J. Aust. Math. Soc. **82** (2007), 429–439

EXTREMAL CHARACTERIZATIONS OF REFLEXIVE SPACES

XIANFU WANG

(Received 20 June 2005; revised 20 September 2005)

Communicated by A. Rubinov

Abstract

Assume that a Banach space has a Fréchet differentiable and locally uniformly convex norm. We show that the reflexive property of the Banach space is not only sufficient, but also a necessary condition for the fulfillment of the proximal extremal principle in nonsmooth analysis.

2000 *Mathematics subject classification*: primary 49J52.
Keywords and phrases: subgradients, proximal extremal principle, proximal fuzzy sum rule, reflexive space.

1. Introduction

Throughout this note, X is a Banach space and X^* is its topological dual. We denote by $\mathbb{B}$ and $\mathbb{B}^*$ the closed unit balls in the space X and X^*, respectively, and let $\mathbb{B}_\epsilon(x)$ denote the closed ball in X, centered at x, with radius $\epsilon > 0$. By $d_\Omega(x)$ we mean the distance from x to a non-empty closed subset $\Omega \subset X$.

In [5], Clarke *et al.* developed proximal subdifferential calculus and its applications in Hilbert spaces. In particular, the fuzzy sum rule and the chain rule hold. In [2, 3, 8], the existence of the proximal subdifferential and the proximal normal formula for Clarke's normal cone

$$N(x;\Omega) = \overline{co}^{w^*}\left\{x^* \in X^* \;\middle|\; \begin{array}{l} x^* = \lim x_n^*,\, x_n \to x,\, x_n^* \text{ is a proximal} \\ \text{normal functional to } \Omega \text{ at } x_n \end{array}\right\},$$

in reflexive Banach spaces with Kadec and Fréchet differentiable norms were given. However, it is not clear whether the proximal fuzzy sum rule and proximal fuzzy chain rule hold on general reflexive spaces.

This research was supported by NSERC.

Inspired by the work of Mordukhovich and Shao [13] and employing similar methods, in this note we characterize the reflexivity of Banach spaces using extremal principles and fuzzy sum rules in terms of proximal normals/subdifferentials. Assume that X has a locally uniformly convex and Fréchet differentiable norm. We show that the proximal extremal principle and fuzzy sum rule hold whenever X is reflexive, and moreover, reflexivity of a Banach space X is equivalent to the fulfillment of each of these principles in X. Consequently, the reflexive space framework gives necessary and sufficient conditions for the usage of proximal-type normal/subdifferential constructions.

2. Constructs in nonsmooth analysis

Let us review some basic concepts of nonsmooth analysis used in the sequel. A linear functional $x^* \in X^*$ is said to be

(a) *proximal normal functional* to Ω at $x \in \Omega$ if there is $u \notin \Omega$ such that

$$\|u - x\| = d_\Omega(u) \quad \text{and} \quad \langle x^*, u - x\rangle = \|x^*\|\,\|u - x\|.$$

(b) *Fréchet normal* to Ω at x if

$$\limsup_{u \to_\Omega x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le 0,$$

where $u \to_\Omega x$ means that $u \to x$ and $u \in \Omega$.

The collections of proximal normals and Fréchet normals are denoted, respectively, by $N_P(x; \Omega)$, $N_F(x; \Omega)$, and they are known as the *proximal normal cone* and the *Fréchet normal cone* to Ω at x. For an extended real-valued function $f : X \to (-\infty, \infty]$, the formulae

$$\partial_F f(x) := \{x^* \in X^* : (x^*, -1) \in N_F((x, f(x)); \operatorname{epi} f)\},$$
$$\partial_p f(x) := \{x^* \in X^* : (x^*, -1) \in N_P((x, f(x)); \operatorname{epi} f)\},$$

define the *Fréchet subdifferential*, and *proximal subdifferential* of f at x, respectively, provided that f is finite at x. Otherwise, these subdifferentials are assumed to be empty. If X is reflexive and the norm of X is Fréchet differentiable, then

$$N_P(x; \Omega) \subset N_F(x; \Omega) \quad \text{for every } x \in \Omega,$$

by [3, Corollary 3.1]. It is easy to check

(2.1) $$0 \in \partial_p f(x) \quad \text{for any local minimizer } x \in \operatorname{dom} f.$$

https://doi.org/10.1017/S144678870003620X Published online by Cambridge University Press

Let Ω_1, Ω_2 be non-empty closed subsets of a Banach space X and let $\bar{x} \in \Omega_1 \cap \Omega_2$. According to Mordukhovich and Shao [13, 9], $\bar{x}$ is a *locally extremal point* of the set system $\{\Omega_1, \Omega_2\}$ if there are sequences $\{a_{ik}\} \subset X$ and a neighborhood U of x such that $a_{ik} \to 0$ as $k \to \infty$ for $i = 1, 2$ and $(\Omega_1 - a_{1k}) \cap (\Omega_2 - a_{2k}) \cap U = \emptyset$ for all $k = 1, 2, \ldots$.

