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C^∞-CONVERGENCE OF CIRCLE PATTERNS TO MINIMAL SURFACES

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Abstract. Given a smooth minimal surface $F : \Omega \to \mathbb{R}^3$ defined on a simply connected region Ω in the complex plane $\mathbb{C}$, there is a regular SG circle pattern Q_Ω^ϵ. By the Weierstrass representation of F and the existence theorem of SG circle patterns, there exists an associated SG circle pattern P_Ω^ϵ in $\mathbb{C}$ with the combinatoric of Q_Ω^ϵ. Based on the relationship between the circle pattern P_Ω^ϵ and the corresponding discrete minimal surface $F^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ defined on the vertex set V_Ω^ϵ of the graph of Q_Ω^ϵ, we show that there exists a family of discrete minimal surface $\Gamma^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$, which converges in $C^\infty(\Omega)$ to the minimal surface $F : \Omega \to \mathbb{R}^3$ as $\epsilon \to 0$.

§1. Introduction

The theory of discrete differential geometry is presently emerging on the border of differential and discrete geometry, which studies geometric shapes with a finite number of elements (polyhedra) and aims at a development of discrete equivalents of the geometric notions and methods of surface theory (see [1], [2], [3], [4], [8], [12], etc.). A smooth geometric shape (such as surface) appears then as a limit of the refinement of the discretization. One of the central problems of discrete differential geometry is to find proper discrete analogues of special classes of surfaces, such as minimal, constant mean curvature, isothermic, etc. In [2], a new discrete model was introduced to investigate conformal discretizations of minimal surface, i.e., the analogous discrete minimal surfaces consisting of touching spheres, and of circles which intersect the spheres orthogonally in their points of touch. It is proved that the discrete minimal surfaces converge to the smooth ones. The advantages of the discretizations are that they respect conformal properties of surfaces, possess a maximum principle, etc. Here, we are concerned with the C^∞-convergence of discrete minimal surfaces given in terms of circles

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and spheres, that is, the convergence of discrete minimal surface discussed in [2] is extended to C^∞-convergence.

For each $\epsilon > 0$, let SG^ϵ denote the square grid with mesh $\epsilon > 0$. The vertices of SG^ϵ form the square lattice $V^\epsilon = \{n\epsilon + m\epsilon i : (n, m) \in \mathbb{Z} \times \mathbb{Z}\}$, and an edge connects any two vertices of SG^ϵ at distance ϵ. The 1-skeleton of SG^ϵ is the graph of regular SG circle pattern Q^ϵ each circle of which has radius equal to $\epsilon/\sqrt{2}$. Aussume that Ω is a simply-connected domain in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$. Let Q^ϵ_Ω be the largest sub-pattern of Q^ϵ that is contained in Ω, and let SG^ϵ_Ω be the sub-complex of SG^ϵ whose 1-skeleton is equal to the graph of Q^ϵ_Ω. The vertex set of SG^ϵ_Ω is denoted by V^ϵ_Ω. Suppose that $F : \Omega \to \mathbb{R}^3$ is a minimal immersion without umbilic points in conformal curvature line coordinates. First, by the Weierstrass representation of F and the local theory of SG circle patterns (see [12, §6]), there is an associated SG circle pattern P^ϵ_Ω in $\mathbb{C}$ with the combinatoric of Q^ϵ_Ω. In the meantime, one gets a discrete minimal surface $F^\epsilon : V^\epsilon_\Omega \to \mathbb{R}^3$ corresponding to F, which consists of spheres and circles. Secondly, in terms of Möbius invariants of circle pattern P^ϵ_Ω, we define the discrete Schwarzians of P^ϵ_Ω and show that they are uniformly bounded in $C^\infty(\Omega)$. Thirdly, we construct a Möbius transformation T^ϵ through circle pattern P^ϵ_Ω such that they can be expressed by the discrete Schwarzians. By the C^∞-boundedness of the Schwarzians, we will prove that T^ϵ converges in $C^\infty(\Omega)$ to some Möbius transformation T as $\epsilon \to 0$. Lastly, using the relation between the discrete minimal surface F^ϵ and circle pattern P^ϵ_Ω, and combining with the definition and the C^∞-convergence of T^ϵ, we will show that there exists a family of discrete surface $\Gamma^\epsilon : V^\epsilon_\Omega \to \mathbb{R}^3$, obtained by scaling appropriately the centers of spheres and circles in F^ϵ, which converges in $C^\infty(\Omega)$ to the minimal surface $F : \Omega \to \mathbb{R}^3$ as $\epsilon \to 0$.

This paper is organized as follows. For a given smooth minimal surface $F : \Omega \to \mathbb{R}^3$, we will give the associated SG circle patterns P^ϵ_Ω, the corresponding discrete minimal surfaces F^ϵ and the relation between them in Section 2. In Section 3, we first give the definitions of C^∞-convergence and C^∞-boundedness for discrete functions. Then by using Möbius invariants of P^ϵ_Ω, we define the discrete Schwarzians of P^ϵ_Ω and show that they are uniformly bounded in $C^\infty(\Omega)$. In Section 4, we construct the Möbius transformations T^ϵ through circle pattern P^ϵ_Ω such that they can be expressed by the discrete Schwarzians. Then it is proved that T^ϵ converges in $C^\infty(\Omega)$ to some Möbius transformation T as $\epsilon \to 0$. In Section 5, by the relation between the discrete minimal surface F^ϵ and the circle pattern P^ϵ_Ω, we will show that there exists a family of discrete minimal surface $\Gamma^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ which converges in $C^\infty(\Omega)$ to $F : \Omega \to \mathbb{R}^3$ as $\epsilon \to 0$.

§2. Circle patterns and discrete minimal surfaces

In this section, for any smooth minimal surface, we apply its Weierstrass representation to yield associated circle patterns with the combinatorics of square grid and define corresponding discrete minimal surfaces. Moreover, we give the relationship between the circle patterns and the discrete minimal surfaces (also see [12], [2] for more details).

For each positive number $\epsilon > 0$, let SG^ϵ be the cell complex whose vertices form the square lattice $V^\epsilon = \epsilon\mathbb{Z} + i\epsilon\mathbb{Z} = \{v = \epsilon n + i\epsilon m : (n, m) \in \mathbb{Z} \times \mathbb{Z}\}$, whose edges are the pair $[v, v']$ such that $|v - v'| = \epsilon$ and $v, v' \in V^\epsilon$, and whose 2-cells are the squares $\{v + x + iy : x, y \in [0, \epsilon]\}$, $v \in V^\epsilon$.

An indexed collection $P^\epsilon = \{P^\epsilon(v) : v \in V^\epsilon\}$ of oriented circles in the Riemann sphere $\hat{\mathbb{C}}$ is said to be a circle pattern with the combinatorics of square grid SG^ϵ (or a SG circle pattern) if the following three conditions hold.

(a) Whenever v, v' are neighbors in SG^ϵ, the corresponding circles $P^\epsilon(v)$, $P^\epsilon(v')$ intersect orthogonally.

(b) If v_1, v_2 are neighbors of a vertex v in SG^ϵ, and they belong to the same square of SG^ϵ, then the circles $P^\epsilon(v_1)$, $P^\epsilon(v_2)$ are distinct and tangent.

