1 are orthogonal trajectories of the tangent planes of Z. We have the following lemma concerning this situation.

Lemma. *Let Z be a cylinder and let its normal section be a plane curve C. Suppose the family of tangent planes of Z is $\{\Pi_s\}$, $a \leqslant s \leqslant b$, $a < b, a, b \in \mathbf{R}$. If the orthogonal trajectory is a curve Γ, then:*

(1) *Γ is a plane curve, and the plane Π which includes the curve Γ is orthogonal to the direction of the generator of Z. In other words, the plane Π is orthogonal to a fixed direction.*

(2) *Γ is a involute of intersection $Z \cap \Pi$ of Z and the plane Π, and the intersection is congruent to a curve C. So we can say that Γ is an involute of the curve C.*

Proof: We can choose a frame $Oijk$ or $Oxyz$ in E^3 such that for the cylinder Z:

$$m(s, z) = m(s) + zk, \tag{3.24}$$

where for the curve C:

$$m(s) = x(s)i + y(s)j, \tag{3.25}$$

and

$$m' = \alpha,$$

$$\alpha' = \kappa\beta, \quad \beta' = -\kappa\alpha, \tag{3.26}$$

$$\alpha k = 0, \quad \beta k = 0. \tag{3.27}$$

The family of the tangent planes of Z is $\{\Pi_s\}$, where

$$\begin{gathered} \Pi_s\colon \rho_s(u, v) = m(s) + u\alpha(s) + vk, \\ -\infty < u, \quad v < +\infty, \end{gathered} \tag{3.28}$$

where u, v are the parameters of points on the plane Π_s.

Now we can express the orthogonal trajectory Γ of the $\{\Pi_s\}$ by

$$\Gamma\colon \rho(s) = m(s) + u(s)\alpha(s) + v(s)k. \tag{3.29}$$

Taking the derivative of (3.29) using (3.26), we get

$$\rho' = (1 + u')\alpha + \kappa u\beta + v'k. \tag{3.30}$$

Since Γ is the orthogonal trajectory, its tangent line is along the normal of the plane Π_s, so

$$\rho'\alpha = 0, \quad \rho' k = 0. \tag{3.31}$$

Inserting (3.30) into (3.31), in view of (3.27), we have

$$1 + u' = 0, \quad v' = 0.$$

From the above equations

$$u = s_0 - s, \quad v = v_0 \ (= \text{ constant}). \tag{3.32}$$

Inserting (3.32) into (3.29), we get

$$\Gamma\colon \rho(s) = m(s) + (s_0 - s)\alpha(s) + v_0 k. \tag{3.33}$$

It follows that the curve Γ is on the plane $\Pi\colon z = v_0$, and it is an involute of the intersection

$$Z \cap \Pi\colon \rho(s) = m(s) + v_0 k. \tag{3.34}$$

So we obtain the conclusions (1) and (2) in the lemma. $\square$

Using Theorem 2 and the lemma, we get the following Theorem 3.

THEOREM 3. *Two families of lines of curvature on the M-surface are:*

(1) *Second family $\{\Gamma_2\}$; every curve Γ_2 is a plane curve such that pairs of curves are congruent. Its plane is parallel to a fixed direction which is the generator of Z.*

(2) *First family $\{\Gamma_1\}$; every curve Γ_1 is a plane curve such that pairs are not congruent, in general. Its plane is orthogonal to a fixed direction which is the generator of Z. Every curve Γ_1 is an involute of the curve C.*

EXAMPLE. Surfaces of revolution are special examples of Molding surfaces. In this case, the cylinder Z becomes a straight line. The first family of lines of curvature $\{\Gamma_1\}$ is a set of parallels. Γ_1 is a circle, and it is the "involute" of the "point". The second family of lines of curvature $\{\Gamma_2\}$ is a set of meridians. They are congruent to each other.

COROLLARY. *Let a cylinder Z be given by*

$$\rho(s, v) = m(s) + vk, \tag{3.35}$$

$$km = 0, \quad \alpha = m', \tag{3.36}$$

and a plane curve C on some tangent plane Π_0 of Z, and C be given by

$$\rho(t) = m(s_0) + u(t)\alpha(s_0) + v(t)k. \tag{3.37}$$

Then we have the Molding surface M:

$$\rho(s, t) = m(s) + (s_0 - s)\alpha(s) + u(t)\alpha(s_0) + v(t)k. \tag{3.38}$$

4. LC-SURFACES OF THIRD TYPE

Now we study LC-surfaces of the third type. From (3.1)–(3.3) we are given

$$\theta_1 = k'\omega_{13}, \quad \theta_2 = h'\omega_{23}, \tag{4.1}$$

$$\omega_{12} = h'\omega_{13} + k'\omega_{23}. \tag{4.2}$$

Let

$$\sigma = h'k'. \tag{4.3}$$

We denote the differentials of h' and k' by

$$\begin{aligned} dh' &= \alpha\omega_{13} + \beta\omega_{23} \\ dk' &= \gamma\omega_{13} + \delta\omega_{23} \end{aligned} \tag{4.4}$$

