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NOTES ON *K*-TOPOLOGICAL GROUPS AND HOMEOMORPHISMS OF TOPOLOGICAL GROUPS

HANFENG WANG and WEI HE ✉

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Abstract

In this paper, it is shown that there exists a connected topological group which is not homeomorphic to any ω-narrow topological group, and also that there exists a zero-dimensional topological group G with neutral element e such that the subspace $X = G \setminus \{e\}$ is not homeomorphic to any topological group. These two results give negative answers to two open problems in Arhangel'skii and Tkachenko [*Topological Groups and Related Structures* (Atlantis Press, Amsterdam, 2008)]. We show that if a compact topological group is a K-space, then it is metrisable. This result gives an affirmative answer to a question posed by Malykhin and Tironi ['Weakly Fréchet–Urysohn and Pytkeev spaces', Topology Appl. 104 (2000), 181–190] in the category of topological groups. We also prove that a regular K-space X is a weakly Fréchet–Urysohn space if and only if X has countable tightness.

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1. Introduction

This paper consists of two parts. In the first part, we consider two open problems posed by Arhangel'skii and Tkachenko in [2]. The first (Open Problem 3.4.6) asks whether every connected topological group is homeomorphic to an ω-narrow topological group. The second (Open Problem 1.4.1) asks whether for a zero-dimensional topological group G with neutral element e, the space $X = G \setminus \{e\}$ is homeomorphic to a topological group. In this paper we will show that there exists a connected topological group which is not homeomorphic to any ω-narrow topological group, and this gives a negative answer to the first open problem. We show that there exists a zero-dimensional topological group G with neutral element e such that the subspace $X = G \setminus \{e\}$ is not homeomorphic to any topological group, and this gives a negative answer to the second open problem.

In the second part, we consider an open problem posed by Malykhin and Tironi in [9] for topological spaces. This asks whether a compact K-space X must have countable tightness. We restrict our attention to the category of topological groups,

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and prove that a compact topological group which is a K-space must be metrisable. This gives an affirmative answer to the open problem in the category of topological groups. Moreover, we prove a stronger result that every locally compact topological group which is a K-space must be metrisable.

Recall that a topological group G is a group G with a topology such that the product mapping of $G \times G$ into G is jointly continuous and the inverse mapping of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. Obviously, each topological group is homogeneous. Thus, to define a topological group topology on a group G, it is enough to define a local base at the identity e of G and then translate it to all points in G.

A topological group G is called ω-narrow [5] if and only if for every open neighbourhood V of the neutral element e in G, there exists a countable subset A of G such that $AV = G$. The class of ω-narrow topological groups contains all Lindelöf topological groups and all topological groups with countable cellularity. Also, ω-narrow topological groups are characterised as subgroups of topological products of families of second countable topological groups (see [5]).

A topological space X is called a K-space [9] if, for every $x \in X$ and $B \subset X$ satisfying $x \in \overline{B} \setminus B$, there exists a sequence $\zeta = \{K_i : i \in \omega\}$ of disjoint compact subsets of B such that, for every neighbourhood U of x, $\{i \in \omega : K_i \cap U = \emptyset\}$ is finite. For convenience, we denote this relation of x and ζ by $x(K)\zeta$.

We say that the tightness of a topological space X is countable if, for each $x \in X$ and $A \subset X$ satisfying $x \in \overline{A}$, there exists a countable subset B of A such that $x \in \overline{B}$.

Recall that a topological space X is called a weakly Fréchet–Urysohn space [9] if, for each $x \in X$ and $A \subset X$ satisfying $x \in \overline{A} \setminus A$, there exists a sequence $\zeta = \{F_i : i \in \omega\}$ consisting of disjoint finite subsets of A such that for every neighbourhood U of x, $\{i \in \omega : F_i \cap U = \emptyset\}$ is finite. For convenience, we denote this relation of x and ζ by $x(F)\zeta$.

By the definitions, every weakly Fréchet–Urysohn space is a K-space. We will show that a regular K-space X is a weakly Fréchet–Urysohn space if and only if X has countable tightness.