(c) Whenever the situation is as in (b) and v_2 is neighbor of v, which is one step counterclockwise from v_1, the circular order of the triplet of points $P^\epsilon(v) \cap P^\epsilon(v_1) - P^\epsilon(v_2)$, $P^\epsilon(v_1) \cap P^\epsilon(v_2)$, $P^\epsilon(v) \cap P^\epsilon(v_2) - P^\epsilon(v_1)$ agrees with the orientation of $P^\epsilon(v)$.

Clearly, the 1-skeleton of SG^ϵ is the graph of regular SG circle pattern Q^ϵ each circle of which has radius equal to $\epsilon/\sqrt{2}$.

Suppose that Ω is a simply connected domain in the complex plane $\mathbb{C}$. Without loss of generality, we may assume $0 \in \Omega$. Let $\tilde{Q}_\Omega^\epsilon$ be the largest subpattern of Q^ϵ that is contained in Ω. Let Q_Ω^ϵ be the connected component of $\tilde{Q}_\Omega^\epsilon$ that contains 0, and let SG_Ω^ϵ be the cell complex whose 1-skeleton is equal to the graph of Q_Ω^ϵ. The set of vertices of SG_Ω^ϵ is denoted by V_Ω^ϵ, and the set of centers of squares of SG_Ω^ϵ by $\hat{V}_\Omega^\epsilon$.

A smooth immersed surface in $\mathbb{R}^3$ is said to be isothermic if it admits a conformal curvature line parametrization in a neighborhood of every non-umbilic. An isothermic immersion is a minimal surface if and only if the dual immersion is contained in a sphere. In that case the dual immersion is

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in fact the Gauss map of the minimal surface, up to scale and translation. We first give the following lemma about the Weierstrass representation of minimal surfaces, which follows from [2].

LEMMA 1. *Suppose that $F : \Omega \to \mathbb{R}^3$ is a minimal immersion without umbilic points in conformal curvature line coordinates. Then*

$$F = \left(\operatorname{Re}\int \frac{1-f(z)^2}{f'(z)}\,dz, \operatorname{Re}\int \frac{i(1+f(z)^2)}{f'(z)}\,dz, \operatorname{Re}\int \frac{2f(z)}{f'(z)}\,dz\right), \tag{1}$$

where $f : \Omega \to \mathbb{C}$ is a locally injective meromorphic function. □

For the locally injective meromorphic function f in Lemma 1, set

$$\nu_\epsilon^{(1)}(v) = 1 + \epsilon^2 \operatorname{Re} S_f(v) \tag{2}$$

for each boundary vertex $v \in \partial V_\Omega^\epsilon$, where S_f denotes the Schwarzian derivative of f. Then SG^ϵ Dirichlet principle [12, Theorem 6.2] implies that $\nu_\epsilon^{(1)}$ is extended to a solution of the SG^ϵ-Dirichlet problem on SG_Ω^ϵ. Let $\nu_\epsilon^{(2)}$ be the companion of $\nu_\epsilon^{(1)}$ in the solution of the $SG^\epsilon - CR$ equation [12, §5], i.e.,

$$\frac{\nu_\epsilon^{(2)}(v+\omega_0)}{\nu_\epsilon^{(2)}(v+\omega_1)} = \left(\frac{\nu_\epsilon^{(1)}(v+i\epsilon)^{-1}+1}{\nu_\epsilon^{(1)}(v)^{-1}+1}\right)^2 \tag{3}$$

and

$$\frac{\nu_\epsilon^{(2)}(v+\omega_0)}{\nu_\epsilon^{(2)}(v+\omega_3)} = \left(\frac{\nu_\epsilon^{(1)}(v+\epsilon)+1}{\nu_\epsilon^{(1)}(v)+1}\right)^2 \tag{4}$$

for any $v \in V_\Omega^\epsilon$, such that

$$\nu_\epsilon^{(2)}(\omega_0) = 1 + \epsilon^2 \operatorname{Im} S_f(\omega_0), \tag{5}$$

where $\omega_j = i^j(\epsilon/2 + i\epsilon/2)$ $(j = 0, 1, 2, 3)$.

Based on the local theory of SG circle patterns [12, Theorem 6.1], there exists an SG circle pattern $P_\Omega^\epsilon = \{P^\epsilon(v) : v \in V_\Omega^\epsilon\}$ for SG_Ω^ϵ in the complex $\mathbb{C}$ that has $\nu_\epsilon^{(1)}$ and $\nu_\epsilon^{(2)}$ as its Möbius invariants. That is, $\nu_\epsilon^{(1)}(v) = -\operatorname{cr}[q_0(v), q_2(v); q_3(v), q_1(v)]$ for any $v \in V_\Omega^\epsilon$ and $\nu_\epsilon^{(2)}(u) = -\operatorname{cr}[q_j(v+\epsilon), q_j(v-\epsilon); q_j(v-i\epsilon), q_j(v+i\epsilon)$ for any $u = v + \omega_j \in \hat{V}_\Omega^\epsilon$, where $q_j(v)$

($j = 0, 1, 2, 3$) denotes the point of contact with circles $P^\epsilon(v)$ and $P^\epsilon(v+2\omega_j)$ and $\mathrm{cr}[q_1, q_2, q_3, q_4]$ denotes the cross ratio of the four points $q_1, q_2, q_3, q_4 \in \hat{\mathbb{C}}$.

On the other hand, we further suppose that there are two kinds of vertices v_s and v_c in V_Ω^ϵ such that each edge of SG_Ω^ϵ has vertices of different kinds. Then a discrete isothermic surface is a mapping

$$F^\epsilon : V_\Omega^\epsilon \longrightarrow \mathbb{R}^3$$

such that images $F^\epsilon(v_s)$ and $F^\epsilon(v_c)$ of v_s and v_c are spheres and circles respectively. Spheres $F^\epsilon(v_s)$ and circles $F^\epsilon(v_c)$ intersect orthogonally if v_s and v_c belong to the same edge of SG_Ω^ϵ, and spheres $F^\epsilon(v_s^{(1)})$ and $F^\epsilon(v_s^{(2)})$ (respectively, the circles $F^\epsilon(v_c^{(1)})$ and $F^\epsilon(v_c^{(2)})$) are distinct and tangent if $v_s^{(1)}$ and $v_s^{(2)}$ (respectively, $v_c^{(1)}$ and $v_c^{(2)}$) belong to the same face of SG_Ω^ϵ.

Let $\hat{F}^\epsilon(v)$ be the center of sphere (or circle) $F^\epsilon(v)$ for each $v \in V_\Omega^\epsilon$, $p_j(v_s)$ be the intersection points of spheres $F^\epsilon(v_s)$ and $F^\epsilon(v_s + \omega_j)$ ($j = 0, 1, 2, 3$), and let $p_j(v_s) = \hat{F}^\epsilon(w_s) + b_j$. Then a discrete isothermic surface $F^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ is called a discrete minimal surface if it satisfies any one of the equivalent conditions below.

(a) The points $\hat{F}^\epsilon(v_s) + (-1)^j b_j$ lie on a circle.

(b) There is a $d \in \mathbb{R}^3$ such that $(-1)^j (b_j, d)$ is the same for $j = 0, 1, 2, 3$.