We say that the *F-extremal principle* holds in X if for any extremal system of closed sets $\Omega_i \subset X$, $i = 1, 2$, and any local extremal point $\bar{x}$ of $\{\Omega_1, \Omega_2\}$ and $\epsilon > 0$ there exist $x_i \in \Omega_i \cap \mathbb{B}_\epsilon(\bar{x})$ and $x_i^* \in X^*$ such that

$$\begin{aligned} &x_i^* \in N_F(x_i; \Omega_i) \quad \text{for } i = 1, 2, \\ &\|x_1^* + x_2^*\| < \epsilon, \quad 1 + \epsilon \geq \|x_i^*\| \geq 1/2 - \epsilon. \end{aligned} \tag{2.2}$$

This principle is the driving force of the variational theory and its applications developed in [12].

We say that the *F-fuzzy sum rule* holds if for any $\delta > 0$ and $\epsilon > 0$, any functions $\phi_1, \phi_2 : X \to (-\infty, +\infty]$, and $\bar{x} \in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$ such that ϕ_1 is lower semicontinuous, and ϕ_2 is Lipschitz continuous around $\bar{x}$, one has

$$\begin{aligned} &\partial_F(\phi_1 + \phi_2)(\bar{x}) \\ &\quad \subset \bigcup \left\{ \sum_{i=1}^{2} \partial_F \phi_i(x_i) \;\middle|\; \begin{array}{l} x_i \in \mathbb{B}_\delta(\bar{x}),\ |\phi_i(x_i) - \phi_i(\bar{x})| \leq \delta, \\ i = 1, 2 \end{array} \right\} + \epsilon \mathbb{B}^*. \end{aligned} \tag{2.3}$$

If $N_F(x; \Omega)$ is replaced by $N_P(x; \Omega)$ in (2.2), and ∂_F by ∂_p in (2.3), we call them the *proximal extremal principle*, and *proximal fuzzy sum rule*, respectively.

Let us summarize the well-known subdifferential characterizations of Asplund spaces.

THEOREM 2.1. *Let X be a Banach space. Then the following are equivalent.*

(a) *X is Asplund.*
(b) *The F-extremal principle holds in X.*
(c) *The F-fuzzy sum rule holds in X.*
(d) *For each non-empty closed subset Ω of X, the set of points $x \in \operatorname{bd}\Omega$ such that $N_F(x; \Omega) \neq \{0\}$ is dense in the boundary* $\operatorname{bd}\Omega$ *of* Ω.

While (a) if and only if (b) is due to Mordukhovich and Shao [13], (a) if and only if (c) is due to Fabian [6], and (a) if and only if (d) by Fabian and Mordukhovich [7], in which one may find a complete subdifferential characterizations of Asplund spaces. As such, we now know that the Asplund space framework gives necessary and sufficient conditions for the usage of Fréchet-type normal/subdifferential constructions. All of these have been documented in Mordukhovich [12].

We believe that it is interesting to investigate what happens to Theorem 2.1 if one uses proximal subdifferentials and proximal normal cones instead.

3. Nonsmooth characterizations of reflexive spaces

We shall work in Banach spaces with a locally uniformly convex and Fréchet differentiable norm. Recall that a norm $\|\cdot\|$ on X is said to be *locally uniformly convex* if, for each $x \in X$, one has $\|x_n - x\| \to 0$ whenever $\|x_n\| \to \|x\|$ and $\|x + x_n\|/2 \to \|x\|$, as $n \to \infty$. It follows from the parallelogram law that the norm of a Hilbert space is locally uniformly convex. Every reflexive Banach space can be given an equivalent Fréchet differentiable and locally uniformly convex (hence Kadec) norm. The following Lau's nearest point theorem [1, 10] will be needed on our way.

LEMMA 3.1. *If X is a reflexive Banach space with a Kadec norm, and Ω is a closed subset of X, then the set of those points that have a nearest point in Ω is dense in X.*

Now we can state and prove our main equivalence result.

