Symplectic Geometry of the Moduli Space of Flat Connections on a Riemann Surface: Inductive Decompositions and Vanishing Theorems

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Abstract. This paper treats the moduli space $\mathcal{M}_{g,1}(\Lambda)$ of representations of the fundamental group of a Riemann surface of genus g with one boundary component which send the loop around the boundary to an element conjugate to $\exp \Lambda$, where Λ is in the fundamental alcove of a Lie algebra. We construct natural line bundles over $\mathcal{M}_{g,1}(\Lambda)$ and exhibit natural homology cycles representing the Poincaré dual of the first Chern class. We use these cycles to prove differential equations satisfied by the symplectic volumes of these spaces. Finally we give a bound on the degree of a nonvanishing element of a particular subring of the cohomology of the moduli space of stable bundles of coprime rank k and degree d.

1 Introduction

Let Σ be a compact Riemann surface of genus g, and G a compact connected Lie group with maximal torus T. The moduli space $\mathcal{M}(\Sigma) = \mathrm{Hom}\big(\pi_1(\Sigma), G\big)/G$ of equivalence classes (under conjugation) of representations of the fundamental group of Σ in G has an open dense subset equipped with the structure of a smooth symplectic manifold of dimension $(2g-2)(\dim G)$. This space has an alternative characterization in algebraic geometry, where it appears as a moduli space of semistable holomorphic bundles over Σ.

This paper continues our study of Hamiltonian group actions on appropriate subsets of $\mathcal{M}(\Sigma)$, which was initiated in [19] and continued in [20] and [21]: our previous work focused on the case $G = \mathrm{SU}(2)$. The theme of this paper is to study the functions $S_{g,1}(\cdot)$ encoding the volumes[1] of the moduli spaces $\mathcal{M}_{g,1}(\Lambda)$ of representations of the fundamental group of a Riemann surface of genus g with one boundary component, where the loop around the boundary is constrained to lie in the conjugacy class of $\exp(\Lambda)$. The volume $S_{g,1}(\Lambda)$ of $\mathcal{M}_{g,1}(\Lambda)$ is a piecewise polynomial function of Λ: this follows straightforwardly from the Duistermaat-Heckman theorem [5], since $\mathcal{M}_{g,1}(\Lambda)$ is the reduced space at the orbit $\mathcal{O}_\Lambda$ of an extended moduli space $\tilde{\mathcal{M}}_{g,1}$ equipped with a Hamiltonian action of G (see [14]). One would like to know the degree in Λ of the piecewise polynomial function $S_{g,1}(\Lambda)$. This degree may be studied by observing that there is a differential operator $\mathcal{L}$

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[1]In this paper the term 'volume' refers to the symplectic volume, or more generally to the integral of the top exterior power of a presymplectic form.

of degree n_+ (the number of positive roots of G) such that $\mathcal{L}^2 S_{g,1}$ is equal to $S_{g-1,1}$ times a constant depending only on the group G.

The operator $\mathcal{L}$ is studied using results of one of us [31], in which it was observed that $\mathcal{M}_{g,1}(\Lambda)$ is equipped with a natural torus bundle $V_{g,1}^{(1)}(\Lambda)$ and homology cycles representing the Poincaré dual to line bundles associated to $V_{g,1}^{(1)}(\Lambda)$ were identified. This method is developed further in Section 3 below.

Finally we use information about the degree of the function $S_{g,1}$ to obtain information about the cohomology of the moduli space $M(k,d)$ of semistable holomorphic vector bundles of coprime rank k and degree d and fixed determinant over Σ_0^g: this moduli space may equivalently be described as the space of representations of $\pi_1(\Sigma_1^g)$ into G which send the element of $\pi_1(\Sigma_1^g)$ represented by a loop around the boundary to the central element $e^{2\pi i d/k}\,\mathrm{diag}(1,\dots,1)$ (generating the center of SU(k)). The rational cohomology of the space $M(k,d)$ has a natural collection of generators $\{a_r, b_r^j, f_r\}$ (for $r = 2,\dots,k$ and $j = 1,\dots,2g$) [1]; the classes a_r, b_r^j and f_r have degrees $2r$, $2r-1$ and $2r-2$ respectively. Explicit formulas for the intersection numbers of powers of these generators were discovered by Witten [33] and proved recently by one of us in joint work with F. Kirwan [18]; nevertheless alternative approaches may shed considerable light on the origins and consequences of these intersection formulas. Knowledge of the degree of $S_{g,1}$ turns out to imply vanishing theorems for certain subrings of the rational cohomology of $M(k,d)$.

Our proof of the vanishing theorems is achieved using an argument due to Donaldson and Witten (spelled out in the case of rank 2 bundles in [4, Section 6, after (18)]): there is a suitable range of Λ for which $\mathcal{M}_{g,1}(\Lambda)$ fibers over $M(k,d)$ with fiber a coadjoint orbit $\mathcal{O}_{\Lambda-\tilde{c}}$ (for suitable $\tilde{c} \in \mathbf{t}$). The symplectic form on $\mathcal{M}_{g,1}(\Lambda)$ is equal to the pullback of the symplectic form on $M(k,d)$ plus a term linear in $\Lambda - \tilde{c}$. One finds that the coefficients in the polynomial in $\Lambda - \tilde{c}$ which encodes the symplectic volume of $\mathcal{M}_{g,1}(\Lambda)$ enable one to extract information about certain intersection pairings in the cohomology of $M(k,d)$. In particular, knowledge of the degree of the function $S_{g,1}$ enables us to conclude:

Theorem 4.11 *The product*

$$a_2^{m_2}\cdots a_k^{m_k}$$

vanishes if $\sum_{r\geq 2} r m_r > (2g-2)n_+$.

Theorem 4.22 *There is a nonvanishing element β in $H^*\big(M(k,d)\big)$ of the form $a_2^{m_2}\cdots a_k^{m_k}$ with $\sum_{r\geq 2} r m_r = (2g-2)n_+$.*

Theorem 4.22 tells us that Theorem 4.11 gives a *sharp* bound on the degree of a nonvanishing class of the form $\prod_r a_r^{m_r}$.

The ring generated by the $a_2,\dots,a_k$ has as a subring the *Pontrjagin ring*, in other words the ring generated by the Pontrjagin classes of the tangent bundle of $M(k,d)$. Theorem 4.22 provides a counterexample to a conjecture of Ne'eman [26], which is that the Pontrjagin ring vanishes above dimension $2gk^2 - 4g(k-1) + 2$. We prove that there is a nonvanishing element β of degree $(2g-2)k(k-1)$: this exceeds Ne'eman's conjectured bound when g is chosen sufficiently large. In fact one may find a nonvanishing element of the required degree which is in the Pontrjagin ring: see Remark 4.23.

Remark Earl and Kirwan have independently obtained the above vanishing theorem for the Pontrjagin ring and the sharpness of the bound, by directly using the formulas for intersection numbers in the cohomology of $M(k, d)$ given in [18]. We are grateful to them for providing us with a prepublication copy of their paper [7]. A different approach to these spaces, using heat kernel methods, is due to K. Liu [24], whose results give a different proof of the vanishing of the pairings of elements of the Pontrjagin ring with powers of the symplectic form ([24, Corollary 1, p. 575]; our Proposition 4.10 below).

The layout of this paper is as follows. In Section 2 we define the moduli spaces of representations that form the subject of this paper, and their construction via reduction of extended moduli spaces. In Section 3 we construct natural line bundles on $\mathcal{M}_{g,1}(\Lambda)$ (following [31]): in a construction similar to the construction given in [31], we exhibit natural homology cycles representing the Poincaré duals to the first Chern classes of these line bundles. Characterization of these cycles enables us to prove differential equations satisfied by the functions $S_{g,1}(\Lambda) = \mathrm{vol}\big(\mathcal{M}_{g,1}(\Lambda)\big)$: these are given at the end of Section 3, and generalize corresponding differential equations proved in [21] for the case $G = \mathrm{SU}(2)$. Finally in Section 4 we give a bound on the degree of a nonvanishing element of the ring generated by the classes $a_2, \ldots, a_k$ in the cohomology of $M(k, d)$, and show that there exists a nonvanishing element of this degree (in other words the bound is sharp).