In this paper, a topological group G always means a Tychonoff space. Also, ω, ω_1 and c denote the first infinite cardinality, the first uncountable cardinality and the cardinality of the continuum, respectively. Further, $\omega(X)$ and $c(X)$ denote the weight and the cellularity of the space X, respectively. For other terms and symbols we refer to [2] or [4].

2. The answers to two questions of homeomorphisms of topological groups

As a generalisation of the Lindelöf property in topological groups, the ω-narrow property has many interesting results (see [2]). Typical examples of ω-narrow topological groups include the product topological groups $\mathbb{R}^\kappa$ which is connected and $\mathbb{Z}^\kappa$ which is not connected, where $\mathbb{R}$ and $\mathbb{Z}$ denote the additive groups of reals and integers with the usual topology, respectively. In [2], some open problems about ω-narrow topological groups were left. The following is one of them.

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PROBLEM 2.1. Is every connected topological group homeomorphic to an ω-narrow topological group?

To answer this question, we first recall an interesting topological group constructed by Hartman and Mycielski in [6] from a given topological group G.

Let $(G, \cdot)$ be a topological group with identity e. Hartman and Mycielski constructed a new topological group as follows. Take $\dot{G}$ to be the set of all functions f defined on the interval $J = [0, 1)$ with values in G such that, for some sequence $0 = a_0 < a_1 \cdots < a_n = 1$, f is constant on $[a_k, a_{k+1})$ for each $k = 0, \ldots, n-1$. A binary operation $*$ is defined on $\dot{G}$ such that $(f * g)(x) = f(x) \cdot g(x)$, for all $f, g \in \dot{G}$ and $x \in J$. Then every element $f \in \dot{G}$ has a unique inverse $f^{-1} \in \dot{G}$ defined by $(f^{-1})(r) = f(r)^{-1}$ for each $r \in J$. It is easy to see that $(\dot{G}, *)$ is a group with identity $\dot{e}$, where $\dot{e}(r) = e$ for each $r \in J$. For any open neighbourhood V of e in G and a real number $\varepsilon > 0$, we define a subset $O(V, \varepsilon)$ of $\dot{G}$ by $O(V, \varepsilon) = \{f \in \dot{G} : \mu(\{r \in J : f(r) \notin V\}) < \varepsilon\}$, where μ is the usual Lebesgue measure on J. Let $\mathcal{N}(e)$ be a base for G at e and $\mathcal{N}(\dot{e}) = \{O(V, \varepsilon) \mid V \in \mathcal{N}(e), \varepsilon > 0\}$. Then $(\dot{G}, *)$ becomes a Hausdorff topological group with $\mathcal{N}(\dot{e})$ being a local base at the identity of $\dot{G}$.

The following important result is due to Hartman and Mycielski [6].

THEOREM 2.2. *Let $(G, \cdot)$ be a topological group. Then the topological group $(\dot{G}, *)$ is pathwise connected and G is topologically isomorphic to a closed subgroup of $\dot{G}$. If G is metrisable, then $\dot{G}$ is metrisable.*

Now we use the topological group $(\dot{G}, *)$ constructed above to give a negative answer to Problem 2.1 [2, Open Problem 3.4.6].

THEOREM 2.3. *Let G be the additive group $(\mathbb{R}, +)$ of all real numbers with the discrete topology. Then the topological group $\dot{G}$ is not homeomorphic to any ω-narrow topological group.*

PROOF. Obviously, G is a metrisable topological group. By Theorem 2.2, $\dot{G}$ is a connected metrisable topological group and $\omega(\dot{G}) \geq \omega(G) = c$.

According to [2, Proposition 3.4.5], a first countable ω-narrow topological group has a countable base. Thus, if $\dot{G}$ is homeomorphic to some ω-narrow topological group H, then $\omega(H) = \omega$ since $\dot{G}$ is first countable. It follows that $\omega(\dot{G}) = \omega$, which contradicts $\omega(\dot{G}) \geq c$. □

Recall the definition of balanced groups [2, p. 69]. Assume that G is a topological group. A subset A of G is said to be invariant if $xAx^{-1} = A$ for each $x \in G$. A topological group G is called balanced if it has a local base at the neutral element consisting of invariant subsets.