THEOREM 3.2. *Let X be a Banach space with a locally uniformly convex and Fréchet differentiable norm. Then the following are equivalent.*

(i) *X is a reflexive space.*
(ii) *The proximal extremal principle holds in X.*
(iii) *For each non-empty closed subset Ω of X, the set of points $x \in \operatorname{bd}\Omega$ such that $N_P(x;\Omega) \neq \{0\}$ is dense in the boundary $\operatorname{bd}\Omega$ of Ω.*
(iv) *The proximal fuzzy sum rule holds in X.*

PROOF. We prove the theorem by following the scheme (i) implies (ii) implies (iii) implies (i), then (ii) if and only if (iv).

(i) implies (ii): First, when X is a Banach space with a Fréchet differentiable norm, the F-extremal principle holds. Although this was done in Ioffe [8, Lemma 2], the full version of the F-extremal principle in Asplund spaces was established by Mordukhovich and Shao [13]. For further details and ϵ-Fréchet normal versions, see [9, 12].

That is, if x is an extremal point of the set system $\{\Omega_1, \Omega_2\}$ in X with a Fréchet differentiable norm, then for any $\epsilon > 0$ there exist x_i, x_i^*, $i = 1, 2$, such that

$$\begin{gathered} x_i \in \Omega_i, \quad \|x_i - x\| \le \epsilon, \quad 1 + \epsilon \ge \|x_i^*\| \ge 1/2 - \epsilon, \quad \text{and} \\ x_i^* \in N_F(x_i;\Omega_i) \quad \text{and} \quad \|x_1^* + x_2^*\| < \epsilon. \end{gathered} \tag{3.1}$$

Next we observe that in any reflexive Banach space with a Fréchet differentiable and locally uniformly convex norm one can always approximate Fréchet normals in the norm topology by proximal ones at base points near the point of interest.

PROPOSITION 3.3. *Suppose that X is a reflexive Banach space with a norm which is Fréchet differentiable and locally uniformly convex. Suppose further that Ω is a closed subset of X and $u \in \Omega$. Then for every $\epsilon > 0$, $\delta > 0$, one has*

$$N_F(u;\Omega) \subset \bigcup \left\{ N_P(v;\Omega) : v \in \mathbb{B}_\delta(u) \cap \Omega \right\} + \epsilon \mathbb{B}^*.$$

This will be proved if we show that given $u^* \in N_F(u;\Omega)$, there are sequences $u_n \in \Omega$ and u_n^* such that $u_n \to u$, $u_n^* \in N_P(u_n;\Omega)$ and $u_n^* \to u^*$ (in norm). Let $u^* \in N_F(u;\Omega)$. We may, of course, suppose that $\|u^*\| = 1$. By the definition

$$\langle u^*, w-u\rangle \le r(\|w-u\|)\|w-u\| \quad \text{for all } w \in \Omega, \tag{3.2}$$

where $r(t) \to 0$ at $t \to 0$.

Take $h \in X$, $\|h\| = 1$ such that

$$\langle u^*, h\rangle = \|u^*\| \cdot \|h\| = 1, \tag{3.3}$$

which is possible since X is reflexive. Then, as follows from (3.2), $u + th \notin \Omega$ for sufficiently small $t > 0$.

Since X is reflexive, and the given norm on X is locally uniformly convex, by Lemma 3.1, for sufficiently small $t > 0$ there is h_t such that $\|h_t - h\| < t$ and $u + th_t \notin \Omega$ has a nearest point in Ω; that is to say, $u_t \in \Omega$ such that

$$\|u + th_t - u_t\| = d_\Omega(u + th_t).$$

We set $u_t = u + tv_t$. Then $u_t \to u$ as $t \downarrow 0$, and $t\|v_t - h_t\| = d_\Omega(u+th_t) \le t\|h_t\|$, so that $\|v_t - h_t\| \le \|h_t\|$ and $\|v_t\| \le 2\|h_t\|$. As $u_t \in \Omega$, by (3.2),

$$\langle u^*, v_t\rangle \le r(t\|v_t\|)\|v_t\| \le r(t\|v_t\|)2(1+t),$$

we get

$$\limsup_{t\downarrow 0} \langle u^*, v_t\rangle \le 0. \tag{3.4}$$

Set further $w_t := (h - v_t)/(1+2t)$. Then

$$\|w_t\| \le \frac{\|h - h_t\| + \|h_t - v_t\|}{1+2t} \le \frac{t + \|h_t\|}{1+2t} \le \frac{t+1+t}{1+2t} \le 1,$$

and $|\langle u^*, w_t\rangle| \le \|u^*\|\|w_t\| \le 1$. Using (3.4), we have

$$\begin{aligned} 1 &\ge \limsup_{t\downarrow 0}\langle u^*, w_t\rangle \ge \liminf_{t\downarrow 0}\langle u^*, w_t\rangle = \liminf_{t\downarrow 0} \frac{\langle u^*, h\rangle - \langle u^*, v_t\rangle}{1+2t} \\ &\ge \langle u^*, h\rangle - \limsup_{t\downarrow 0}\langle u^*, v_t\rangle \ge \langle u^*, h\rangle = 1, \end{aligned}$$

therefore

$$\lim_{t\downarrow 0}\langle u^*, w_t\rangle = \langle u^*, h\rangle = 1. \tag{3.5}$$