(c) There is a plane through $\hat{F}^\epsilon(v_s)$ such that the points $\{p_j(v_s) \mid j = 0, 2\}$ and the points $\{p_j(v_s) \mid j = 1, 3\}$ lie in planes which are parallel to it at the same distance on opposite sides.

From the definition above, it is easy to see that a discrete isothermic surface is a discrete minimal surface, if and only if the dual discrete isothermic surface corresponds to a Koebe polyhedron (see [2, §4]). Let $A^\epsilon(v)$ denote the center of circle $P^\epsilon(v)$ in the circle pattern P_Ω^ϵ for each $v \in V_\Omega^\epsilon$. Then the following lemma, which can follow from [2, §5], gives the relation between the discrete minimal surfaces F^ϵ and the circle patterns P_Ω^ϵ.

LEMMA 2. *For any vertices $v_s^{(1)}$ and $v_s^{(2)}$ that belong to the same square of SG_Ω^ϵ, let $\hat{F}^\epsilon(v_s^{(1)})$ and $\hat{F}^\epsilon(v_s^{(2)})$ denote the centers of spheres $F^\epsilon(v_s^{(1)})$ and $F^\epsilon(v_s^{(2)})$ in the discrete minimal surface F^ϵ, respectively. Then*

$$\begin{aligned}(6)\quad & \hat{F}^\epsilon(v_s^{(1)}) - \hat{F}^\epsilon(v_s^{(2)}) \\ & = \mathrm{Re}\left(\frac{R(v_s^{(1)}) + R(v_s^{(2)})}{1 + |q|^2} \, \frac{\overline{A(v_s^{(1)})} - \overline{A(v_s^{(2)})}}{|A(v_s^{(1)}) - A(v_s^{(2)})|} (1 - q^2, i(1 + q^2), 2q) \right),\end{aligned}$$

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where q denotes the point of contact with circles $P^\epsilon(v_s^{(1)})$ and $P^\epsilon(v_s^{(2)})$ in the circle pattern P_Ω^ϵ and the radii $R(v_s^{(j)})$ of the sphere $F^\epsilon(v_s^{(j)})$ $(j = 1, 2)$ are

$$R(v_s^{(j)}) = \left| \frac{1 + |A(v_s^{(j)})|^2 - |A(v_s^{(j)} - q|^2}{2|A(v_s^{(j)} - q|} \right|. \tag{7}$$

□

For the discrete minimal surface F^ϵ, we define a discrete surface $\tilde{F}^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ by $\tilde{F}^\epsilon(v) = \hat{F}^\epsilon(v)$ for each $v \in V_\Omega^\epsilon$. Then $\tilde{F}$ is called a discrete minimal surface comprised of points and F^ϵ is called one consisting of spheres and circles for distinction. Next, we will show that after scaling appropriately, $\tilde{F}^\epsilon$ converges in $C^\infty(\Omega)$ to $F : \Omega \to \mathbb{R}^3$.

§3. The C^∞-boundedness of discrete Schwarzians

In this section we first give some definitions and notations related to discrete differential operators (also see [8]). Then using the Möbius invariants $\nu_\epsilon^{(1)}$ and $\nu_\epsilon^{(2)}$ given in Section 2, we define the discrete Schwarzians of circle pattern P_Ω^ϵ and show that they are uniformly bounded in $C^\infty(\Omega)$.

Let W be a subset of V_Ω^ϵ. A vertex $v \in W$ is said to be an interior vertex of W if all its neighboring vertices are contained in W. Let $int\, W$ denote the set of the interior vertices of W. Given a function $\rho : W \to \mathbb{R}$, we define the discrete directional derivative $\partial\rho_j : int\, W \to \mathbb{R}$ by

$$\partial_j^\epsilon \rho(v) = (\rho(v + i^j\epsilon) - \rho(v))/\epsilon,$$

for each $j \in \mathbb{Z}_4$. The discrete Laplacian of a function $\rho : W \to \mathbb{R}$ is a function in $int\, W$ defined by the formula

$$\Delta^\epsilon \rho(v) = 1/(4\epsilon^2) \sum_{j=0}^{3} (\rho(v + i^j\epsilon) - \rho(v)).$$

For any differentiable function $H : \Omega \to \mathbb{R}$, let $\partial_j H$ denote the directional derivate

$$\partial_j H(z) = \lim_{s\to 0} \frac{H(z + i^j s) - H(z)}{s},$$

where $j = 0, 1, 2, 3$. Let $f : \Omega \to \mathbb{C}^d$ be a function defined in Ω. For each $\epsilon > 0$, let f^ϵ be a function defined on some set of vertices $\tilde{V}^\epsilon \subset V_\Omega^\epsilon$, with

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values in $\mathbb{C}^d$. Assume that for each $z \in \Omega$, there are some $\delta_1, \delta_2 > 0$ such that $\{v \in V^\epsilon_\Omega : |v - z| < \delta_2\} \subset \tilde{V}^\epsilon$ whenever $\epsilon \in (0, \delta_1)$.

If for every $z \in \Omega$ and every $\delta > 0$, there are some $\delta_1, \delta_2 > 0$ such that $|f(z) - f^\epsilon(v)| < \delta$, for every $\epsilon \in (0, \delta_1)$ and every $v \in V^\epsilon_\Omega$ with $|v - z| < \delta_2$, then we say f^ϵ converges to f, locally uniformly in Ω.

Let $n \in \mathbb{N}$, and suppose that f is C^n-smooth. If for every sequence $j_1, j_2, \ldots, j_k \in \mathbb{Z}_4$ with $k \leq n$ we have $\partial^\epsilon_{j_k} \partial^\epsilon_{j_{k-1}} \cdots \partial^\epsilon_{j_1} f^\epsilon \to \partial_{j_k} \partial_{j_{k-1}} \cdots \partial_{j_1} f$ locally uniformly in Ω, then we say that f^ϵ converges to f in $C^n(\Omega)$. If that holds for all $n \in \mathbb{N}$, then the convergence is C^∞. The functions f^ϵ are said to be uniformly bounded in $C^n(\Omega)$ provided that for every compact $K \subset \Omega$ there is some constant $C(K, n)$ such that

$$\|\partial^\epsilon_{j_k} \partial^\epsilon_{j_{k-1}} \cdots \partial^\epsilon_{j_1} f^\epsilon\|_{K \cap V^\epsilon_\Omega} < C(K, n)$$

whenever $k \leq n$, and ϵ is sufficiently small, where $\|\cdot\|$ denotes the L^∞-norm. The functions f^ϵ are uniformly bounded in $C^\infty(\Omega)$, if they are uniformly bounded in $C^n(\Omega)$ for every $n \in \mathbb{N}$.

For a smooth vector function $F : \Omega \to \mathbb{R}^3$ and a discrete vector function $F^\epsilon : V^\epsilon_\Omega \to \mathbb{R}^3$, it is said that F^ϵ converges in $C^\infty(\Omega)$ to F if every component of F^ϵ converges in $C^\infty(\Omega)$ to the corresponding one of F, and that F^ϵ are uniformly bounded in $C^\infty(\Omega)$ if all components of F^ϵ are uniformly bounded in $C^\infty(\Omega)$.