THEOREM 2.4. *Suppose that G is a balanced topological group such that for each open neighbourhood U of the neutral element e, there exists a countable subset M of G satisfying $UMU = G$. Then G is an ω-narrow topological group.*

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Proof. Take an arbitrary open neighbourhood V of e. Since G is a balanced topological group, we can choose a neighbourhood W of e such that $W^2 \subset V$ and $xWx^{-1} = W$ for each $x \in G$. According to the assumption, there exists a countable subset M of G satisfying $WMW = G$. Since $xWx^{-1} = W$, that is, $xW = Wx$, for each $x \in G$, it follows that $WM = MW$. Hence, $G = WMW = MW^2 \subset MV$, which means that G is an ω-narrow topological group. □

Theorem 2.4 gives a partial answer to [2, Open Problem 5.1.12].

Since every abelian topological group is balanced, the following result is obvious.

Corollary 2.5. *Suppose that G is a topological group such that for each open neighbourhood U of the neutral element e, there exists a countable subset M of G satisfying $UMU = G$. If G is an abelian group, then G is an ω-narrow topological group.*

We now consider the second problem. We know that, for a topological group G with neutral element e, its subspace $G \setminus \{e\}$ can fail to be homogeneous. A typical example is the product topological group $\mathbb{R} \times \mathbb{Z}^\omega$ of the topological group $\mathbb{R}$ of reals and the topological group $\mathbb{Z}^\omega$. But if G is a zero-dimensional topological group with the identity e, then the space $G \setminus \{e\}$ is homogeneous (see [2, p. 36]).

Taking account of the homogeneity for every topological group, Arhangel'skii and Tkachenko posed the following question [2, Open Problem 1.4.1].

Problem 2.6. Let G be a zero-dimensional topological group with neutral element e. Must the space $X = G \setminus \{e\}$ be homeomorphic to a topological group?

We give a negative answer to this question as follows.

Theorem 2.7. *There exists a zero-dimensional topological group G with neutral element e such that the subspace $X = G \setminus \{e\}$ is not homeomorphic to any topological group.*

Proof. Let G be the product topological group D^{ω_1}, where D is the two-element topological group $\{0, 1\}$. Obviously, G is zero-dimensional. We claim that the subspace $X = G \setminus \{e\}$ is not homeomorphic to any topological group.

Assume the contrary, that is, there exists a topological group H which is homeomorphic to X. Since X is open in the compact topological group G, it is locally compact, which implies that H is locally compact. According to [2, Corollary 3.1.4], a locally compact topological group is paracompact, so that H is paracompact, which implies that X is paracompact. By the Hewitt–Marczewski–Pondiczery theorem in [4] we know that D^{ω_1} is separable, that is, G is separable. It follows that X is separable since X is open in G. Therefore, the Souslin number of X is countable. Taking into account that X is paracompact and $c(X) = \omega$, we conclude that X is Lindelöf.

For each ordinal $\alpha < \omega_1$, put

$$K_\alpha = \{(x_\beta) \in G : x_\beta = 0, \beta \leq \alpha\};$$

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then K_α is closed in G for each $\alpha < \omega_1$. Let $F_\alpha = K_\alpha \cap X$; then F_α is closed in X for each $\alpha < \omega_1$. Therefore, we have a family $\{F_\alpha : \alpha < \omega_1\}$ consisting of decreasing nonempty closed subsets of X. Since $\bigcap_{\alpha<\omega_1} K_\alpha = \{e\}$ and $e \notin X$, we know that $\bigcap_{\alpha<\omega_1} F_\alpha = \emptyset$, which implies that the family $\{X \setminus F_\alpha : \alpha < \omega_1\}$ is an open cover of X. Since X is Lindelöf, $\{X \setminus F_\alpha : \alpha < \omega_1\}$ has a countable subcover $\{X \setminus F_{\alpha_i} : i \in \omega\}$, that is, $\bigcap_{i\in\omega} F_{\alpha_i} = \emptyset$. Taking into account that ω_1 is a regular cardinality, we can find an ordinal number γ such that $\gamma < \omega_1$ and $\gamma > \alpha_i$ for each $i \in \omega$. Since $\{F_\alpha : \alpha < \omega_1\}$ is a decreasing family, we conclude that $F_\gamma \subset \bigcap_{i\in\omega} F_{\alpha_i} = \emptyset$, which is contradiction. □

3. Some results on K-topological groups

If a topological group G is a K-space, then we will call it a K-topological group.