2 Moduli Spaces of Representations

2.1 Preliminaries

Let G denote a compact connected semisimple Lie group with maximal torus T. The Lie algebras of G and T will be denoted $\mathbf{g}$ and $\mathbf{t}$ respectively, and their duals $\mathbf{g}^*$ and $\mathbf{t}^*$; we introduce the Weyl group W. Let $d = \dim(G)$ and $l = \dim(T)$. Let us fix a (closed) fundamental Weyl chamber $\mathbf{t}_+$ in $\mathbf{t}$, and correspondingly the dual fundamental Weyl chamber $\mathbf{t}_+^*$ in $\mathbf{t}^*$. Their interiors are denoted $\mathbf{t}_+^o$ and $(\mathbf{t}_+^*)^o$.

The integer lattice Λ^I in $\mathbf{t}$ is the kernel of the exponential map $\exp\colon \mathbf{t} \to T$. Its dual lattice is the lattice Λ^w in $\mathbf{t}^*$ (the *weight lattice*).

We introduce the standard alternating character $\epsilon\colon W \to \{\pm 1\}$ on W.

The canonical pairing $\mathbf{t} \otimes \mathbf{t}^* \to \mathbf{R}$ is denoted $(\cdot, \cdot)$.

Definition 2.1 (**The Open Alcove**)   Let D_+^o be the interior of the *fundamental alcove* in $\mathbf{t}_+^o$: D_+^o is the subset of $\mathbf{t}$ cut out by the intersection of the half spaces

$$\{\xi : (\alpha_j, \xi) > 0\},$$
$$\{\xi : (\alpha_{\max}, \xi) < 1\}.$$

Here, α_j run over the simple roots of G, and $\alpha_{\max}$ is the *highest root* (the highest weight of the adjoint representation).

Definition 2.2 (**The Closed Alcove**)   Let $D_+ \subset \mathbf{t}_+$ denote the closure of D_+^o in $\mathbf{t}$.

Definition 2.3   An element $\Lambda \in D_+$ is regular if it is in D_+^o.

The alcove is a fundamental domain for the action on $\mathbf{t}$ of the *affine Weyl group* W_{aff}, which is the semidirect product of the ordinary Weyl group and the integer lattice Λ^I. (See [29, Section 5.1].)

Definition 2.4 (Inversion map) If $\Lambda \in D_+^o$ we define $\bar{\Lambda}$ to be the element of D_+^o for which

$$\exp \bar{\Lambda} = (\exp \Lambda)^{-1} = \exp(-\Lambda).$$

Often we shall implicitly identify D_+ with $\mathbf{t}/W_{\mathrm{aff}}$ (in other words with T/W), and the equivalence class of $\bar{\Lambda}$ in $\mathbf{t}/W_{\mathrm{aff}}$ is the equivalence class of $-\Lambda$.

The following are immediate:

Lemma 2.5

(a) Let q denote the quotient map $T \to T/W$, and let $\exp\colon \mathbf{t} \to T$ denote the exponential map. Then the map $r\colon D_+ \to T/W$ defined by $r = q \circ \exp$ is surjective.
(b) The restriction of the map r to D_+^o is a local diffeomorphism.

Part (b) follows from the formula for the differential of the exponential map (see [11, Theorem II.1.7]); this formula shows that the differential at Λ is surjective if $\alpha_i(\Lambda) \notin \mathbf{Z}$ for all roots α_i, which is true for $\Lambda \in D_+^o$. The set of conjugacy classes of G is in bijective correspondence with T/W; the conjugacy classes of *regular* elements in G (those in a unique maximal torus) are in bijective correspondence with D_+^o, or equivalently with its image in T/W under the exponential map.

We shall introduce an Ad-invariant inner product $\langle \cdot, \cdot \rangle\colon \mathbf{g} \otimes \mathbf{g} \to \mathbf{R}$ whose restriction to a Weyl-invariant inner product on $\mathbf{t}$ will also be denoted $\langle \cdot, \cdot \rangle$. We shall sometimes use this inner product to identify a subset of $\mathbf{t}_+^*$ with $D_+ \subset \mathbf{t}_+$. If G is simple, the invariant inner product $\langle \cdot, \cdot \rangle$ is unique up to multiplication by a positive constant.

Remark 2.6 Throughout this paper we shall make use of a collection of parameters $\Lambda \in D_+$. For the purposes of Section 2, these parameters could equivalently be replaced by their images $\exp \Lambda$ under the exponential map, which parametrize conjugacy classes in G. Later in the paper, however, we shall recall that the presymplectic structure defined on a class of presymplectic spaces (the extended moduli spaces: see Section 2.3) naturally identifies the parameter $\Lambda \in D_+$ with the value of the moment map for a torus action on these moduli spaces; from Section 3 onward we shall focus on the symplectic volumes of reduced spaces with respect to these torus actions, which are *piecewise polynomial* functions of the value of the moment map $\Lambda \in D_+$. We shall be particularly interested in the degrees of these polynomials. For this reason it is convenient for us to work with parameters Λ taking values in the fundamental alcove D_+, a subset of the vector space $\mathbf{t}$ (so that volumes of reduced spaces are polynomial functions on open regions which are the complement of finitely many hyperplanes in D_+) and not in terms of their images under the exponential map: we keep this notation throughout the paper.

2.2   Definitions of Moduli Spaces

Let Σ_n^g denote an oriented two-manifold of genus g with n oriented boundary components $S_1, \ldots, S_n$.

Definition 2.7   Let $\boldsymbol{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be a collection of n values in D_+. The moduli space of representations is defined by

$$\mathcal{M}_{g,n}(\boldsymbol{\Lambda}) = \mathcal{R}_{g,n}(\boldsymbol{\Lambda})/G,$$

where

$$(2.1) \qquad \mathcal{R}_{g,n}(\boldsymbol{\Lambda}) = \{\rho \in \mathrm{Hom}\big(\pi_1(\Sigma_n^g), G\big) : \rho([S_a]) \in \mathrm{Cl}(\exp \Lambda_a), a = 1, \ldots, n\}$$

and G acts on $\mathcal{R}_{g,n}(\boldsymbol{\Lambda})$ by conjugation. Here, $\mathrm{Cl}(\exp \Lambda_a)$ denotes the conjugacy class of $\exp \Lambda_a$ in G.

The fundamental group of Σ_n^g is the free group on $2g + n$ generators with one relation:

$$\pi_1(\Sigma_n^g) = \Big\langle x_1, \ldots, x_{2g}, c_1, \ldots, c_n : \prod_{j=1}^{g}[x_j, x_{j+g}] = \prod_{r=1}^{n} c_r \Big\rangle.$$

Thus we have

$$\mathcal{M}_{g,n}(\boldsymbol{\Lambda}) = \Big\{ (h_1, \ldots, h_{2g}, \beta_1, \ldots, \beta_n) \in G^{2g+n} :$$

$$(2.2) \qquad\qquad \prod_{j=1}^{g}[h_j, h_{j+g}] = \prod_{r=1}^{n} \beta_r, \beta_r \in \mathrm{Cl}(\exp \Lambda_r) \Big\}/G.$$

The *regular set* $\mathcal{M}_{g,n}^o(\boldsymbol{\Lambda})$ is the quotient by the G action of the subset $\mathcal{R}_{g,n}^o(\boldsymbol{\Lambda})$ of $\mathcal{R}_{g,n}(\boldsymbol{\Lambda})$ where the action of $G/Z(G)$ is free. We remark that the set $\mathcal{M}_{g,n}^o(\boldsymbol{\Lambda})$ is a smooth manifold, and is an open dense subset of $\mathcal{M}_{g,n}(\boldsymbol{\Lambda})$. In this article we shall chiefly be concerned with the case $n = 1$.