It follows that

$$1 \geq \frac{\|w_t + h\|}{2} \geq \left\langle u^*, \frac{w_t + h}{2}\right\rangle \to 1, \quad \text{and}$$
$$1 \geq \|w_t\| \geq \langle u^*, w_t\rangle \to 1 = \|h\|.$$

Since the norm is locally uniformly convex, w_t norm converges to h, hence $\|v_t\| \to 0$.

For any $t > 0$ there is a unique $u_t^* \in X^*$ such that

$$\|u_t^*\| = 1, \quad \langle u_t^*, u + th_t - u_t\rangle = \|u + th_t - u_t\| = d_\Omega(u + th_t). \tag{3.6}$$

Then $u_t^* \in N_P(u_t; \Omega)$ and $\langle u_t^*, h_t - v_t\rangle = \|h_t - v_t\|$. Now

$$|\langle u_t^*, h - h_t\rangle| \leq \|h - h_t\| \to 0, \quad \text{and} \quad |\langle u_t^*, v_t\rangle| \leq \|v_t\| \to 0,$$

when $t \downarrow 0$. This gives

$$\begin{aligned}\lim_{t\downarrow 0}\langle u_t^*, h\rangle &= \lim_{t\downarrow 0}[\langle u_t^*, h - h_t\rangle + \langle u_t^*, h_t - v_t\rangle + \langle u_t^*, v_t\rangle] \\ &= \lim_{t\downarrow 0}\langle u_t^*, h_t - v_t\rangle = \lim_{t\downarrow 0}\|h_t - v_t\| = \|h\| = 1.\end{aligned}$$

Since the norm in X is Fréchet differentiable at h, by (3.3), $u^* \in \mathbb{B}^*$ is *weak* strongly exposed* by h. We conclude that u_t^* must converge in norm to u^*. See [14, Proposition 5.11].

Now combining the F-extremal principle and Proposition 3.3, by (3.1) there exist y_i, y_i^*, $i = 1, 2$, such that $y_i^* \in N_P(y_i; \Omega_i)$ and $\|y_i - x_i\| < \epsilon$, $\|y_i^* - x_i^*\| < \epsilon$. Then $\|y_i - x\| \leq 2\epsilon$, $1 + 2\epsilon \geq \|y_i^*\| \geq 1/2 - 2\epsilon$, and $\|y_1^* + y_2^*\| \leq 3\epsilon$. This is the extremal principle in proximal normal functionals.

(ii) implies (iii): Assume the extremal principle holds in X. We prove the following: Let $\Omega \subset X$ be a non-empty closed subset of X. Then the set of points

$$x \in \Omega \quad \text{and} \quad N_P(x; \Omega) \neq \{0\}, \tag{3.7}$$

is dense in the boundary of Ω.

To this end, we follow the proof of [13, Corollary 3.4]. Indeed, if $\bar{x}$ is a boundary point of the set Ω, then it is a locally extremal point of the system $\{\Omega_1, \Omega_2\}$ in X, where $\Omega_1 = \Omega$ and $\Omega_2 = \{\bar{x}\}$. Applying (ii) with $0 < \epsilon < 1/2$ we find $x_1 \in \Omega$ and $x_1^*, x_2^* \in X^*$ such that $x_1 \in \mathbb{B}_\epsilon(\bar{x})$, $x_1^* \in N_P(x_1; \Omega)$, and $1 + \epsilon \geq \|x_i^*\| \geq 1/2 - \epsilon$, $\|x_1^* + x_2^*\| < \epsilon$. This implies that $\|x_1^*\| > 0$ and the cone $N_P(\cdot; \Omega)$ is nontrival at x_1, which is a boundary point of Ω within ϵ of $\bar{x}$. Therefore, the set (3.7) is norm dense in the boundary of Ω.