For $\nu^{(1)}_\epsilon$ and $\nu^{(2)}_\epsilon$ given in Section 2, we define the two discrete Schwarzians of P^ϵ_Ω as follows: let

$$h^{(1)}_\epsilon(v) = \epsilon^{-2}(\nu^{(1)}_\epsilon(v) - 1) \tag{8}$$

for each vertex $v \in V^\epsilon_\Omega$ and let

$$h^{(2)}_\epsilon(u) = \epsilon^{-2}(\nu^{(2)}_\epsilon(u) - 1) \tag{9}$$

for every $u \in \hat{V}^\epsilon_\Omega$. It is easy to see that $h^{(1)}_\epsilon = h^{(2)}_\epsilon \equiv 0$ if P^ϵ_Ω is a regular SG circle pattern, because $\nu^{(1)}_\epsilon = \nu^{(2)}_\epsilon \equiv 1$ for regular SG circle patterns.

Let SG^ϵ_h denote $(1/2)SG^\epsilon_\Omega + (1 + i)\epsilon/4$, and the set of vertices of SG^ϵ_h is denoted by V^ϵ_h. For any $w \in V^\epsilon_h$, let $v \in V^\epsilon_\Omega$ be the unique vertex of SG^ϵ_Ω that is closest to w. Let M_w be the Möbius transformation that takes ∞, 0, 1 to $q_0(v)$, $q_1(v)$, $q_3(v)$, respectively. Set

$$M_{[w_1,w_2]} = M^{-1}_{w_1} \circ M_{w_2}$$

for each directed edge $[w_1, w_2]$ of SG^ϵ_h. Write $e_j(v) = [v + \omega_j/2, v + \omega_{j+1}/2]$ for any $v \in V^\epsilon_\Omega \cup \hat{V}^\epsilon_\Omega$ and for any $j \in \mathbb{Z}_4$, then we have

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LEMMA 3. *There hold the following equalities*

$$M_{e_0(u)}(z) = M_{e_2(u)}(z) = 1 - i\epsilon(h_\epsilon^{(2)}(u)+1)^{1/2}z, \tag{10}$$

$$M_{e_1(u)}(z) = M_{e_3(u)}(z) = 1 - i(1/\epsilon)(h_\epsilon^{(2)}(u)+1)^{-1/2}z \tag{11}$$

for each $u \in \hat{V}_\Omega^\epsilon$ *and*

$$M_{e_0(v)}(z) = M_{e_2(v)}(z) = (\epsilon^{-2}(h_\epsilon^{(1)}(v)+1)^{-1}+1)^{-1}/(1-z), \tag{12}$$

$$M_{e_1(v)}(z) = M_{e_3(v)}(z) = (\epsilon^{2}(h_\epsilon^{(1)}(v)+1)+1)^{-1}/(1-z) \tag{13}$$

for every $v \in V_\Omega^\epsilon$.

Proof. For any $u \in \hat{V}_\Omega^\epsilon$, we first consider the directed edge $e_2(u) = [u+\omega_2/2, u+\omega_3/2] \in SG_h^\epsilon$. By the definition of $M_{[w_1,w_2]}$, we get that $M_{[w_1,w_2]}$ does not change if we apply a Möbius transformation to P_Ω^ϵ. So we may assume that $M_{u+\omega_2/2}$ is the identity. It follows from the definition of M_w that

$$q_1(v) = \infty, \quad q_1(v-\epsilon) = 0, \quad q_1(v-i\epsilon) = 1,$$

where $v \in V_\Omega^\epsilon$ is the unique vertex of SG_Ω^ϵ that is closest to $u+\omega_3/2$. Hence we deduce that the four points $q_1(v\pm\epsilon), q_1(v\pm i\epsilon)$ form the vertices of a rectangle. From the definition of $\nu_\epsilon^{(2)}$ in Section 2, we get

$$\nu_\epsilon^{(2)}(u) = -\left(\frac{q_1(v+\epsilon)-q_1(v-i\epsilon)}{q_1(v-i\epsilon)-q_1(v-\epsilon)}\right)^2,$$

which implies

$$q_1(v+\epsilon) = q_1(v-i\epsilon) - i(\nu_\epsilon^{(2)})^{1/2}(q_1(v-i\epsilon)-q_1(v-\epsilon)) = 1 - i(\nu_\epsilon^{(2)})^{1/2}.$$

Since $M_{e_2(u)}$ takes ∞, 0, 1 to $q_1(v)$, $q_1(v-i\epsilon)$, $q_1(v+i\epsilon)$, respectively, we conclude from (9) that

$$M_{e_2(u)}(z) = 1 - i(\nu_\epsilon^{(2)})^{1/2}z = 1 - i\epsilon(h_\epsilon^{(2)}(u)+1)^{1/2}z.$$

Similarly, we deduce that $M_{e_0(u)}(z) = 1 - i\epsilon(h_\epsilon^{(2)}(u)+1)^{1/2}z$. So (10) holds. Identical to the above arguments, we conclude that (11) holds.

Next, we consider the directed edge $e_0(v) = [v+\omega_0/2, v+\omega_1/2] \in SG_h^\epsilon$ for any $v \in V_\Omega^\epsilon$. We assume with no loss of generality that $M_{v+\omega_0/2}$ is the identity. Then we obtain

$$q_0(v) = \infty, \quad q_1(v) = 0, \quad q_1(v) = 1.$$

By the definition of $\nu_\epsilon^{(1)}$, we get $q_2(v) = 1/(\nu_\epsilon^{(1)}(v) + 1)$. Note that $M_{e_0(v)}$ takes ∞, 0, 1 to $q_1(v)$, $q_1(v)$, $q_1(v)$, respectively, combining with (8), we deduce

$$M_{e_0(v)} = ((\nu_\epsilon^{(1)}(v))^{-1} + 1)^{-1}/(1 - z)$$
$$= (\epsilon^{-2}(h_\epsilon^{(1)}(v) + 1)^{-1} + 1)^{-1}/(1 - z).$$

Similarly, we get

$$M_{e_2(v)}(z) = (\epsilon^{-2}(h_\epsilon^{(1)}(v) + 1)^{-1} + 1)^{-1}/(1 - z).$$

Thus (12) holds. With the same arguments as above, we get that (13) holds. □

LEMMA 4. (i) *The equality*

$$h_\epsilon^{(1)}(v) = \operatorname{Re} S_f(v) + \epsilon^2 \cdot O(1) \tag{14}$$

holds for each $v \in V_\Omega^\epsilon$.

(ii) *The equality*

$$h_\epsilon^{(2)}(u) = \operatorname{Im} S_f(u) + \delta_\epsilon(u)\epsilon^2 \cdot O(1) \tag{15}$$

holds for each $u \in \hat{V}_\Omega^\epsilon$*, where* $\delta_\epsilon(u)$ *denotes the combinatorial distance in* SG_Ω^ϵ *from* u *to* ω_0*, i.e., the least* l *such that there is a sequence* $\{u_1, u_2, \ldots, u_l = u\} \subset \hat{V}_\Omega^\epsilon$ *such that* $u_1 = \omega_0$ *and* $|u_{j+1} - u_j| = \epsilon$ *for* $j = 1, 2, \ldots, l - 1$.