In [9], Malykhin and Tironi investigated weakly Fréchet–Urysohn spaces and Pytkeev spaces. The following open problem was posed in [9, Question 6.4].

Problem 3.1. Must a compact K-space X have countable tightness?

We now consider this problem for compact topological groups; equivalently, we ask whether a compact K-topological group must have countable tightness. To answer this question, we first recall a theorem in [2, Theorem 4.2.1].

Theorem 3.2. *If G is a nonmetrisable compact topological group of weight τ, then the space D^τ is homeomorphic to a subspace of G, where D is the two-element topological group* $\{0, 1\}$.

Theorem 3.2 is an easy corollary from a famous theorem (every compact topological group G is a dyadic compactum) in [7] and a general result of Engelking on dyadic compacta in [3]. From Theorem 3.2 we can obtain the important theorem in [1]: every compact topological group with countable tightness is metrisable.

We now show that for compact topological groups, Problem 3.1 has an affirmative answer.

Theorem 3.3. *Suppose that G is a compact topological group. If G is a K-topological group, then G is metrisable.*

Proof. Assume the contrary, that is, that G is not a metrisable topological group. Then there exists a cardinal number τ such that $\omega(X) = \tau$ and $\tau \geq \omega_1$. According to Theorem 3.2, the space D^τ is homeomorphic to a subspace of G. Since D^{ω_1} is homeomorphic to a subspace of D^τ, it follows that D^{ω_1} is homeomorphic to a subspace of G.

Let Y be the subspace of D^{ω_1} consisting of all elements of (x_α) such that, for some successor ordinal $\beta < \omega_1$, the αth coordinate x_α is 0 whenever $\alpha < \beta$, and all other coordinates of (x_α) are 1. Obviously, the cardinality of Y is ω_1. We claim that Y is a discrete subspace of D^{ω_1}. Indeed, for an arbitrary element $y = (y_\alpha)$ of Y, suppose that γ is the first coordinate of y which equals 1. Then, by the choice of Y, γ is a successor

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ordinal. We denote the predecessor of γ by $\gamma - 1$. Put

$$U = \{(x_\alpha) \in D^{\omega_1} : x_\gamma = 1, x_{\gamma-1} = 0\};$$

then the subset U is an open neighbourhood of y in D^{ω_1} and $U \cap Y = \{y\}$. Therefore, Y is a discrete subspace of D^{ω_1}.

It is easy to see that the element $x = (0_\alpha)$ of D^{ω_1} satisfies $x \in \overline{Y} \setminus Y$ where $0_\alpha = 0$ for each ordinal $\alpha < \omega_1$. Since G is a K-space, then it follows from the hereditariness of K-spaces that D^{ω_1} is a K-space. Thus, there exists a sequence $\zeta = \{K_i : i \in \omega\}$ of disjoint compact subsets of Y satisfying $x(K)\zeta$. Since each K_i is compact and Y is discrete, it follows that each K_i is a finite subset. Therefore, $\bigcup_{i\in\omega} K_i$ is a countable subset of $D^{\omega_1} \setminus \{x\}$. It follows from the choice of Y that there exists a $\beta < \omega_1$ such that the βth coordinate of each element of $\bigcup_{i\in\omega} K_i$ is 1. Put $V = \{(t_\alpha) \in D^{\omega_1} : t_\beta = 0\}$; then V is an open neighbourhood of x. However, $V \cap (\bigcup_{i\in\omega} K_i) = \emptyset$, which contradicts $x(K)\zeta$.

Hence, D^{ω_1} is not a K-space. Since K-spaces are hereditary, we know that G is not a K-space, which is a contradiction. Thus, G is metrisable. □

From the proof of Theorem 3.3 we can see the topological group D^{ω_1} is not a K-space. Thus the following result is obvious.