(iii) implies (i): If X is not reflexive, by James' theorem there exists $x^* \in X^*$ with $1 = \|x^*\| > \langle x^*, y\rangle$ for each $y \in \mathbb{B}$. Let $\Omega := \mathbb{B} \cap \{x \in X : \langle x^*, x\rangle \leq 0\}$, and $U := (1/3)\mathbb{B} \cap \{x \in X : \langle x^*, x\rangle > 0\}$. Then $d_\Omega(x) = \langle x^*, x\rangle$ for each $x \in U$.

To see this, choose $y_n \in \mathbb{B}$ such that $\langle x^*, y_n\rangle \to 1$ so that we may assume $\langle x^*, y_n\rangle > 1/2$ for all n. If $x \in U$, set

$$z_n = x - \frac{\langle x^*, x\rangle}{\langle x^*, y_n\rangle} y_n.$$

Then $\|z_n\| \leq 1/3 + 2|\langle x^*, x\rangle|\|y_n\| \leq 1/3 + 2/3 \leq 1$, and $\langle x^*, z_n\rangle = 0$. Therefore, $z_n \in \Omega$ and

$$d_\Omega(x) \leq \liminf_{n\to\infty} \|x - z_n\| \leq \liminf_{n\to\infty} \left\| \frac{\langle x^*, x\rangle}{\langle x^*, y_n\rangle} y_n \right\| \leq \langle x^*, x\rangle,$$

if $\langle x^*, x\rangle > 0$. On the other hand, if $\langle x^*, y\rangle \leq 0$ for $y \in \Omega$, this gives

$$d_\Omega(x) = \inf_{y\in\Omega} \|x - y\| \geq \inf_{y\in\Omega} \langle x^*, x - y\rangle \geq \langle x^*, x\rangle.$$

If we can show that for each $y \in (1/3)\mathbb{B} \cap \operatorname{bd}\Omega$, $N_P(y;\Omega) = \{0\}$, this will be a contradiction to (iii). Suppose not, that is, $N_P(y;\Omega) \neq \{0\}$ for some $y \in (1/3)\mathbb{B} \cap \operatorname{bd}\Omega$. Then there exists a point $x \in U$ having $y \in \Omega$ as a nearest point. This means $\|x - y\| = d_\Omega(x) = \langle x^*, x\rangle = \langle x^*, x - y\rangle$, since $\langle x^*, y\rangle = 0$. This contradicts the fact that x^* does not attain its norm.

(ii) implies (iv): As shown above, (ii) implies that X is reflexive. We need the following simple result, which generalizes Proposition 4.5 [5, page 138] (thanks go to a referee for pointing out that this also follows from Loewen [11, Theorem 5.5]).

PROPOSITION 3.4. *Assume that a Banach space X is reflexive and has a locally uniformly convex and Fréchet differentiable norm, and that $f : X \to (-\infty, +\infty]$ is lower semicontinuous. If $x^* \in \partial_F f(x)$, then for any $\epsilon > 0$ there exist $y \in \mathbb{B}_\epsilon(x)$ and $y^* \in \partial_p f(y)$ such that $|f(y) - f(x)| < \epsilon$ and $\|x^* - y^*\| < \epsilon$.*

Indeed, if $x^* \in \partial_F f(x)$, then $(x^*, -1) \in N_F((x, f(x)); \operatorname{epi} f)$. Note that $\operatorname{epi} f$ is a closed subset of $X \times \mathbb{R}$, whose norm we take to be $\|(x, r)\| := \sqrt{\|x\|^2 + r^2}$. Clearly, this canonical norm on $X \times \mathbb{R}$ is locally uniformly convex and Fréchet differentiable whenever the norm on X is. By Proposition 3.3, there exists $(y^*, \beta) \in N_P((y, r); \operatorname{epi} f)$ such that $(y, r) \in \operatorname{epi} f$, $\|y - x\| < \epsilon$, $|r - f(x)| < \epsilon$ and

$$\|x^* - y^*\| + |-1 - \beta| < \nu.$$

https://doi.org/10.1017/S144678870003620X Published online by Cambridge University Press

For $\nu > 0$ sufficiently small, we have $|\beta| > 1/2$, and this forces $r = f(y)$. Then $(y^*, \beta) \in N_P((y, f(y)); \operatorname{epi} f)$ gives $-y^*/\beta \in \partial_p f(y)$, and

$$\left\| \frac{-y^*}{\beta} - x^* \right\| = \left\| \frac{-y^* + x^*}{\beta} - x^* \left(1 + \frac{1}{\beta}\right) \right\|$$
$$\leq \frac{\| - y^* + x^* \|}{|\beta|} + \|x^*\| \frac{|1+\beta|}{|\beta|} \leq 2\nu + 2\nu \|x^*\| < \epsilon,$$

if ν is sufficiently small.