Proof. First, for any $v \in \mathit{int}\, V_\Omega^\epsilon$, expending S_f in power series about v and noting that $\sum_{k=0}^{3} i^{jk} = 0$ for $j = 1, 2, 3$, we obtain

$$\Delta^\epsilon \operatorname{Re} S_f = \operatorname{Re}\left(1/(4\epsilon^2) \sum_{j=0}^{3} [S_f(v + i^j \epsilon) - S_f(v)]\right) = O(\epsilon^2).$$

Consider the function

$$g_1(v) = h_\epsilon^{(1)}(v) - \operatorname{Re} S_f + \beta |v|^2,$$

where $\beta \in (0, \epsilon^2)$ is some function of ϵ. Similar to the proof of [12, Lemma 9.2], we deduce from the Taylor's formula and the properties of Möbius invariant $h_\epsilon^{(1)}$ that g_1 has no maxima in $\mathit{int}\, V_\Omega^\epsilon$ if β is chosen as $\beta = C\epsilon^2$ with $C > 0$ a sufficiently large constant. By the assumption (2), it follows that $g_1(v) = \beta|v|^2 = O(\epsilon^2)$ on $\partial V_\Omega^\epsilon$. Hence we obtain $g_1(z) \le O(\epsilon^2)$ in V_Ω^ϵ, which implies

$$h_\epsilon^{(1)}(v) \le \operatorname{Re} S_f(v) + O(\epsilon^2).$$

On the other hand, if we let $g_2(v) = h_\epsilon^{(1)}(v) - \operatorname{Re} S_f(v) - \beta|v|^2$, then similar to the above arguments, we conclude that

$$h_\epsilon^{(1)}(v) \ge \operatorname{Re} S_f(v) + O(\epsilon^2).$$

So it follows that

$$h_\epsilon^{(1)}(v) = \operatorname{Re} S_f(v) + O(\epsilon^2),$$

which implies (i).

Next, by (3), the relationship between $\nu_\epsilon^{(1)}$ and $h_\epsilon^{(1)}$ and Taylor's formula, we get

$$\begin{aligned}
&\log \nu_\epsilon^{(2)}(v+\omega_0) - \log \nu_\epsilon^{(2)}(v+\omega_1)\\
&\quad = 2\log(\nu_\epsilon^{(1)}(v+i\epsilon)^{-1}+1) - 2\log(\nu_\epsilon^{(1)}(v)^{-1}+1)\\
&\quad = 2\log((1+\epsilon^2 h_\epsilon^{(1)}(v+i\epsilon))^{-1}+1) - 2\log((1+\epsilon^2 h_\epsilon^{(1)}(v))^{-1}+1)\\
&\quad = 2\log(2-\epsilon^2 h_\epsilon^{(1)}(v+i\epsilon)) - 2\log(2-\epsilon^2 h_\epsilon^{(1)}(v)) + O(\epsilon^4)\\
&\quad = \epsilon^2 h_\epsilon^{(1)}(v) - \epsilon^2 h_\epsilon^{(1)}(v+i\epsilon) + O(\epsilon^4).
\end{aligned}$$

Hence it follows from (14) and Taylor's formula

$$\begin{aligned}
\text{(16)}\qquad &\log \nu_\epsilon^{(2)}(v+\omega_0) - \log \nu_\epsilon^{(2)}(v+\omega_1)\\
&\quad = \epsilon^2 \operatorname{Re}(S_f(v) - S_f(v+i\epsilon)) + O(\epsilon^4)\\
&\quad = \epsilon^2 \operatorname{Re}(i\epsilon S_f'(v)) + O(\epsilon^4)\\
&\quad = \epsilon^3 \operatorname{Im} S_f'(v) + O(\epsilon^4)\\
&\quad = \epsilon^2 \operatorname{Im} S_f(v+i\epsilon) - \epsilon^2 \operatorname{Im} S_f(v) + O(\epsilon^4).
\end{aligned}$$

Similarly, we conclude from (4) that

$$\begin{aligned}
\text{(17)}\qquad &\log \nu_\epsilon^{(2)}(v+\omega_0) - \log \nu_\epsilon^{(2)}(v+\omega_3)\\
&\quad = \epsilon^2 \operatorname{Im} S_f(v+\epsilon) - \epsilon^2 \operatorname{Im} S_f(v) + O(\epsilon^4).
\end{aligned}$$

By (5) and Taylor's formula, we deduce that

$$\text{(18)}\qquad \log \nu_\epsilon^{(2)}(\epsilon/2 + i\epsilon/2) = \epsilon^2 \operatorname{Im} S_f(\epsilon/2 + i\epsilon/2) + O(\epsilon^4).$$

So we conclude from (16), (17) and (18) that

$$\log \nu_\epsilon^{(2)}(u) = \epsilon^2 \operatorname{Im} S_f(u) + \delta_\epsilon(u) O(\epsilon^2),$$

By the relationship between $\nu_\epsilon^{(2)}$ and $h_\epsilon^{(2)}$ and Taylor's formula, we obtain

$$h_\epsilon^{(2)}(u) = \operatorname{Im} S_f(u) + \delta_\epsilon(u)\epsilon^2 \cdot O(1),$$

which implies (ii). This completes the proof of the lemma. ∎

LEMMA 5. *Let $v \in V_\Omega^\epsilon$, $u \in \hat{V}_\Omega^\epsilon$, and suppose that the distance δ from v (respectively, u) to $\partial\Omega$ is greater than 2ϵ. Then*

$$h_\epsilon^{(1)}(v) \le C, \quad h_\epsilon^{(2)}(u) \le C$$

for some constant $C = C(\delta, f)$ which depends only on δ and f.

Proof. Note that $\operatorname{Re} S_f$ and $\operatorname{Im} S_f$ are bounded on compact subset $K \subset \Omega$ and $\delta_\epsilon = O(1/\epsilon)$ on K. By Lemma 4, we conclude that the lemma holds. ∎

Similar to the case of regular hexagonal lattice (see [8, §7]), for regular square lattice we have

LEMMA 6. *Suppose that* (i) *W is a subset of V_Ω^ϵ (or $\hat{V}_\Omega^\epsilon$);* (ii) *$u \in int\, W$;* (iii) *δ is the distance from u to $V_\Omega^\epsilon - W$ (or $\hat{V}_\Omega^\epsilon - W$). If $\psi : W \to \mathbb{R}$ is any function, then the inequality*

$$\delta|\partial_j^\epsilon \psi(u)| < 5\|\psi\| + (1/2)\delta^2 \|\Delta^\epsilon \psi\|_{int\, W} \tag{19}$$

holds for any $j = 0, 1, 2, 3$.

Proof. The proof of the Lemma is similar to that of [8, Lemma 7.1] where regular hexagonal lattices were investigated. ∎

THEOREM 1. *Let n be an integer, and let V_δ^ϵ (respectively, $\hat{V}_\delta^\epsilon$) be the set of vertices of V_Ω^ϵ (respectively, $\hat{V}_\Omega^\epsilon$) whose distance to $\partial\Omega$ is at least δ for each $\delta > 0$. Then there are constants $C = C(n, \delta)$ and $\mu = \mu(n, \delta) > 0$ such that*

$$\|\partial_{j_n}^\epsilon \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1} h_\epsilon^{(1)}\|_{V_\delta^\epsilon} < C, \quad \|\partial_{j_n}^\epsilon \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1} h_\epsilon^{(2)}\|_{\hat{V}_\delta^\epsilon} < C \tag{20}$$

hold for each $\epsilon < \mu$, and $j_0, j_1, \ldots, j_n \in \mathbb{Z}_4$.