2.3   Extended Moduli Spaces

The spaces $\mathcal{M}_{g,1}(\Lambda)$ arise as reduced spaces of a class of spaces equipped with G-actions, the *extended moduli spaces* [14] (see also [10], [12], [15] and references in these articles): the parameter Λ is the value of the moment map for a torus action on the extended moduli space.

2.3.1   The Definition of Extended Moduli Spaces

The definition of the extended moduli spaces is as follows.

Definition 2.8 (**[14, (5.6)]**)   The extended moduli space associated to the Riemann surface Σ_1^g is

$$\tilde{\mathcal{M}}_{g,1} = \left\{ (h_1, \ldots, h_{2g}, \Lambda) \in G^{2g} \times \mathbf{g} : \prod_{j=1}^{n} [h_j, h_{j+g}] = e^{\Lambda} \right\}.$$

The action of G is defined in (5.9) and (5.10) of [14]. An element σ in G acts on $\tilde{\mathcal{M}}_{g,1}$ by sending

$$h_j \mapsto \sigma h_j \sigma^{-1} \quad \text{for } j = 1, \ldots, 2g$$

$$\Lambda \mapsto \sigma \Lambda \sigma^{-1}.$$

2.3.2   Abstract Moment Maps and Presymplectic Forms

Recent work of Karshon [23] has made it possible to enlarge the class of manifolds equipped with Hamiltonian group actions to include manifolds equipped with a 2-form which is closed but not necessarily nondegenerate. This subsection summarizes some of her results pertinent to the study of extended moduli spaces.

Let M be a smooth manifold equipped with the action of a compact group G.

Definition 2.9 (**Karshon**)   An abstract moment map for the action of G is a smooth function $\Phi \colon M \to \mathbf{g}^*$ satisfying the following conditions:

(i) Φ is G-equivariant (where G acts on $\mathbf{g}^*$ by the coadjoint action).
(ii)   For any subgroup H of G, inducing a projection $\pi_H \colon \mathbf{g}^* \to \mathbf{h}^*$, the value of the map $\pi_H \circ \Phi$ is constant on each component of the fixed point set $M^H \subset M$.

Karshon studies manifolds M equipped with an abstract moment map. She proves

Lemma 2.10 (**[23, Lemma 7.1]**)   *Let M be a smooth manifold equipped with the action of a torus T and an abstract moment map $\Phi \colon M \to \mathbf{t}^*$. Then if ξ is a regular value of Φ, the action of T on $\Phi^{-1}(\xi)$ is locally free.*

Thus it is possible to define reduced spaces $M_\xi = \Phi^{-1}(\xi)/T$ whenever M satisfies the hypotheses of the Lemma; if additionally the moment map Φ is proper, these reduced spaces are compact orbifolds for all regular values ξ.

An important example is the case when M is equipped with a 2-form ω which is closed but not necessarily nondegenerate: such a 2-form is called a *presymplectic structure* or *presymplectic form* on M.

Definition 2.11   Suppose M is a smooth manifold equipped with the action of a compact Lie group G, an abstract moment map $\Phi \colon M \to \mathbf{g}^*$, and a closed 2-form ω. Then Φ is Hamiltonian with respect to ω if for every element $X \in \mathbf{g}$ generating a vector field $X^{\#}$ on M, we have

(2.3) $$\iota_{X}^{\#}\omega = d\Phi_X,$$

where ι denotes the interior product and

$$\Phi_X(m) = \big(\Phi(m), X\big).$$

Remark 2.12 The idea of presymplectic reduction occurs also in [28] (Proposition 11).

Remark 2.13 If M is a smooth manifold equipped with a presymplectic form ω, a G-action and a smooth G-equivariant map $\Phi\colon M \to \mathbf{g}^*$ for which (2.3) is satisfied with respect to ω, then condition (ii) of Definition 2.9 is an immediate consequence of (2.3). In other words, G-equivariant maps that are Hamiltonian with respect to a presymplectic form ω are always abstract moment maps, even when ω is degenerate.

If the abstract moment map Φ is Hamiltonian with respect to a closed 2-form ω, then $\omega + \Phi$ is an equivariantly closed 2-form and hence defines an equivariant cohomology class (see [2, Chapter 7]).

If M is a presymplectic manifold equipped with the action of a torus T and a proper abstract moment map Φ, Karshon proves [23, Proposition 11.1] that the formula expressing the symplectic volume $v(\xi)$ of the reduced space M_ξ in terms of fixed point data generalizes to this situation. This formula shows that under these hypotheses, the volume is a piece-wise polynomial function of ξ with discontinuities at the images under Φ of the set in M where the action of T is not locally free. Karshon's proof shows that the result that $v(\xi)$ is piecewise polynomial (which is one part of the Duistermaat-Heckman theorem) depends neither on the assumption that M is compact nor on the assumption that ω is nondegenerate: it results from a local argument in a neighbourhood of a regular value ξ of Φ, and requires only that there be a neighbourhood of $\Phi^{-1}(\xi)$ in M on which Φ is Hamiltonian with respect to ω.

2.3.3 Extended Moduli Spaces Viewed as Presymplectic Manifolds

The results of Section 2.3.2 may straightforwardly be applied to the case of extended moduli spaces. We define a map $J\colon \tilde{\mathcal{M}}_{g,1} \to \mathbf{g}$ to be minus the projection onto $\mathbf{g}$: in other words, if we denote the points in $\tilde{\mathcal{M}}_{g,1}$ by $(\mathbf{h}, \Lambda)$ where $\mathbf{h} = (h_1, \ldots, h_{2g})$, then

$$J(\mathbf{h}, \Lambda) = -\Lambda.$$

One may additionally define spaces

$$\tilde{\mathcal{M}}_{g,1}^T = J^{-1}(\mathbf{t}_+);$$

$\tilde{\mathcal{M}}_{g,1}^T$ is equipped with an action of T.

Let $\tilde{\mathcal{M}}_{g,1}^{(s)} \subset G^{2g} \times \mathbf{g}$ be the smooth locus of the extended moduli space $\tilde{\mathcal{M}}_{g,1}$.

Lemma 2.14 *The space $\tilde{\mathcal{M}}_{g,1}^{(s)}$ is equipped with a smooth closed 2-form ω with respect to which the map $J\colon \tilde{\mathcal{M}}_{g,1}^{(s)} \to \mathbf{g}$ is Hamiltonian.*

Proof For the construction of the 2-form ω and the proofs that ω is closed and that the map J is Hamiltonian with respect to ω, see [15] and [16] (in the case $n = 1$); the generalization to Riemann surfaces with more then one boundary component is given in [13]. ∎

Lemma 2.15 *The function J is an abstract moment map in the sense of Definition 2.9.*

Proof Equivariance of J is immediate from the definition. By Remark 2.13, the result follows from Lemma 2.14. ∎

The condition for a point in $\tilde{\mathcal{M}}_{g,n}$ to be smooth is a straightforward consequence of the regular value theorem: when $n = 1$ it takes the following form [14, Theorem 5.1].