Whenever the proximal extremal principle holds in X, of course the F-extremal principle holds. The latter implies that the F-fuzzy calculus holds in X, by Theorem 2.1 or Ioffe [8, Lemma 2]. That is, if f_1, f_2 are lower semicontinuous near x and that one of them is Lipschitz continuous at x, then for any $\epsilon > 0$ one has

$$\partial_F (f_1 + f_2)(x) \subset \bigcup \left\{ \sum_{i=1}^{2} \partial_F f_i(x_i) + \epsilon \mathbb{B}^* \;\middle|\; \begin{array}{l} x_i \in \mathbb{B}_\epsilon(x), |f_i(x_i) - f_i(x)| < \epsilon, \\ i = 1, 2 \end{array} \right\}.$$

Now let $x^* \in \partial_p (f_1 + f_2)(x)$. Since $\partial_p \subset \partial_F$, we have $x^* \in \partial_F (f_1 + f_2)(x)$, so there exist $x_i^* \in \partial_F f_i(x_i)$ such that $x_i \in \mathbb{B}_\epsilon(x)$, $|f_i(x_i) - f_i(x)| < \epsilon$ and $x^* \in x_1^* + x_2^* + \epsilon \mathbb{B}^*$. By Proposition 3.4, choose $y_i^* \in \partial_p f_i(y_i)$ such that $\|y_i^* - x_i^*\| < \epsilon/2$ and $y_i \in \mathbb{B}_{\epsilon/2}(x_i)$, $|f_i(y_i) - f_i(x_i)| < \epsilon/2$. This implies that $x^* \in y_1^* + y_2^* + 2\epsilon \mathbb{B}^*$. Therefore

$$\partial_p (f_1 + f_2)(x) \subset \bigcup \left\{ \sum_{i=1}^{2} \partial_p f_i(y_i) + 2\epsilon \mathbb{B}^* \;\middle|\; \begin{array}{l} y_i \in \mathbb{B}_{2\epsilon}(x), |f_i(y_i) - f_i(x)| < 2\epsilon, \\ i = 1, 2 \end{array} \right\},$$

which is the proximal fuzzy sum rule in X.

(iv) implies (ii): Assume that the proximal fuzzy sum rule holds in X. Let Ω_1 and Ω_2 be two closed sets in X, which form an extremal system, and let $\bar{x} \in \Omega_1 \cap \Omega_2$ be a local extremal point of $\{\Omega_1, \Omega_2\}$. We need to prove that for any $\epsilon > 0$ there exist $x_i \in \Omega_i \cap \mathbb{B}_\epsilon(\bar{x})$ and $x_i^* \in N_P(x_i; \Omega_i)$, $i = 1, 2$ such that

$$\|x_1^* + x_2^*\| \leq \epsilon, \quad \text{and} \quad 1 + \epsilon > \|x_i^*\| \geq 1/2 - \epsilon. \tag{3.8}$$

To proceed, we use ideas from [13, Lemma 4.1]. According to the definition of locally extremal points, for a given $\epsilon > 0$, we can choose $a \in X$ such that $\|a\| < \epsilon^2/16$ and $(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset$ for some neighborhood U of $\bar{x}$. For simplicity we take $U = X$. Thus considering the function $f(u, v) := \|u - v + a\|/2$, we conclude that $f(u, v) > 0$ for any $u \in \Omega_1$ and $v \in \Omega_2$ while $f(\bar{x}, \bar{x}) < \epsilon^2/16$.

Now apply Ekeland's variational principle to the function f on the complete metric space $E := \Omega_1 \times \Omega_2$ whose metric is induced by canonical norm

$$\|(u, v)\| := \sqrt{\|u\|^2 + \|v\|^2} \quad \text{on } X^2.$$

With $\epsilon := \varepsilon^2/16$ and $\lambda := \epsilon/4$, we find points $(\tilde{x}_1, \tilde{x}_2) \in E$ such that $\|\tilde{x}_i - \bar{x}\| \le \lambda$, $i = 1, 2$, and $f(\tilde{x}_1, \tilde{x}_2) \le f(u, v) + (\epsilon/4)\|(u - \tilde{x}_1, v - \tilde{x}_2)\|$, for all $(u, v) \in E$. This means that the function

$$\phi(u, v) := f(u, v) + \frac{\epsilon}{4}\sqrt{\|u - \tilde{x}_1\|^2 + \|v - \tilde{x}_2\|^2} + \delta((u, v), \Omega_1 \times \Omega_2),$$