Proof. The proof proceeds by induction on n. In the case $n = 0$, by Lemma 5, we get that (20) holds. So we suppose that $n > 0$, and that (20) holds for $n = 0, 1, 2, \ldots, n-1$. Let

$$\varphi_1 = \partial_{j_{n-1}} \cdots \partial_{j_1} h_\epsilon^{(1)}, \quad \varphi_2 = \partial_{j_{n-1}} \cdots \partial_{j_1} h_\epsilon^{(2)}.$$

Note that the operators Δ^ϵ and ∂_j can commute with each other, so we have

$$\Delta^\epsilon \varphi_1 = \Delta^\epsilon \partial_{j_{n-1}} \cdots \partial_{j_1} h_\epsilon^{(1)} = \partial_{j_{n-1}} \cdots \partial_{j_1} \Delta^\epsilon h_\epsilon^{(1)},$$

and

$$\Delta^\epsilon \varphi_2 = \Delta^\epsilon \partial_{j_{n-1}} \cdots \partial_{j_1} h_\epsilon^{(2)} = \partial_{j_{n-1}} \cdots \partial_{j_1} \Delta^\epsilon h_\epsilon^{(2)}.$$

By Lemma 4, it follows that

$$\Delta^\epsilon h_\epsilon^{(1)} = \Delta^\epsilon \operatorname{Re} S_f + \Delta^\epsilon(\epsilon^2 O(1)), \quad \Delta^\epsilon h_\epsilon^{(2)} = \Delta^\epsilon \operatorname{Im} S_f + \Delta^\epsilon(\delta_\epsilon \epsilon^2 O(1)).$$

Since $\Delta^\epsilon S_f = O(\epsilon^2)$ and $\delta_\epsilon = O(1/\epsilon)$ on compact subset $K \subset \Omega$, we have

$$\Delta^\epsilon h_\epsilon^{(1)} = O(\epsilon^2) + O(1), \quad \Delta^\epsilon h_\epsilon^{(2)} = O(\epsilon^2) + \Delta^\epsilon(\epsilon O(1))$$

on compact subset $K \subset \Omega$. Note that $O(1) \in C^\infty(\Omega)$, we deduce that there exists a constant $C_1 = C_1(\delta, n)$ such that

$$\|\Delta^\epsilon \varphi_1(v)\|_{V_\delta^\epsilon} \le C_1, \quad \|\Delta^\epsilon \varphi_2(v)\|_{\hat{V}_\delta^\epsilon} \le C_1.$$

Since $|\varphi_1|$ and $|\varphi_2|$ are bounded on V_δ^ϵ and $\hat{V}_\delta^\epsilon$, respectively, it follows from Lemma 4 that $|\partial\varphi_1|$ and $|\partial\varphi_2|$ are bounded on V_δ^ϵ and $\hat{V}_\delta^\epsilon$, respectively, which completes the induction step. So (20) holds for any integer n and any $j_0, j_1, \ldots, j_n \in \mathbb{Z}_4$. ∎

§4. The Möbius transformations of circle patterns

In this section, we construct the Möbius transformations T^ϵ through circle pattern P_Ω^ϵ such that they can be expressed by the discrete Schwarzians $h_\epsilon^{(1)}$ and $h_\epsilon^{(2)}$. By the boundedness of $h_\epsilon^{(1)}$ and $h_\epsilon^{(2)}$, we will prove that T^ϵ converges in $C^\infty(\Omega)$ to some Möbius transformation T as $\epsilon \to 0$ and obtain the relation between T and f.

Note that $0 \in V_\Omega^\epsilon$, so we may suppose that P_Ω^ϵ is normalized by Möbius transformations so that

$$q_0(0) = f(\omega_0), \;\; q_1(0) = f(\omega_1), \;\; q_3(0) = f(\omega_3), \tag{21}$$

where $\omega_j = i^j(1+i)\epsilon/2$ $(j = 0, 1, 2, 3)$. Then we have

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THEOREM 2. *For each $v \in int\, V^{\epsilon}_{\Omega}$, let $T^{\epsilon} = T^{\epsilon}(v)$ be the Möbius transformation that takes the three points ω_0, ω_1, ω_3 to the points $q_0(v)$, $q_1(v)$, $q_3(v)$, respectively. Then*

(i) *the limit*

$$T(z) = \lim_{\epsilon \to 0} T^{\epsilon}(v) \tag{22}$$

exists for any $\epsilon \to 0$, and the convergence is in $C^{\infty}(\Omega)$.

(ii) $T(z)(0) = f(z)$.

Proof. (i) Let $B = B^{\epsilon}$ be the Möbius transformation that takes ∞, 0, 1 to ω_0, ω_1, ω_3, respectively. Then we have

$$B(z) = \frac{\epsilon z - (1+i)\epsilon/2}{(1-i)z + i}. \tag{23}$$

Recall the definition of M_w and $M_{[w_1,w_2]}$ as in Section 3, we deduce that

$$T^{\epsilon}(v) \circ B = M_{v+(1+i)\epsilon/4}.$$

and

$$T^{\epsilon}(v)^{-1} \circ T^{\epsilon}(v+\epsilon) = B \circ M_{[v+\omega_0/2, v+\omega_0/2+\epsilon/2]} \circ M_{[v+\omega_0+\epsilon/2, v+\omega_0+\epsilon]} \circ B^{-1}.$$

Using the usual matrix notation for Möbius transformations and the fact that $M_{[w_1,w_2]} = M^{-1}_{[w_2,w_1]}$, we obtain from (10), (12) and (23) that

$$\begin{aligned} &T^{\epsilon}(v)^{-1} \circ T^{\epsilon}(v+\epsilon) \\ &= \begin{pmatrix} \epsilon & -(1+i)\epsilon/2 \\ 1-i & i \end{pmatrix} \circ \begin{pmatrix} -i\epsilon(h^{(2)}_{\epsilon}(v+\omega_0)+1)^{1/2} & 1 \\ 0 & 1 \end{pmatrix} \\ &\quad \circ \begin{pmatrix} 0 & (\epsilon^{-2}(h^{(1)}_{\epsilon}(v+\epsilon)+1)^{-1}+1)^{-1} \\ -1 & -1 \end{pmatrix}^{-1} \circ \begin{pmatrix} \epsilon & -(1+i)\epsilon/2 \\ 1-i & 1 \end{pmatrix}^{-1}. \end{aligned}$$

By Lemma 4 and noting that S_f is Lipschitz, we deduce that

$$T^{\epsilon}(v)^{-1} \circ T^{\epsilon}(v+\epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon S_f(v)/2 & 1 \end{pmatrix} + O(\epsilon^2) = I + \epsilon C(v) + O(\epsilon^2),$$

where I denotes the identity matrix and

$$C(v) = \begin{pmatrix} 0 & 1 \\ -S_f(v)/2 & 0 \end{pmatrix}.$$

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This implies

$$T^{\epsilon}(v+\epsilon) = T^{\epsilon}(v) + \epsilon T^{\epsilon}(v)C(v) + T^{\epsilon}(v)O(\epsilon^2). \tag{24}$$