Corollary 3.4. *A compact topological group need not be a K-space. In particular, an ω-narrow topological group need not be a K-space.*

In Theorem 3.3 the compactness of G cannot be replaced by countable compactness. A suitable example is the Σ-product of ω_1 copies of a two-element topological group D, which is countably compact. We denote this group by H. Then H is a Fréchet–Urysohn space, which implies that H is a K-space. However, H is nonmetrisable. Another example is the σ-product of ω_1 copies of D, which is a σ-compact space. We denote this group by M. Since M is a subspace of H, it is a K-space. Taking account of the fact that M is dense in H, we have $\chi(M) = \chi(H)$, so that M is nonmetrisable either. Therefore the compactness of G in Theorem 3.3 cannot be replaced by σ-compactness.

It turns out that Theorem 3.3 remains valid if one replaces the compactness of G by Čech-completeness. To prove this, we first need an auxiliary result.

Recall that a topological group G is feathered [2, p. 235] if it contains a nonempty compact subset K of countable character in G, that is, K has a countable neighbourhood base in G.

Theorem 3.5. *A feathered topological group G is metrisable if and only if it is a K-topological group.*

Proof. Necessity is obvious. It remains to verify sufficiency.

Since G is feathered, according to [2, Lemma 4.3.19], there exists a compact subgroup H of G such that the left coset space G/H is metrisable. By the condition that G is a K-topological group, we know that H is a K-topological group. Then, by Theorem 3.3, H is metrisable. Since H and G/H are both first countable,

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by [2, Corollary 1.5.21], we know that G is first countable. Therefore, G is metrisable. □

Since every Čech-complete topological group is feathered, the following result is obvious.

Corollary 3.6. *Every Čech-complete K-topological group is metrisable. In particular, each locally compact K-topological group is metrisable.*

Since every sequential space is a K-space, we have the following result.

Corollary 3.7. *Every Čech-complete sequential topological group is metrisable. In particular, each locally compact sequential topological group is metrisable.*

Now we consider K-topological groups with countable pseudocharacter. We will show the following result.

Theorem 3.8. *Every K-topological group with countable pseudocharacter has countable tightness.*

The proof of this theorem will follow from the next two results. First, we recall the definition of G_δ-diagonal. A topological space X is said to have a G_δ-diagonal if there exists a sequence $\{\mathcal{V}_i : i \in \omega\}$ of open covers of X such that $\bigcap_{i\in\omega} st(x, \mathcal{V}_i) = \{x\}$ for every $x \in X$, where $st(x, \mathcal{V}_i) = \bigcup\{V \in \mathcal{V}_i : x \in V\}$.

Lemma 3.9. *A regular K-space X with a G_δ-diagonal has countable tightness.*

Proof. Suppose that $x \in X$, $A \subset X$ and $x \in \overline{A} \setminus A$. Since X is a K-space, there exists a sequence $\zeta = \{K_i : i \in \omega\}$ consisting of disjoint compact subsets of A such that $x(K)\zeta$. In particular, $x \in \overline{\bigcup_{i\in\omega} K_i}$. Since X has a G_δ-diagonal, for each $i \in \omega$, the subspace K_i also has a G_δ-diagonal. Then, for each $i \in \omega$, K_i is a separable and metrisable, which follows from the fact that K_i is compact and has a G_δ-diagonal. Choose a countable dense subset C_i of K_i for every $i \in \omega$. Then $B = \bigcup_{i\in\omega} C_i$ is a countable subset of A and $x \in \overline{B}$. Hence, X has countable tightness. □

Since every submetrisable space has a G_δ-diagonal, the following result is obvious.

Corollary 3.10. *Every regular submetrisable K-space has countable tightness.*

Lemma 3.11. *Every topological group G with countable pseudocharacter has a G_δ-diagonal.*

Proof. Since G has countable pseudocharacter, there exists a sequence $\{U_i : i \in \omega\}$ of open subsets of G such that $\bigcap_{i\in\omega} U_i = \{e\}$, where e is the identity of G. Taking account of the fact that G is a topological group, we can find another sequence $\{V_i : i \in \omega\}$ of open symmetric neighbourhoods of e such that $V_{i+1}^2 \subset V_i \cap U_i$ for each $i \in \omega$. By virtue of $\{V_i : i \in \omega\}$ we have a sequence $\{\mathcal{V}_i : i \in \omega\}$ of open covers of G such that $\mathcal{V}_i = \{xV_i : x \in G\}$ for each $i \in \omega$.