Proposition 2.16 *A point* $(\mathbf{h}, \Lambda)$ *of* $\tilde{\mathcal{M}}_{g,1}$ *is smooth when*

$$z(\mathbf{h})^\perp \otimes \mathbf{C} + \mathbf{t} \otimes \mathbf{C} + \bigoplus_{\alpha \in \triangle_\Lambda} X_\alpha = \mathbf{g} \otimes \mathbf{C}.$$

Here, $z(\mathbf{h})$ *is the Lie algebra of the centralizer of* $\mathbf{h}$ *under the conjugation action of G, while* $\triangle_\Lambda$ *is the set of roots α of* $\mathbf{g}$ *for which either* $\alpha(\Lambda) = 0$ *or* $\exp 2\pi i \alpha(\Lambda) \neq 1$, *and* $X_\alpha \subset \mathbf{t}^\perp \otimes \mathbf{C}$ *is the eigenspace of the adjoint action of T corresponding to the root α.*

A similar (though stronger) condition guarantees smoothness for $J^{-1}(\mathbf{t}_+) \stackrel{\text{def}}{=} \tilde{\mathcal{M}}_{g,1}^T$ (see [14, Theorem 5.2]):

Proposition 2.17 *A point* $(\mathbf{h}, \Lambda)$ *of* $\tilde{\mathcal{M}}_{g,1}^T$ *is smooth when*

$$z(\mathbf{h})^\perp + \mathbf{t} = \mathbf{g}.$$

It follows that if Λ is regular, $(\mathbf{h}, \Lambda)$ is smooth (since the stabilizer of Λ is T, so the stabilizer of $\mathbf{h}$ must be contained in T).

The 2-form ω is nondegenerate on an open dense set of $\tilde{\mathcal{M}}_{g,1}^{(s)}$, but we are unable to find a sufficient condition for nondegeneracy at the point $(\mathbf{h}, \Lambda)$ which is weaker than the condition (which is a sufficient condition in the case $n = 1$) that the stabilizer of $\mathbf{h}$ under the conjugation action of G is finite. In view of Karshon's results, however, we need not restrict our analysis to points where ω is nondegenerate, but only to the smooth locus $\tilde{\mathcal{M}}_{g,1}^{(s)}$ of $\tilde{\mathcal{M}}_{g,1}$. We shall refer to the space $J^{-1}(\mathcal{O}_\Lambda)/G$ as the *reduced space* of $\tilde{\mathcal{M}}_{g,1}$ at $\mathcal{O}_\Lambda$ provided that $J^{-1}(\mathcal{O}_\Lambda) \subset \tilde{\mathcal{M}}_{g,1}^{(s)}$, even when ω is degenerate at some points in $J^{-1}(\mathcal{O}_\Lambda)/G$. Likewise we shall refer to the evaluation

$$\int_{\mathcal{M}_{g,n}(\mathbf{\Lambda})} \exp(\omega_{\mathbf{\Lambda}})$$

as the *volume* of $\mathcal{M}_{g,n}(\mathbf{\Lambda})$ even if ω is degenerate for some points in the preimage $J^{-1}(\Lambda) \subset \tilde{\mathcal{M}}_{g,1}^T$.

Proposition 2.18 *If* $J^{-1}(\mathcal{O}_\Lambda) \subset \tilde{\mathcal{M}}_{g,1}^{(s)}$, *then the reduced space*

$$J^{-1}(\mathcal{O}_\Lambda)/G$$

of the space $\tilde{\mathcal{M}}_{g,1}$ *is given by* $\mathcal{M}_{g,1}(\Lambda)$; *if Λ is regular it is also given by the reduced space* $J^{-1}(\Lambda)/T$ *of the space* $\tilde{\mathcal{M}}_{g,1}^T$.

3 Line Bundles on Moduli Spaces

In this section we shall define certain natural line bundles over the moduli space $\mathcal{M}_{g,n}(\mathbf{\Lambda})$ when $(\Lambda_1, \ldots, \Lambda_n) \in (D_+)^n$ satisfy appropriate hypotheses. We shall then study the Poincaré dual of the first Chern class of such a line bundle, by identifying a homology cycle representing it. This will be done by identifying the Poincaré dual of the first Chern class with the zero locus of a transversal section of the line bundle: this zero locus is then equipped with a canonical orientation. The procedure for specifying the orientation is reviewed in Section 3.1. In Section 3.2 we construct line bundles over the moduli spaces, while in Sections 3.3 and 3.4 we identify homology cycles representing the Poincaré dual to the first Chern classes of these line bundles. Section 3.4 contains the construction for general compact semisimple G, while Section 3.3 contains an alternative but more straightforward construction in the case $G = \mathrm{SU}(k)$, which avoids the use of roots and weights. (In fact the intersection of the cycles representing the Poincaré duals of the line bundles constructed in Section 3.4 is the same as that for the line bundles constructed in Section 3.3: see Remark 3.13.) Finally in Sections 3.5 and 3.6 we identify the intersections of homology cycles representing Poincaré duals of Chern classes of the line bundles constructed in Section 3.2.

3.1 Identification of the Chern Class Using the Zero Locus of a Transversal Section

It is well known (see for instance Section 12 of [3]) that the Poincaré dual of the top Chern class of a C^∞ vector bundle E (over a compact manifold M) with fibres $E_x \cong \mathbf{C}^m$ is given by the homology class of the zero locus of a C^∞ section $s \colon M \to E$ provided that $s(M)$ (viewed as a submanifold of the total space E) is transverse to the zero section of E.

Proposition 3.1   *Let M and N be compact oriented manifolds and let $f \colon M \to N$ be a C^∞ map. Let y be a regular value of f. Then $f^{-1}(y)$ is an oriented manifold whose orientation is determined by the exact sequence*

$$(3.1) \qquad\qquad 0 \to T_x\big(f^{-1}(y)\big) \to T_x M \xrightarrow{df_x} T_{f(x)}N \to 0.$$

The exact sequence (3.1) determines a canonical isomorphism

$$(3.2) \qquad\qquad \bigwedge^{\max} T_x M \cong \bigwedge^{\max} T_x\big(f^{-1}(y)\big) \otimes \bigwedge^{\max} T_{f(x)}N.$$

Recall that a section s of a complex vector bundle $E \to M$ is transverse to the zero section if and only if in any oriented local trivialization

$$E|_U \cong U \times \mathbf{C}^m$$

for which we may write s locally as a map $s|_U \colon U \to \mathbf{C}^m$, we suppose that zero is a regular value of $s|_U$.

Corollary 3.2 ([**3, Proposition 12.8**])   *Let M be an oriented manifold and let $E \to M$ be an (oriented) complex vector bundle of dimension m; suppose s is a section of E transverse to the zero section. Then the zero locus of s is an oriented submanifold of M oriented via the conditions of Proposition 3.1, which represents the Poincaré dual of $c_m(E)$.*

Remark 3.3 The bundle E appearing in Corollary 3.2 is assumed simply to be a C^∞ bundle: we are not assuming the base space is a complex manifold, much less that the bundle carries a holomorphic structure.

In fact we shall make use of the following special case:

Corollary 3.4 *Suppose M_1 and M_2 are compact oriented manifolds and $f\colon M_1 \to M_2$ a C^∞ map. Let U be an open neighbourhood of $f^{-1}(m)$ for some regular value m of f, and let $V \to U \subset M_1$ be a family of principal T-bundles with fiber a torus T and $s\colon V \to \mathbf{C}^N$ a family of T-equivariant maps, where T acts on $\mathbf{C}$ by a collection of weights $\alpha_j \in \mathrm{Hom}\big(T, U(1)\big)$, $j = 1, \ldots, N$. Suppose 0 is a regular value of $s\colon V \to \mathbf{C}^N$. Then the zero locus of the corresponding section of the complex line bundle $V \times_T \mathbf{C}^N$ is the base space of the principal T-bundle*

$$s^{-1}(0) \cap f^{-1}(0) \to s^{-1}(0) \cap f^{-1}(0)/T,$$

where the orientation of $s^{-1}(0) \cap f^{-1}(0) \subset V$ is given by Proposition 3.1 and Corollary 3.2 using

$$(df, ds)\colon T_x V \to T_{f(x)} M_2 \oplus \mathbf{C}^N.$$

A choice of orientation on T then yields an orientation on $s^{-1}(0) \cap f^{-1}(0)/T$.

3.2 Line Bundles and Homology Cycles in Moduli Spaces

Let $\mathcal{M}_{g,n}(\mathbf{\Lambda})$ denote the moduli space of representations of the fundamental group of a two-manifold with n boundary components, defined as in Section 2.2. This moduli space is equipped with a collection of natural complex line bundles (equivalently, principal circle bundles) which were constructed in [31]: we summarize this construction here.