Similarly, we have

$$T^{\epsilon}(v+i^j\epsilon) = T^{\epsilon}(v) + i^j\epsilon T^{\epsilon}(v)C(v) + T^{\epsilon}(v)O(\epsilon^2) \tag{25}$$

for $j = 1, 2, 3$. We assume, without loss of generality, that

$$f(0) = 0, \;\; f'(0) = 1, \;\; f''(0) = 0,$$

because the statement of Theorem 2 is Möbius invariant. Thus we get from Taylor's formula that

$$f(i^j(1+i)\epsilon/2) = i^j(1+i)\epsilon/2 + O(\epsilon^3)$$

for $j = 0, 1, 2, 3$. From (21) and the definition of $T^{\epsilon}(v)$, it follows that

$$T^{\epsilon}(0)(i^j(1+i)\epsilon/2) = i^j(1+i)\epsilon/2 + O(\epsilon^3)$$

for $j = 0, 1, 3$, which implies

$$T^{\epsilon}(0) = I + O(\epsilon).$$

By (24) and (25), we deduce that the matrices $T^{\epsilon}(v)$ $(v \in \mathit{int}\, V_{\Omega}^{\epsilon})$ are bounded in compact subsets of Ω, independently of ϵ. On the other hand, (24) and (25) imply

$$\partial_j^{\epsilon} T^{\epsilon}(v) = i^j T^{\epsilon}(v)\cdot C(v) + T^{\epsilon}(v)\cdot O(\epsilon) = T^{\epsilon}(v)\cdot O(1) \tag{26}$$

for $j = 0, 1, 2, 3$, where $O(1) = i^j C(v) + O(\epsilon)$. It follows from Theorem 1 that $h_{\epsilon}^{(1)}$ and $h_{\epsilon}^{(2)}$ are C^{∞}-bounded, so $O(1)$ is bounded in $C^{\infty}(\Omega)$. By repeating differentiation of (26), we conclude that $T^{\epsilon}(v)$ is bounded in $C^{\infty}(\Omega)$ uniformly in ϵ. By the properties of C^{∞}-convergence of functions (see [8, Lemma 2.1]), we obtain that (22) holds for some subsequence of $\epsilon \to 0$, and the convergence is $C^{\infty}(\Omega)$.

In the following we will show that (22) is also valid for every sequence of $\epsilon \to 0$. Indeed, let $D(v)$ be the matrix solution of the differential equation

$$D'(v) = D(v)C(v) \tag{27}$$

with initial condition $D(0) = I$, then we have

$$D(v + i^j\epsilon) = D(v) + i^j\epsilon D(v)C(v) + O(\epsilon^2)$$

for $j = 0, 1, 2, 3$. From (24) and (25), we obtain

$$\begin{aligned}
&|T^\epsilon(v + i^j\epsilon) - D(v + i^j\epsilon)| \\
&\quad \le |T^\epsilon(v) - D(v)| + \epsilon|(T^\epsilon(v) - D(v))C(v)| + (1 + |T^\epsilon(v)|)O(\epsilon^2) \\
&\quad \le |T^\epsilon(v) - D(v)|(1 + O(\epsilon)) + (1 + |T^\epsilon(v)|)O(\epsilon^2).
\end{aligned}$$

Note that $T^\epsilon(0) = I + O(\epsilon) = C(0) + O(\epsilon)$, we deduce

$$|T^\epsilon(v) - D(v)| = (\delta(v)O(\epsilon^2) + O(\epsilon))(1 + O(\epsilon))^{\delta(v)},$$

where $\delta(v)$ is the combinatorial distance from v to 0 in SG^ϵ. Hence we deduce

$$|T^\epsilon(v) - D(v)| \le O(\epsilon)e^{O(1)} = O(\epsilon)$$

on a compact subset $K \in \Omega$, because $\delta(v) = O(1/\epsilon)$ on K. So $T^\epsilon(v) = D(v) + O(\epsilon)$, which implies that (22) holds for every $\epsilon \to 0$.

In equation (26), taking a limit as $\epsilon \to 0$, we obtain

$$\partial_j T(z) = i^j T(z)C(z). \tag{28}$$

Hence, we get

$$\partial_j T(z) = -\partial_{j+2}T(z),$$

which implies that $T(z)$ is a matrix-valued analytic function of z. In addition, it follows from (28) that the determinant of $T(z)$ is constant in Ω. Note that at $z = 0$ this determinant is 1. So $T(z)$ is a Möbius transformation for each $z \in \Omega$.

(ii) Let

$$T(z) = \begin{pmatrix} a(v) & b(v) \\ c(v) & d(v) \end{pmatrix}.$$

Then $T(z)$ satisfies the differential equation (27). By the definitions of Schwarzian derivative and $C(v)$, we deduce that

$$S_{b/d} = S_f.$$

Note that $b/d(0) = f(0) = 0$, $(b/d)'(0) = f'(0) = 1$, $(b/d)''(0) = f''(0) = 0$. So we conclude that $T(z)(0) = b(z)/d(z) = f(z)$. □

§5. C^∞-convergence to minimal surfaces

We will show that there exists a family of discrete minimal surface $\Gamma^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ which converges in $C^\infty(\Omega)$ to the smooth minimal surface $F : \Omega \to \mathbb{R}^3$ in this section.

We first give some properties of C^∞-convergence of discrete functions, that is

LEMMA 7. *Suppose that* f^ϵ, g^ϵ *converge in* $C^\infty(\Omega)$ *to functions* $f, g : \Omega \to \mathbb{C}$, *respectively. Then the following statements hold in* $C^\infty(\Omega)$: (i) $f^\epsilon + g^\epsilon \to f + g$; (ii) $f^\epsilon g^\epsilon \to fg$; (iii) $1/f^\epsilon \to 1/f$ *if* $f \neq 0$ *in* Ω; (iv) $\sqrt{f^\epsilon} \to \sqrt{f}$ *if* $f^\epsilon > 0$; (v) $|f^\epsilon| \to |f|$.

Proof. Using the definition of discrete directional derivative as well as the definition of C^∞-convergence of discrete function, we deduce easily that the lemma holds. □

For each $v \in V_\Omega^\epsilon$, let $f^\epsilon(v)$ and $r^\epsilon(v)$ denote the center and radius of circle $P^\epsilon(v)$ in the circle pattern P_Ω^ϵ, respectively, and let $g^\epsilon(v)$ denote the intersection point $q_0(v)$ of circles $P^\epsilon(v)$ and $P^\epsilon(v + 2\omega_0)$ in P_Ω^ϵ. Then we have

LEMMA 8. *The discrete functions* $f^\epsilon(v)$ *and* $g^\epsilon(v)$ *converge in* $C^\infty(\Omega)$ *to the locally injective meromorphic function* $f : \Omega \to \mathbb{C}$, *and* $r^\epsilon(v)/\epsilon$ *converges in* $C^\infty(\Omega)$ *to* $|f'|$ *as* $\epsilon \to 0$.