We claim that $\bigcap_{i\in\omega} st(x, \mathcal{V}_i) = \{x\}$ for every $x \in G$. Assuming the contrary, there exist two distinct points y and z such that $z \in \bigcap_{i\in\omega} st(y, \mathcal{V}_i)$. Then, for each $i \in \omega$, there exists a point $x_i \in G$ such that $\{y, z\} \subset x_i V_i$, that is, we can find two points u_i, v_i in V_i such that $y = x_i u_i$ and $z = x_i v_i$. It follows that $x_i = y u_i^{-1}$, which implies that

$$z = y u_i^{-1} v_i \in y V_i^{-1} V_i = y V_i^2 \subset y U_i \quad \text{for every } i \in \omega.$$

Since $\bigcap_{i\in\omega} U_i = \{e\}$ and G is homogeneous, $\bigcap_{i\in\omega} yU_i = \{y\}$ which implies $y = z$. This is a contradiction. Hence, G has a G_δ-diagonal. □

Lemma 3.11 fails to be valid in the category of topological spaces, even for compact spaces. A suitable example is the Alexandroff double circle [4, Example 3.1.26] X which is compact and first countable. However, the Souslin number of X is c, which means that X does not have a G_δ-diagonal. Otherwise, X would be separable and metrisable, which implies that the Souslin number of X is countable. This is a contradiction.

Proof of Theorem 3.8. The theorem follows directly from Lemmas 3.9 and 3.11. □

We recall an interesting result given by Arhangel'skii and Tkachenko [2, Lemma 3.3.22].

Lemma 3.12. *The following conditions are equivalent for a topological group G:*

(a) *every compact subspace of G is first countable;*
(b) *every compact subspace of G is metrisable.*

Theorem 3.13. *Assume that G is a K-topological group. If every compact subspace of G is first countable, then G has countable tightness.*

Proof. Suppose that $x \in G$, $A \subset G$ and $x \in \overline{A} \setminus A$. Since G is a K-space, there exists a sequence $\zeta = \{K_i : i \in \omega\}$ of disjoint compact subsets of A such that $x(K)\zeta$. By virtue of the assumption, we know that K_i is first countable for every $i \in \omega$. According to Lemma 3.12, each K_i is metrisable, so that the compact subspace K_i is separable. Therefore, the subspace $\bigcup_{i\in\omega} K_i$ is separable. From $x \in \overline{\bigcup_{i\in\omega} K_i}$ we can conclude that G has countable tightness. □

We know that every weakly Fréchet–Urysohn space is a K-space but the converse is not true. So it is interesting to ask under what conditions a K-space is a weakly Fréchet–Urysohn space. The following result gives a complete answer to this question.

Theorem 3.14. *A regular K-space X is a weakly Fréchet–Urysohn space if and only if X has countable tightness.*

Proof. We begin with necessity. By the definition of weakly Fréchet–Urysohn spaces, every weakly Fréchet–Urysohn space has countable tightness.

We now prove sufficiency. Assume $x \in X$ and $A \subset X$ satisfying $x \in \overline{A} \setminus A$. Since X has countable tightness, there exists a countable subset B of A satisfying $x \in \overline{B} \setminus B$. It follows from the fact X is a K-space that there exists a sequence $\zeta = \{K_i : i \in \omega\}$

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consisting of disjoint compact subsets of B satisfying $x(K)\zeta$. Since each K_i is compact and countable, according to [4, Theorem 3.1.19], it is separable and metrisable. By [8, Lemma 13.2], each K_i, as a countable compact metrisable space, has isolated points, where by an isolated point we mean it is isolated in the subspace K_i. In addition, each accumulation point (if it exists) on K_i is a limit point of some countable isolated points on K_i for every $i \in \omega$. If there exists an infinite subfamily η of ζ such that each element of η contains only isolated points, then there is nothing to prove, since a compact subset consisting of isolated points is a finite subset.