For $m = 1, \ldots, n$ we define

(3.3)
$$V_{g,n}^{(m)}(\mathbf{\Lambda}) = \{\rho \in \mathrm{Hom}\big(\pi_1(\Sigma_n^g), G\big) : \rho([S_j]) \in \mathrm{Cl}\big(\exp(\Lambda_j)\big) \text{ and } \rho([S_m]) = \exp(\Lambda_m)\}.$$

Definition 3.5 The space $V_{g,n}^{(m)}(\mathbf{\Lambda})$ is *T-regular* if the action of $T/Z(G)$ by conjugation is free at all points of $V_{g,n}^{(m)}(\mathbf{\Lambda})$.

If $V_{g,n}^{(m)}(\mathbf{\Lambda})$ is T-regular, then it fibers over $\mathcal{M}_{g,n}(\mathbf{\Lambda})$ with fiber $T/Z(G)$.

Example If $n = 1$, the set of Λ for which $V_{g,1}^{(1)}(\Lambda)$ is T-regular is the complement of a finite set of hyperplanes in D_+: it includes some points of the boundary of D_+.

Let $\alpha \in \mathbf{t}^*$ be a weight of T, in other words $\alpha(\Lambda^I) \subseteq \mathbf{Z}$, so α exponentiates to $\tilde{\alpha} \in \mathrm{Hom}\big(T, U(1)\big)$. Then we may use α to define a complex line bundle $V_{g,n,\alpha}^{(m)}(\mathbf{\Lambda})$ over $\mathcal{M}_{g,n}(\mathbf{\Lambda})$:

(3.4)
$$V_{g,n,\alpha}^{(m)}(\mathbf{\Lambda}) = V_{g,n}^{(m)}(\mathbf{\Lambda}) \times_{T,\alpha} \mathbf{C}$$

where T acts on $V_{g,n}^{(m)}(\mathbf{\Lambda})$ on the right by conjugation, and $t \in T$ acts on $\mathbf{C}$ on the left via $\tilde{\alpha}(t^{-1})$.

The following is well known (see for instance [9]):

Proposition 3.6   *Suppose L is the total space of a principal T-bundle $L \to L/T$. Then the element $\kappa(\alpha)$ of $H^2(L/T)$ representing the equivariant cohomology class $\alpha \in H^2_T(\mathrm{pt})$ in $H^2_T(L)$ is the first Chern class of the complex line bundle*

$$L \times_{T,\alpha} \mathbf{C} \to L/T$$

associated to L by the weight α.

Here, we have used the map[2]

(3.5) $\kappa\colon H^*_T(\mathrm{pt}) \to H^*_T(L) \cong H^*(L/T).$

To determine homology cycles representing the Poincaré dual of $c_1\big(V^{(m)}_{g,n,\alpha}(\mathbf{\Lambda})\big)$ we choose one of the generators $Y \in \{x_1, \dots, x_{2g}, c_1, \dots, \hat{c_m}, \dots, c_n\}$ and identify a submanifold of real codimension 2 which is the zero locus of a transversal section of $V^{(m)}_{g,n,\alpha}(\mathbf{\Lambda})$. We shall pay particular attention to the case when the weight α is a root.

3.3   Poincaré Duals to Chern Classes: The Case $G = \mathrm{SU}(k)$

Let $G = \mathrm{SU}(k)$, in its standard representation as $k \times k$ unimodular complex matrices of determinant 1; we represent its maximal torus T as diagonal $k \times k$ complex matrices of determinant 1 with values in $U(1)$.

Throughout this section we shall assume all subsets $V^{(m)}_{g,n}(\mathbf{\Lambda})$ are T-regular. We shall treat the case when the weight α introduced in the previous subsection is a root γ_{ij} of G, given for $(v_1, \dots, v_k) \in \mathbf{t}$ by

$$\gamma_{ij}(v_1, \dots, v_k) = v_i - v_j$$

or the corresponding element $\tilde{\gamma}_{ij} \in \mathrm{Hom}\big(T, U(1)\big)$:

$$\tilde{\gamma}_{ij}(u_1, \dots, u_k) = u_i u_j^{-1}$$

(for $(u_1, \dots, u_k) \in T$: in other words the u_j are complex numbers of norm 1 and $u_1 \cdots u_k = 1$). Correspondingly we get complex line bundles $L_{ij} = V^{(m)}_{g,n,\gamma_{ij}}(\mathbf{\Lambda})$ over $\mathcal{M}_{g,n}(\mathbf{\Lambda})$, as in Section 3.2.

In this section we shall determine homology cycles representing the Poincaré dual to the first Chern class of these line bundles. Essentially this construction was first given in [31, Section 5]: the details of the presentation here are slightly different.

Proposition 3.7   *The homology cycle $D_{m,\gamma_{ij}}(Y)$ corresponding to the line bundle $V^{(m)}_{g,n,\gamma_{ij}}(\mathbf{\Lambda})$ lifts to the following cycle $\tilde{D}_{m,\gamma_{ij}}(Y)$ in $V^{(m)}_{g,n}(\mathbf{\Lambda})$, for any $Y \in \{x_1, \dots, x_{2g}, c_1, \dots, \hat{c_m}, \dots, c_n\}$:*

$$\tilde{D}_{m,\gamma_{ij}}(Y) = \{\rho \in V^{(m)}_{g,n}(\mathbf{\Lambda}) : \rho(Y)_{ij} = 0\}.$$

The orientation of $\tilde{D}_{m,\gamma_{ij}}(Y)$ is determined as in Corollary 3.2.

Proof For any $Y \in \{x_1, \dots, x_{2g}, c_1, \dots, \hat{c_m}, \dots, c_n\}$, a section $s^{(Y)}_{\gamma_{ij}}$ of $V^{(m)}_{g,n,\gamma_{ij}}(\mathbf{\Lambda})$ is defined which has $D_{m,\gamma_{ij}}(Y)$ as its zero locus: the section $s^{(Y)}_{\gamma_{ij}}$ is specified by the T-equivariant map s from $V^{(m)}_{g,n}(\mathbf{\Lambda})$ to $\mathbf{C}$ sending $\rho \in V^{(m)}_{g,n}(\mathbf{\Lambda})$ to $\rho(Y)_{ij}$. ∎

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[2] In this paper all cohomology groups and equivariant cohomology groups will have complex coefficients.

3.3.1 Normalization and Orientation of the Zero Locus of a Transversal Section

In this section let $G = \mathrm{SU}(k)$; we continue to assume $V_{g,n}^{(m)}(\mathbf{\Lambda})$ is T-regular. We identify the Poincaré dual of the Chern class of the line bundle

$$V_{g,n,\gamma_{ij}}^{(m)}(\mathbf{\Lambda})$$

as the zero locus $D_{m,\gamma_{ij}}(Y)$ of a section of a line bundle, in the case when Y is one of the generators $x_1, \ldots, x_{2g}$. The definition of the moduli space $\mathcal{M}_{g,n}(\mathbf{\Lambda})$ given at (2.2) is $\mathcal{M}_{g,n}(\mathbf{\Lambda}) = F^{-1}(\exp \Lambda_1)/T$ where

$$(3.6) \qquad\qquad F = F_1 F_2$$

and F_1 and F_2 are maps $G^{2g} \times \mathrm{Cl}(\exp \Lambda_2) \times \cdots \times \mathrm{Cl}(\exp \Lambda_n) \to G$ defined by

$$(3.7) \qquad\qquad F_1 \colon (\bar{h}, \bar{\beta}) \mapsto \prod_{j=1}^{g} [h_{2j-1}, h_{2j}]$$

and

$$(3.8) \qquad\qquad F_2 \colon (\bar{h}, \bar{\beta}) \mapsto \beta_n^{-1} \cdots \beta_2^{-1}.$$

Here, $\bar{h} = (h_1, \ldots, h_{2g}) \in G^{2g}$ while

$$\bar{\beta} = (\beta_2, \ldots, \beta_n) \in \mathrm{Cl}(\exp \Lambda_2) \times \cdots \times \mathrm{Cl}(\exp \Lambda_n).$$

For any values $(\Lambda_2, \ldots, \Lambda_n) \in D_+^{n-1}$, it follows from Sard's theorem that the set of Λ_1 for which $\exp(\Lambda_1)$ is a regular value of F is dense in D_+.