Proof. First, we write $T^\epsilon(v)$ and $T(z)$ as matrices

$$T^\epsilon(v) = \begin{pmatrix} a^\epsilon(v) & b^\epsilon(v) \\ c^\epsilon(v) & d^\epsilon(v) \end{pmatrix}, \quad T(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Then it follows from Theorem 2(i) that $\omega_0 c^\epsilon + d^\epsilon$ and $\omega_0 a^\epsilon + b^\epsilon$ converge to d and b in $C^\infty(\Omega)$, respectively. By Theorem 2(ii), we obtain that $b(z)/d(z) = f(z)$. Since the determinant of $T(z)$ is nonzero, d is nonzero in Ω. By Lemma 7, we get that $(\omega_0 a^\epsilon + b^\epsilon)/(\omega_0 c^\epsilon + d^\epsilon) \to b/d$ in $C^\infty(\Omega)$. Note that $g^\epsilon(v) = T^\epsilon(v)(\omega_0)$, so we conclude that the discrete function $g^\epsilon \to f$ in $C^\infty(\Omega)$ as $\epsilon \to 0$.

Next, let c be the circle that contains the three points ω_0, ω_1, ω_3. Since $T^\epsilon(v)$ maps the three points ω_0, ω_1, ω_3 to $q_0(v)$, $q_1(v)$, $q_3(v)$, respectively, and since the three points $q_0(v)$, $q_1(v)$, and $q_3(v)$ lie in the circle $P^\epsilon(v)$, it follows that $T^\epsilon(v)$ maps the circle c onto $P^\epsilon(v)$. Hence it follows that $T^\epsilon(v)(0) = f^\epsilon(v)$. Similar to the above arguments, we conclude that $f^\epsilon(v) = T^\epsilon(v)(0)$ converges in $C^\infty(\Omega)$ to f as $\epsilon \to 0$.

Lastly, note that $\sqrt{r^\epsilon(v) + r^\epsilon(v+\epsilon)} = |f^\epsilon(v) - f^\epsilon(v+\epsilon)| = \epsilon|\partial_0^\epsilon f^\epsilon|$, $r^\epsilon(v) + r^\epsilon(v+2\omega_0) = |f^\epsilon(v) - f^\epsilon(v+2\omega_0)| = \epsilon|\partial_0^\epsilon f^\epsilon(v) + \partial_1 f^\epsilon(v+\epsilon)|$, and $\sqrt{r^\epsilon(v+\epsilon) + r^\epsilon(v+2\omega_0)} = |f^\epsilon(v+\epsilon) - f^\epsilon(v+2\omega_0)| = \epsilon|\partial_1^\epsilon f^\epsilon(v+\epsilon)|$. So we deduce

$$2r^\epsilon(v)/\epsilon = |\partial_0^\epsilon f^\epsilon(v) + \partial_1 f^\epsilon(v+\epsilon)| + \epsilon(|\partial_0^\epsilon f^\epsilon|^2 - |\partial_1^\epsilon f^\epsilon(v+\epsilon)|^2).$$

Because $\partial_j f^\epsilon$ converges in $C^\infty(\Omega)$ to f', it follows from Lemma 7 that $|\partial_j^\epsilon f|$ converges in $C^\infty(\Omega)$ to $|f'|$. Hence we obtain that r^ϵ/ϵ converges $C^\infty(\Omega)$ to $|f'|$. ∎

Now we give the following C^∞-convergence theorem for smooth minimal surfaces.

THEOREM 3. *Suppose that $F : \Omega \to \mathbb{R}^3$ is a minimal immersion without umbilic points in conformal curvature line coordinates. Then there exists a family of discrete surface $\Gamma^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ that converges in $C^\infty(\Omega)$ to F as $\epsilon \to 0$.*

Proof. It follows from Lemma 1 that the smooth minimal surface $F : \Omega \to \mathbb{R}^3$ is expressed by (1). Thus, to prove that there is a discrete minimal surface Γ^ϵ that converges in $C^\infty(\Omega)$ to F, it suffices to show that each component of Γ^ϵ converges in $C^\infty(\Omega)$ to the corresponding one of F.

For any $\epsilon > 0$, let $F^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ be the discrete minimal surface corresponding to F, which consists of spheres and circles, and $\tilde{F}^\epsilon : V_\Omega^\epsilon \to \mathbb{R}^3$ be the discrete minimal surface which is comprised of the centers of spheres and circles in F^ϵ. Then it follows from (6) that

(29)

$$\begin{aligned}
&\tilde{F}^\epsilon(v_s^{(1)}) - \tilde{F}^\epsilon(v_s^{(2)}) \\
&\quad = \mathrm{Re}\left(\frac{R(v_s^{(1)}) + R(v_s^{(2)})}{1+|g^\epsilon|^2}\frac{|f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})|}{f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})}(1-(g^\epsilon)^2, i(1+(g^\epsilon)^2), 2g^\epsilon)\right) \\
&\quad \triangleq \mathrm{Re}(\Lambda_1^\epsilon, \Lambda_2^\epsilon, \Lambda_3^\epsilon)
\end{aligned}$$

for any $v_s^{(1)}$, $v_s^{(2)}$ lying in the same square of SG_Ω^ϵ, where

$$\Lambda_1^\epsilon = \frac{R(v_s^{(1)}) + R(v_s^{(2)})}{1+|g^\epsilon|^2} \frac{|f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})|}{f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})}(1-(g^\epsilon)^2),$$
$$\Lambda_2^\epsilon = \frac{R(v_s^{(1)}) + R(v_s^{(2)})}{1+|g^\epsilon|^2} \frac{|f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})|}{f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})} i(1+(g^\epsilon)^2),$$
$$\Lambda_3^\epsilon = \frac{R(v_s^{(1)}) + R(v_s^{(2)})}{1+|g^\epsilon|^2} \frac{|f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})|}{f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)})}(2g^\epsilon).$$

By Lemma 8, we get that f^ϵ and g^ϵ converge in $C^\infty(\Omega)$ to f, and that $r^\epsilon(v)/\epsilon$ converges in $C^\infty(\Omega)$ to $|f'|$ as $\epsilon \to 0$. Moreover, it is easy to see that $(f^\epsilon(v_s^{(1)}) - f^\epsilon(v_s^{(2)}))/\epsilon$ converges in $C^\infty(\Omega)$ to $2f'$. On the other hand, we get from (7) that

$$R(v_s^{(k)}) = \left| \frac{1 + |f^\epsilon(v_s^{(k)})|^2 - |f^\epsilon(v_s^{(k)}) - g^\epsilon|^2}{2|f^\epsilon(v_s^{(k)}) - g^\epsilon|} \right|$$

for $k = 1, 2$. So we deduce from the definitions of Λ_l^ϵ $(l = 1, 2, 3)$ and Lemma 7 that $\epsilon\Lambda_1^\epsilon$, $\epsilon\Lambda_2^\epsilon$ and $\epsilon\Lambda_3^\epsilon$ converge in $C^\infty(\Omega)$ to $(1-f^2)/f'$, $i(1+f^2)/f'$ and $2f/f'$, respectively.

Set $\Gamma^\epsilon(v) = 2\epsilon^2 \tilde{F}^\epsilon(v)$ for each $v \in V_\Omega^\epsilon$, then we conclude from (29) that the discrete minimal surface Γ^ϵ converges in $C^\infty(\Omega)$ to the smooth minimal surface $F : \Omega \to \mathbb{R}^3$. This completes the proof of Theorem 3. □

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