on X^2 attains its unconditional local minimum at $(\tilde{x}_1, \tilde{x}_2)$. Here $\delta(\cdot, \Omega_1 \times \Omega_2)$ is the indicator function, that is, 0 on $\Omega_1 \times \Omega_2$ and $+\infty$ otherwise. If so, then

$$\phi(u, \tilde{x}_2) := \frac{\|u - \tilde{x}_2 + a\|}{2} + \frac{\epsilon}{4}\|u - \tilde{x}_1\| + \delta(u, \Omega_1) \quad \text{attains a local min at } \tilde{x}_1,$$
$$\phi(\tilde{x}_1, v) := \frac{\|\tilde{x}_1 - v + a\|}{2} + \frac{\epsilon}{4}\|v - \tilde{x}_2\| + \delta(v, \Omega_2) \quad \text{attains a local min at } \tilde{x}_2.$$

Recall that $\|\tilde{x}_1 - \tilde{x}_2 + a\| \ne 0$ and the norm $\|\cdot\|$ on X is Fréchet differentiable. Using (2.1) and (iv), that is, the proximal fuzzy sum rule, for every $0 < \nu < \epsilon/2$ we can find $(x_1, x_2) \in \Omega_1 \times \Omega_2$, $(y_1, y_2) \in X^2$ with $\|x_i - \tilde{x}_i\| < \nu$ and $\|y_i - \tilde{x}_i\| < \nu$, such that $y_1 - \tilde{x}_2 + a \ne 0$, $\tilde{x}_1 - y_2 + a \ne 0$, and

$$0 \in \frac{\nabla_1\|\cdot\|(y_1 - \tilde{x}_2 + a)}{2} + \frac{\epsilon}{4}\mathbb{B}^* + N_P(x_1; \Omega_1) + \frac{\epsilon}{8}\mathbb{B}^*, \tag{3.9}$$

$$0 \in \frac{\nabla_2\|\cdot\|(\tilde{x}_1 - y_2 + a)}{2} + \frac{\epsilon}{4}\mathbb{B}^* + N_P(x_2; \Omega_2) + \frac{\epsilon}{8}\mathbb{B}^*. \tag{3.10}$$

Here, $\nabla_1\|\cdot\|(u - v + a)$ denotes the Fréchet derivative of the norm with respect to u, and $\nabla_2\|\cdot\|$ is similarly defined. Since the norm on X is Fréchet differentiable, $\nabla\|\cdot\| : X \to X^*$ is norm to norm continuous, we may also require $\nu > 0$ sufficiently small such that

$$\left\|\frac{\nabla_1\|\cdot\|(y_1 - \tilde{x}_2 + a)}{2} - \frac{\nabla_1\|\cdot\|(\tilde{x}_1 - \tilde{x}_2 + a)}{2}\right\| < \frac{\epsilon}{8}, \tag{3.11}$$

$$\left\|\frac{\nabla_2\|\cdot\|(\tilde{x}_1 - y_2 + a)}{2} - \frac{\nabla_2\|\cdot\|(\tilde{x}_1 - \tilde{x}_2 + a)}{2}\right\| < \frac{\epsilon}{8}. \tag{3.12}$$

On the other hand, $\nabla_1\|\cdot\|(\tilde{x}_1 - \tilde{x}_2 + a)/2 = -\nabla_2\|\cdot\|(\tilde{x}_1 - \tilde{x}_2 + a)/2$, which by (3.11) and (3.12) implies

$$\left\|\frac{\nabla_1\|\cdot\|(y_1 - \tilde{x}_2 + a)}{2} + \frac{\nabla_2\|\cdot\|(\tilde{x}_1 - y_2 + a)}{2}\right\| < \frac{\epsilon}{4}.$$

It follows from (3.9) and (3.10), that there exists $x_i^* \in N_P(x_i; \Omega_i)$ for $i = 1, 2$ such that

$$x_1^* = -\frac{\nabla_1\|\cdot\|(y_1 - \tilde{x}_2 + a)}{2} + \frac{3\epsilon}{8}b_1^*, \quad x_2^* = -\frac{\nabla_2\|\cdot\|(\tilde{x}_1 - y_2 + a)}{2} + \frac{3\epsilon}{8}b_2^*,$$

for some $b_i^* \in \mathbb{B}^*$. Moreover, $\|x_1^* + x_2^*\| < \epsilon/4 + 6\epsilon/8 = \epsilon$, with

$$\|x_i - \bar{x}\| \leq \|x_i - \tilde{x}_i\| + \|\tilde{x}_i - \bar{x}\| < \nu + \frac{\epsilon}{4} < \epsilon.$$

This implies that x_1^*, x_2^* satisfy (3.8). Hence the proximal extremal principle holds in X. This completes the proof of the theorem. □

REMARK. Complete subdifferential characterizations of Asplund spaces are given by Fabian and Mordukhovich [7]. Similarly, it is possible to enlarge the list of subdifferential characterizations of reflexive spaces in Theorem 3.2. We illustrate this with one example.