The section $s_\gamma^{(x_1)}$ corresponding to the generator $Y = x_1$ and the root $\gamma = \gamma_{ij}$ corresponding to a pair of positive integers $i < j$ are defined by the T-equivariant map $V_{g,n}^{(m)}(\mathbf{\Lambda}) \to \mathbf{C}$ given by

$$(3.9) \qquad\qquad s_{\gamma_{ij}}^{(x_1)}(\bar{h}, \bar{\beta}) = (h_1)_{ij}.$$

We see that zero is a regular value of this map. The exact sequence

$$(3.10) \qquad 0 \to \mathcal{T} \to \mathbf{g}^{2g} \oplus \bigoplus_{j=2}^{n} T(\mathcal{O}_{\Lambda_j}) \xrightarrow{(dF, ds_\gamma^{(x_1)})} \mathbf{g} \oplus \mathbf{C} \to 0$$

is used to orient the tangent space $\mathcal{T}$ to the zero locus $(s_\gamma^{(x_1)})^{-1}(0)$ (as in Proposition 3.1). The map $ds_\gamma^{(x_1)}$ is the projection from the first copy of $\mathbf{g} \cong \mathbf{t} \oplus \mathbf{C}^{n_+}$ onto the copy of $\mathbf{C}$ corresponding to the root $\gamma = \gamma_{ij}$.

This construction may be applied also to the section $s_\gamma^{(x_2)}$ corresponding to $Y = x_2$. This construction will be used in Section 3.7 to orient and normalize the intersection of the zero loci of the sections $s_\gamma^{(x_1)}$ and $s_\gamma^{(x_2)}$ as γ runs over positive roots parametrized by pairs of integers (i, j) with $1 \leq i < j \leq k$: this is a copy of $T \times T \times \mathcal{M}_{g-1,n}(\Lambda)$.

3.4 Poincaré Duals to Chern Classes: The Case of General G

In this section we give a construction of Poincaré duals to Chern classes of certain line bundles over $\mathcal{M}_{g,n}(\Lambda)$, in the case when G is a general compact connected semisimple Lie group. This construction is a generalization of the construction given in the previous section when $G = SU(k)$.

Let n_+ denote the number of positive roots. Let $\mathbf{t}^\perp = \{\sum_{k=1}^{2n_+} s_k \gamma_k, s_k \in \mathbf{R}\}$ be the orthocomplement of $\mathbf{t}$ in $\mathbf{g}$, which is the $\mathbf{R}$-linear span of the (positive and negative) roots. A choice of positive Weyl chamber identifies $\mathbf{t}^\perp$ with the $\mathbf{C}$-linear span of the positive roots γ_j, $j = 1, \ldots, n_+$. Then $\exp(\mathbf{t}^\perp)$ is a subset of G whose image under the natural map $G \to G/T$ is an open dense subset of G/T. This follows from the following two standard results about the *KAN decomposition*, replacing negative roots by positive roots in the obvious way.

Proposition 3.8 ([11, Chap. IX.2, Cor. 1.9])   *Let $\bar{N}$ be the subset of the complexification $G^{\mathbf{C}}$ of the compact semisimple Lie group G consisting of $\exp(\bar{\mathfrak{n}})$ where $\bar{\mathfrak{n}}$ consists of the linear span over $\mathbf{C}$ of the* negative *roots. Let $G^{\mathbf{C}} = GAN$ be the standard[3] decomposition of $G^{\mathbf{C}}$, so that every element $g \in G^{\mathbf{C}}$ is given by*

$$g = k(g) \exp H(g) n(g).$$

Then the map $\bar{N} \to G/T$ defined by $\bar{n} \mapsto k(\bar{n})T$ is a diffeomorphism of $\bar{N}$ onto an open submanifold of G/T whose complement consists of finitely many disjoint manifolds of lower dimension.

Proposition 3.9 ([30, Theorem 3.6.2])   *In the notation of Proposition 3.8, the exponential map*

$$\exp \colon \bar{\mathfrak{n}} \mapsto \bar{N}$$

is a diffeomorphism.

The following is an immediate consequence of the previous two propositions.

Proposition 3.10   *The subset*

$$\left\{ \exp\left(\sum_{k=1}^{n_+} z_k \gamma_k\right) h : z_k \in \mathbf{C}, h \in T \right\}$$

is an open dense subset of G, and this parametrization is unique: in other words, if

$$\exp\left(\sum_k z_k^1 \gamma_k\right) h_1 = \exp\left(\sum_k z_k^2 \gamma_k\right) h_2$$

then $z_k^1 = z_k^2$ for all $k = 1, \ldots, n_+$, and $h_1 = h_2$.   ∎

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[3] This decomposition is usually referred to as the *KAN decomposition*, where K (here denoted G) is the maximal compact subgroup of a complex semisimple Lie group (here denoted $G^{\mathbf{C}}$).

It follows that if $y \in T$ then

$$y\left(\exp\left(\sum_k z_k \gamma_k\right)h\right)y^{-1} = \exp\left(\sum_k z_k \tilde{\gamma}_k(y)\gamma_k\right)h.$$

Definition 3.11 Let Y be one of the generators $\{x_1, \ldots, x_{2g}, c_1, \ldots, \hat{c}_m, \ldots, c_n\}$ of $\pi_1(\Sigma_n^g)$. Let $\tilde{D}_{m,\gamma_j}(Y) \subset V_{g,n}^{(m)}(\boldsymbol{\Lambda})$ be defined by

$$\tilde{D}_{m,\gamma_j}(Y) = \left\{\rho \in V_{g,n}^{(m)}(\boldsymbol{\Lambda}) \subset \mathrm{Hom}\left(\pi_1(\Sigma_n^g), G\right) : \right.$$

$$\left. \rho(Y) = \exp\left(\sum_{k=1}^{n_+} z_k \gamma_k\right)h \text{ for some } h \in T \text{ with } z_j = 0\right\}.$$

Proposition 3.12 *Assume $V_{g,n}^{(m)}(\boldsymbol{\Lambda})$ is T-regular. The image $D_{m,\gamma_j}(Y)$ of $\tilde{D}_{m,\gamma_j}(Y)$ under the quotient map $V_{g,n}^{(m)}(\boldsymbol{\Lambda}) \to \mathcal{M}_{g,n}(\boldsymbol{\Lambda})$ is a submanifold of $\mathcal{M}_{g,n}(\boldsymbol{\Lambda})$ of real codimension 2, which is the zero locus of a transversal section of the line bundle $V_{g,n,\gamma_j}^{(m)}(\boldsymbol{\Lambda})$.*

Proof A section $s_{\gamma_j}^{(Y)}$ of the bundle

$$V_{g,n,\gamma_j}^{(m)}(\boldsymbol{\Lambda}) = V_{g,n}^{(m)}(\boldsymbol{\Lambda}) \times_{T,\gamma_j} \mathbf{C}$$

is equivalent to a T-equivariant map

$$s_{\gamma_j}^{(Y)} \colon V_{g,n}^{(m)}(\boldsymbol{\Lambda}) \to \mathbf{C},$$

defined as follows: if $\rho \in V_{g,n}^{(m)}(\boldsymbol{\Lambda})$ and $\rho(Y) = \exp(\sum_k z_k \gamma_k)h$ for some $h \in T$, we define

$$(3.11) \qquad\qquad\qquad s_{\gamma_j}^{(Y)}(\rho) = z_j. \qquad\qquad\qquad \blacksquare$$

Remark 3.13 We note that if we sum over all the positive roots we obtain a section of a vector bundle which is the direct sum of all the line bundles $V_{g,n,\gamma}^{(m)}(\boldsymbol{\Lambda})$; this section is equivalent to a T-equivariant map $s^{(Y)} \colon V_{g,n}^{(m)}(\boldsymbol{\Lambda}) \to \mathbf{t}^\perp$. It is easy to see that the zero locus $(s^{(Y)})^{-1}(0)$ is the same as the zero locus of the direct sum of the sections $\bigoplus_{i<j} s_{\gamma_{ij}}^{(Y)}$ constructed in Section 3.3 when $G = \mathrm{SU}(k)$. Furthermore the differentials $ds^{(Y)}$ and $\bigoplus_{i<j} ds_{\gamma_{ij}}^{(Y)}$ agree on the zero locus. Our primary purpose in studying these homology cycles is to identify this intersection: we thus see that for this purpose the construction given in the present section generalizes that given in the previous section.