Now we assume that for each $i \in \omega$, K_i has accumulation points. Let $H_i = \{x_{i,m} : m \in \omega\}$ be the subset of accumulation points on K_i. For each $i, m \in \omega$, take a sequence $\{x^k_{i,m} : k \in \omega\} \subset K_i \setminus H_i$ such that $\{x^k_{i,m} : k \in \omega\}$ converges to $x_{i,m}$. Put $F^l_{i,m} = \{x^k_{i,m} : k \geq l\}$, $i, m, l \in \omega$. We consider the following two cases.

Case 1. For each neighbourhood U of x, $\{i \in \omega : U \cap H_i = \emptyset\}$ is finite. Therefore, $\xi = \{F^l_{i,m} : i, m, l \in \omega\}$ is a countable π-network of X at x consisting of infinite subsets. According to [9, Proposition 1.1], there exists a countably infinite sequence λ of finite subsets of B satisfying $x(F)\lambda$.

Case 2. There exists a neighbourhood U of x such that $\{i \in \omega : U \cap H_i = \emptyset\}$ is infinite. Suppose that $M = \{i \in \omega : U \cap H_i = \emptyset\}$. According to the definition of ζ, we have the following conclusion: for each neighbourhood V of x, $\{i \in M : V \cap (K_i \setminus H_i) = \emptyset\} = \{i \in M : V \cap K_i = \emptyset\}$ is finite $(*)$. Choose a neighbourhood O of x satisfying $\overline{O} \subset U$, then $O \cap K_i = O \cap (K_i \setminus H_i)$ for every $i \in M$. According to $(*)$, we can assume that $O \cap (K_i \setminus H_i)$ is not empty for each $i \in M$. Thus, each $\overline{O} \cap K_i = \overline{O} \cap (K_i \setminus H_i)$ is compact and discrete. It follows that $T_i = O \cap (K_i \setminus H_i)$ is finite for each $i \in M$. For any two distinct $m, n \in M$, $T_m \cap T_n \subset K_m \cap K_n = \emptyset$. For each neighbourhood W of x, it follows from $(*)$ that

$$\{i \in M : W \cap T_i = \emptyset\} = \{i \in M : (W \cap O) \cap (K_i \setminus H_i) = \emptyset\}$$

is finite. Therefore, $x(F)\{T_i : i \in M\}$. □

LEMMA 3.15. *Suppose that X is a T_2 K-space, $x \in X$, $A \subset X$ and $x \in \overline{A} \setminus A$. Then there exist two disjoint subsets B, C of A satisfying $x \in \overline{B}$ and $x \in \overline{C}$.*

PROOF. By virtue of the assumption, there exists a sequence $\zeta = \{K_i : i \in \omega\}$ of disjoint compact subsets of A satisfying $x(K)\zeta$. Put

$$B = \bigcup_{i\in\omega} K_{2i} \quad \text{and} \quad C = \bigcup_{i\in\omega} K_{2i+1};$$

then $B \cap C = \emptyset$, $x \in \overline{B}$ and $x \in \overline{C}$, which follow from $x(K)\zeta$. □

The following result gives an example of a countable topological group which is not a K-space, so the countable tightness and countable pseudocharacter cannot make a topological group be a K-space.

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Corollary 3.16. *Suppose that $\beta\omega$ is the Stone–Čech compactification of ω, $p \in \beta\omega \setminus \omega$. Let $X = \omega \cup \{p\}$ be the subspace of $\beta\omega$. Then the countable free topological group $F(X)$ is not a K-space.*

Proof. Obviously, $p \in \overline{\omega} \setminus \omega$. Since p is a free ultrafilter, there do not exist two disjoint subsets B, C of ω such that $x \in \overline{B}$ and $x \in \overline{C}$. By Lemma 3.15, X is not a K-space. Since X is a subspace of the free topological group $F(X)$, $F(X)$ is not a K-space. □

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HANFENG WANG, Institute of Mathematics, Nanjing Normal University,
Nanjing 210046, PR China
and
Department of Mathematics, Shandong Agricultural University,
Taian 271018, PR China
e-mail: whfeng@sdau.edu.cn

WEI HE, Institute of Mathematics, Nanjing Normal University,
Nanjing 210046, PR China
e-mail: weihe@njnu.edu.cn

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