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so that

$$\begin{aligned} d\theta_1 - 2\theta_1 \wedge \theta_2 &= -(\delta + \sigma)*1, \\ d\theta_2 - 2\theta_1 \wedge \theta_2 &= (\alpha - \sigma)*1, \\ *1 &= \omega_{13} \wedge \omega_{23}. \end{aligned} \tag{4.5}$$

From (1.27), using (4.5), we have

$$\alpha + T\delta = (1 - T)\sigma. \tag{4.6}$$

From the Gaussian equation (3.6), using (4.4), it follows that

$$\beta - \gamma = 2\tau \tag{4.7}$$

where

$$2\tau = h'^2 + k'^2 + 1. \tag{4.8}$$

Taking exterior derivatives of (4), we have

$$\begin{aligned} d\alpha \wedge \omega_{13} + d\beta \wedge \omega_{23} + (h'\alpha + k'\beta)*1 = 0, \\ d\gamma \wedge \omega_{13} + d\delta \wedge \omega_{23} + (h'\gamma + k'\delta)*1 = 0, \end{aligned}$$

so there exist functions A, B, $\dots$, F such that

$$\begin{aligned} d\alpha &= A\omega_{13} + (B + h'\alpha)\omega_{23} \\ d\beta &= (B - k'\beta)\omega_{13} + C\omega_{23} \\ d\gamma &= D\omega_{13} + (E + h'\gamma)\omega_{23} \\ d\delta &= (E - k'\delta)\omega_{13} + F\omega_{23}. \end{aligned} \tag{4.9}$$

Taking derivatives of (4.3) and (4.8) and using (4.4), we obtain

$$\begin{aligned} d\sigma &= \sigma_1\omega_{13} + \sigma_2\omega_{23}, \\ \sigma_1 &= h'\gamma + k'\alpha, \quad \sigma_2 = h'\delta + k'\beta, \end{aligned} \tag{4.10}$$

$$\begin{aligned} d\tau &= \tau_1\omega_{13} + \tau_2\omega_{23}, \\ \tau_1 &= h'\alpha + k'\gamma, \quad \tau_2 = h'\beta + k'\delta. \end{aligned} \tag{4.11}$$

Taking derivatives of (4.6), we get

$$(\sigma + \delta)dT + T(d\sigma + d\delta) = d\sigma - d\alpha. \tag{4.12}$$

Using (4.6), (4.10) and (4.11) from (4.12) we have

$$(4.13)\qquad \begin{aligned}(\sigma+\delta)^2 dT + [(\sigma-\alpha)(E-k'\delta+\sigma_1)+(\sigma+\delta)(A-\sigma_1)]\,\omega_{13}\\ +[(\sigma-\alpha)(F+\sigma_2)+(\sigma+\delta)(B+k'\alpha-\sigma_2)]\,\omega_{23}=0.\end{aligned}$$

Inserting (4.6) into (1.26), we have

$$(4.14)\qquad (\sigma+\delta)^2 dT - 2(\sigma-\alpha)(\alpha+\delta)k'\omega_{13} + 2(\sigma+\delta)(\alpha+\delta)h'\omega_{23} = 0.$$

Comparing (4.13) and (4.14), we get

$$\begin{aligned}(\sigma+\delta)A+(\sigma-\alpha)E-(\alpha-\sigma)(2\alpha+\delta)k'-(\alpha+\delta)\sigma_1=0,\\ (\sigma+\delta)B+(\sigma-\alpha)F-(\delta+\sigma)(\alpha+2\delta)h'-(\alpha+\delta)\sigma_2=0.\end{aligned}$$

Using (4.10), the above equations become

$$(4.15)\qquad \begin{aligned}(\sigma+\delta)A+(\sigma-\alpha)E&=\gamma(\alpha+\delta)h'+[\alpha(3\alpha+2\delta)-\sigma(2\alpha+\delta)]k',\\ (\sigma+\delta)B+(\sigma-\alpha)F&=[\delta(2\alpha+3\delta)+\sigma(\alpha+2\delta)]h'+\beta(\alpha+\delta)k'.\end{aligned}$$

Taking derivatives of (4.7) using (4.9) and (4.11), we have

$$(4.16)\qquad \begin{aligned}B-D&=2\alpha h'+(\beta+2\gamma)k',\\ C-E&=(2\beta+\gamma)h'+2\delta k'.\end{aligned}$$

From the above discussion we obtain the following theorem.

THEOREM 4. *Let M be a surface with non-zero Gaussian curvature. The necessary and sufficient condition for it to be an LC-surface of the third type is that the first and second derivatives of h' and k', α, β, γ, δ and $A, B, \ldots, F$ satisfy (4.7), (4.15) and (4.16).*

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