3.5   Intersections of Homology Cycles

In this section we identify the intersection of homology cycles representing the Poincaré duals of the line bundles $V_{g,n,\gamma}^{(m)}(\mathbf{\Lambda})$ as γ ranges over the positive roots. Throughout this section we continue to assume $V_{g,n}^{(m)}(\mathbf{\Lambda})$ is T-regular. Without loss of generality we take $m = 1$.

Define homology cycles $D_\gamma(x_{2g-1})$ by $(s_\gamma^{(x_{2g-1})})^{-1}(0)$ where $s_\gamma^{(x_{2g-1})}$ is defined by (3.11).

Proposition 3.14   *The homology cycle*

$$D_{\mathcal{D}^2} = \bigcap_{\gamma > 0} D_\gamma(x_{2g-1}) \cap D_\gamma(x_{2g}) \subset \mathcal{M}_{g,n}(\mathbf{\Lambda})$$

represents the Poincaré dual of the class

(3.12) $\kappa(\mathcal{D}^2) \in H^*\big(\mathcal{M}_{g,n}(\mathbf{\Lambda})\big)$

which comes from $\mathcal{D}^2 \in H_T^*(\mathrm{pt})$ *via the map* $\kappa \colon H_T^*\big(V_{g,n}^{(m)}(\mathbf{\Lambda})\big) \to H^*\big(\mathcal{M}_{g,n}(\mathbf{\Lambda})\big)$ *defined in (3.5). The orientation of* $(s_\gamma^{(x_{2g-1})})^{-1}(0)$ *is specified by the exact sequence (3.10). We have defined* $\mathcal{D} \in H_T^*(\mathrm{pt}) \cong S(\mathbf{t}^*)$ *by*

$$\mathcal{D} = \prod_{\gamma > 0} \gamma,$$

in other words $\mathcal{D}$ *is the product of the positive roots.*

Proof   This follows since $D_\gamma(x_{2g-1})$ and $D_\gamma(x_{2g})$ are both homology cycles representing the Poincaré dual of $\kappa(\gamma)$, by Proposition 3.7, and since the section $\bigoplus_\gamma (s_\gamma^{(x_{2g-1})}) \oplus \bigoplus_\gamma (s_\gamma^{(x_{2g})})$ is a transversal section.   ∎

Proposition 3.15   *Suppose* $V_{g,n}^{(m)}(\mathbf{\Lambda})$ *is T-regular. Then the homology cycle* $D_{\mathcal{D}^2} \subset \mathcal{M}_{g,n}(\mathbf{\Lambda})$ *is identified with* $\mathcal{M}_{g-1,n}(\mathbf{\Lambda}) \times T \times T$ *by a diffeomorphism which identifies the respective (pre)symplectic forms.*

Proof   By Proposition 3.10, Definition 3.11 and Proposition 3.12, the homology cycle $D_{\mathcal{D}}$ may be represented in $V_{g,n}^{(m)}(\mathbf{\Lambda})$ by

$$\tilde{D}_{\mathcal{D}} = \{\rho \in V_{g,n}^{(m)}(\mathbf{\Lambda}) : \rho(x_{2g-1}) \in T\},$$

with the orientation determined by Corollary 3.4. Thus $\tilde{D}_{\mathcal{D}^2}$ may be represented by

$$\{\rho \in V_{g,n}^{(m)}(\mathbf{\Lambda}) : \rho(x_{2g-1}) \in T, \rho(x_{2g}) \in T\} = V_{g-1,n}^{(m)}(\mathbf{\Lambda}) \times T \times T,$$

again with the orientation determined by Corollary 3.4. The image of $\tilde{D}_{\mathcal{D}^2}$ under the natural quotient map $V_{g,n}^{(m)}(\mathbf{\Lambda}) \to \mathcal{M}_{g,n}(\mathbf{\Lambda})$ is the homology cycle

$$D_{\mathcal{D}^2} = \mathcal{M}_{g-1,n}(\mathbf{\Lambda}) \times T \times T.$$

It is not hard to see that the orientation of $\mathcal{M}_{g-1,n}(\Lambda) \times T \times T$ (where the orientation of $T \times T$ is obtained from the identification of (x_{2g-1}, x_{2g}) as a symplectically dual pair under the intersection form on $H_1(\Sigma_0^g)$ induced by the cup product) agrees with that of $D_{\mathcal{D}^2} = F_g^{-1}\big(\exp(-\Lambda_1)\big) \cap s^{-1}(0)$, where

$$F_g := F \colon G^{2g} \times \mathrm{Cl}(\exp \Lambda_2) \times \cdots \times \mathrm{Cl}(\exp \Lambda_n) \to G$$

was defined by (3.6), (3.7) and (3.8), and

(3.13) $\qquad\qquad s \colon G^{2g} \times \mathrm{Cl}(\exp \Lambda_2) \times \cdots \times \mathrm{Cl}(\exp \Lambda_n) \to \mathbf{C}^{n_+} \oplus \mathbf{C}^{n_+}$

was defined (in Section 3.3.1) by

(3.14) $\qquad s \colon (h_1, \ldots, h_{2g}, \beta_2, \ldots, \beta_n) \mapsto \bigoplus_{i<j} (h_{2g-1})_{ij} \oplus \bigoplus_{i<j} (h_{2g})_{ij},$

or in Section 3.4 (replacing $\mathbf{C}_+^n$ by $\mathbf{t}^\perp$) by

(3.15) $\qquad\qquad s \colon (h_1, \ldots, h_{2g}, \beta_2, \ldots, \beta_n) \mapsto z_{2g-1} \oplus z_{2g}$

if $h_i = t_i \exp(z_i)$ for $i = 2g - 1$ or $i = 2g$. Zero is obviously a regular value of (F, s). The exact sequence (very similar to (3.10))

(3.16) $\qquad\qquad 0 \to \mathcal{T}_g' \to \mathbf{g}^{2g} \oplus (\mathbf{t}^\perp)^{n-1} \xrightarrow{(dF_g, ds)} \mathbf{g} \oplus \mathbf{t}^\perp \oplus \mathbf{t}^\perp \to 0$

determines the orientation of $s^{-1}(0) \cap \mathcal{M}_{g,n}(\Lambda)$; here $\mathcal{T}_g'$ is the tangent space to $s^{-1}(0) \cap F_g^{-1}(0)$. It is related to the direct sum of the three exact sequences

(3.17) $\qquad\qquad 0 \to \mathcal{T}_{g-1} \to \mathbf{g}^{2g-2} \oplus (\mathbf{t}^\perp)^{n-1} \xrightarrow{dF_{g-1}} \mathbf{g} \to 0$

and

$$0 \to \mathbf{t} \oplus \mathbf{t} \to \mathbf{t} \oplus \mathbf{t} \to 0 \to 0$$

and

$$0 \to 0 \to \mathbf{t}^\perp \oplus \mathbf{t}^\perp \to \mathbf{t}^\perp \oplus \mathbf{t}^\perp \to 0;$$

here, $\mathcal{T}_{g-1}$ is the tangent space to $R_{g-1,n}(\Lambda)$ (in the notation of Definition 2.7). This yields an exact sequence