A set $\Omega \subset X$ is *sequentially normally compact* at $\bar{x} \in \Omega$ if for any sequence $((x_n, x_n^*)) \in X \times X^*, n \geq 1$, satisfying

$$x_n^* \in N_P(x_n; \Omega), \quad x_n \to \bar{x}, \quad \text{and} \quad x_n^* \xrightarrow{\text{weak}^*} 0,$$

one has $\|x_n^*\| \to 0$ as $n \to \infty$. Define the *sequential limits*

$$N(\bar{x}; \Omega) := \limsup_{x \to \bar{x}} N_P(x; \Omega).$$

The following statement may be added to the list of Theorem 3.2.

(v) For every locally extremal point $\bar{x} \in \Omega_1 \cap \Omega_2 \subset X$ of the system of closed sets $\{\Omega_1, \Omega_2\}$, one of which is sequentially normally compact at $\bar{x}$, there exists $x^* \neq 0$ such that $x^* \in N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2))$.

REMARK. When X is a Fréchet smooth Banach space, Borwein and Zhu have fully developed the Fréchet subdifferential calculus and applications [4]. By Theorem 3.2, many of the results there can be written in terms of proximal subdifferentials provided the Banach space X is reflexive and has a locally uniformly convex and Fréchet differentiable norm. For a full Fréchet subdifferential calculus in Asplund spaces, see Mordukhovich [12, Chapters 2 and 3].

Acknowledgements

I would like to thank the referees for their suggestions, and for pointing out an inaccuracy of my original proof of Proposition 3.3.

References

[1] J. M. Borwein and S. Fitzpatrick, 'Existence of nearest points in Banach spaces', *Canad. J. Math.* **4** (1989), 702–720.

[2] J. M. Borwein and J. Giles, 'The proximal normal formula in Banach space', *Trans. Amer. Math. Soc.* **302** (1987), 371–381.

[3] J. M. Borwein and H. M. Strojwas, 'Proximal analysis and boundaries of closed sets in Banach space, Part I: Theory', *Canad. J. Math.* **38** (1986), 431–452; 'Part II: Applications', *Canad. J. Math.* **39** (1987), 428–472.

[4] J. M. Borwein and Q. J. Zhu, *Techniques of variational analysis: an introduction* (Springer, Berlin, 2005).

[5] F. H. Clarke, Y. Ledyaev, R. J. Stern and P. R. Wolenski, *Nonsmooth analysis and control theory*, Graduate Texts in Mathematics 178 (Springer, New York, 1998).

[6] M. Fabian, 'Subdifferentiability and trustworthiness in the light of the smooth variational principle of Borwein and Preiss', *Acta Univ. Carolin. Math. Phys.* **30** (1989), 51–56.

[7] M. Fabian and B. S. Mordukhovich, 'Nonsmooth characterizations of Asplund spaces and smooth variational principles', *Set-Valued Anal.* **6** (1998), 381–406.

[8] A. D. Ioffe, 'Proximal analysis and approximate subdifferentials', *J. London Math. Soc. (2)* **41** (1990), 175–192.

[9] A. Y. Kruger and B. S. Mordukhovich, 'Extremal points and the Euler equation in nonsmooth optimization', *Dolk. Akad. Nauk BSSR* **24** (1980), 684–687.

[10] K. S. Lau, 'Almost Chebyshev subsets in reflexive Banach spaces', *Indiana Univ. Math. J.* **27** (1978), 791–795.

[11] P. D. Loewen, 'A mean value theorem for Frechet subgradients', *Nonlinear Anal.* **23** (1994), 1365–1381.

[12] B. S. Mordukhovich, *Variational analysis and generalized differentiation, I: Basic Theory, II: Applications*, Grundlehren Series (Fundamental Principles of Mathematical Sciences) 330 and 331 (Springer, Berlin, 2005).

[13] B. S. Mordukhovich and Y. Shao, 'Extremal characterizations of Asplund spaces', *Proc. Amer. Math. Soc.*, **124** (1996), 197–205.

[14] R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Mathematics 1364 (Springer, Berlin, 1993).

Mathematics, The Irving K. Barber School of Arts and Sciences
UBC Okanagan
3333 University Way
Kelowna BC V1V 1V7
Canada
e-mail: Shawn.Wang@ubc.ca