(3.18)

$$0 \to \mathcal{T}_{g-1} \oplus \mathbf{t} \oplus \mathbf{t} \to \mathbf{g}^{2g-2} \oplus (\mathbf{t} \oplus \mathbf{t}^\perp) \oplus (\mathbf{t} \oplus \mathbf{t}^\perp) \oplus (\mathbf{t}^\perp)^{n-1} \xrightarrow{(dF_{g-1}, ds)} \mathbf{g} \oplus \mathbf{t}^\perp \oplus \mathbf{t}^\perp \to 0.$$

Here, the sequence (3.17) determines the orientation of $\mathcal{M}_{g-1,n}(\Lambda)$. Closer examination reveals that these sequences determine the same orientation on $\mathcal{T}_g' \cong \mathcal{T}_{g-1} \oplus \mathbf{t} \oplus \mathbf{t}$: although the operator dF_g in the exact sequence (3.16) is different from the corresponding operator appearing in (3.18), their difference is a linear map coming from the terms $dF_g(X_{2g-1})$ and $dF_g(X_{2g})$ (where X_{2g-1} and X_{2g} are tangent vectors to the $(2g-1)$-th and $(2g)$-th copies of G in G^{2g}), which takes nonzero values only on the subspace $(\mathbf{t} \oplus \mathbf{t}^\perp) \oplus (\mathbf{t} \oplus \mathbf{t}^\perp)$ (see for instance (3.23) below); and examination of (3.23) shows that addition of this map does not change the sign of the determinant corresponding to the exact sequence. ∎

Remark Proposition 3.15 generalizes Proposition 4.3 and Proposition 3.5 of [31].

3.6   Riemann Surfaces with More than One Boundary Component: Normalization and Orientation of Poincaré Duals of Chern Classes

In this section we shall study the intersection

$$\bigcap_{\gamma>0} D_{m,\gamma}(c_{m'})$$

for some $m' \neq m$. Throughout this section we continue to assume $V_{g,n}^{(m)}(\mathbf{\Lambda})$ is T-regular. Without loss of generality we take $m = 1$ and $m' = 2$.

   We shall prove

Theorem 3.16   *Suppose $V_{g,n}^{(1)}(\mathbf{\Lambda})$ is T-regular. Then we have*

(3.19)
$$\bigcap_{\gamma>0} D_{1,\gamma}(c_2) = \sum_{w_2 \in W} \epsilon(w_2)\epsilon(w_1^{(w_2,\Lambda_1,\Lambda_2)}) \mathcal{M}_{g,n-1}\big(w_1^{(w_2,\Lambda_1,\Lambda_2)}(\Lambda_1 + w_2\Lambda_2), \Lambda_3, \ldots, \Lambda_n\big),$$

where $w_1^{(w_2,\Lambda_1,\Lambda_2)}$ is the element of W for which $\exp w_1^{(w_2,\Lambda_1,\Lambda_2)}(\Lambda_1 + w_2\Lambda_2) \in \exp D_+$. (This element is unique provided $\Lambda_1 + w_2\Lambda_2$ is not fixed by the action of any nontrivial element of the Weyl group.)

Proof   The space $\mathcal{M}_{g,n}(\mathbf{\Lambda})$ is defined as $F^{-1}\big(\exp(-\Lambda_1)\big)$ where

(3.20) $F = F_1 F_2 \colon G^{2g} \times \mathrm{Cl}(\exp \Lambda_2) \times \cdots \times \mathrm{Cl}(\exp \Lambda_n) \to G,$

and as in Section 3.3.1 we have

(3.21) $F_1 \colon (h_1, \ldots, h_{2g}, \beta_2, \ldots, \beta_n) \mapsto \prod_{j=1}^{g} [h_{2j-1}, h_{2j}]$

while

(3.22) $F_2 \colon (h_1, \ldots, h_{2g}, \beta_2, \ldots, \beta_n) \mapsto \beta_n^{-1} \beta_{n-1}^{-1} \cdots \beta_2^{-1}.$

Thus we have

$$dF = dF_1 F_2 + F_1 dF_2 = F_1 F_2 \big(\mathrm{Ad}\, F_2^{-1}(F_1^{-1} dF_1) + F_2^{-1} dF_2\big).$$

We identify the tangent space $T_h G$ with $\mathbf{g}$ by identifying $hX \in T_h G$ with $X \in \mathbf{g}$. Hence we see that

(3.23)
$$F_1^{-1} dF_1 \colon (X_1, \ldots, X_{2g})$$
$$\mapsto \sum_{j=1}^{g} \mathrm{Ad}\Big(\prod_{l>j}[h_{2l-1}, h_{2l}]\Big)^{-1} \big(\mathrm{Ad}(h_{2j}h_{2j-1})(X_{2j} - X_{2j-1})$$
$$+ \mathrm{Ad}(h_{2j}h_{2j-1}h_{2j}^{-1})X_{2j-1} - \mathrm{Ad}(h_{2j})X_{2j}\big).$$

If $\beta_j = \mathrm{Ad}(g_j)e^{\Lambda_j}$, we identify the tangent space to the conjugacy class $\mathrm{Cl}(e^{\Lambda_j})$ at the point β_j (regarding the conjugacy class as a submanifold of G) by identifying $\mathbf{t}^\perp$ with $\beta_j \mathrm{Ad}(g_j)\mathbf{t}^\perp$. (Notice that g_j is well defined up to right multiplication by an element of T, and $\mathrm{Ad}(g_j)\mathbf{t}^\perp$ depends only on the coset $g_j T$.)

We then find that

$$(3.24) \qquad -F_2^{-1}dF_2(Y_2,\ldots,Y_n) = \mathrm{Ad}(\beta_2\cdots\beta_n)Y_n + \cdots + \mathrm{Ad}(\beta_2\beta_3)Y_3 + \mathrm{Ad}(\beta_2)Y_2.$$

(This expression is well defined since the β_j are elements of G and the Y_j are elements of $\mathrm{Ad}(g_j)\mathbf{t}^\perp \subset \mathbf{g}$.)

A section of $\bigoplus_{\gamma>0} V_{g,n,\gamma}^{(1)}(\mathbf{\Lambda})$ is given by the T-equivariant map

$$s\colon (h_1,\ldots,h_{2g},\beta_2,\ldots,\beta_n) \mapsto \sum_{\gamma>0}(\beta_2)_\gamma,$$

where we define $(\beta_2)_\gamma = z_\gamma$ if $\beta_2 = \exp(\sum_{\gamma>0} z_\gamma\gamma)h$ for some $h \in T$ (as in (3.11)) and $z_\gamma \in \mathbf{C}$.

We see that $s(h_1,\ldots,h_{2g},\beta_2,\ldots,\beta_n) = 0$ if and only if β_2 is in T, in other words if and only if $\beta_2 = \exp(w_2\Lambda_2)$ for some $w_2 \in W$. Thus w_2 lifts to an element $g_2 \in N(T)$. In this case $Y_2 \in \mathbf{t}^\perp$ and

$$ds_{(\mathbf{h},\beta_2,\ldots,\beta_n)}(X_1,\ldots,X_{2g},Y_2,\ldots,Y_n) = \mathrm{Ad}(g_2)Y_2.$$

Zero is clearly a regular value of (F,s). We note that $\mathrm{Ad}(g_2)$ changes the orientation of $\mathbf{t}^\perp$ by $\epsilon(w_2)$.

Our Poincaré dual thus decomposes as

$$\coprod_{w_2\in W} \epsilon(w_2)\Big\{ (h_1,\ldots,h_{2g},\beta_3,\ldots,\beta_n) \in G^{2g}\times \mathrm{Cl}(\exp\Lambda_3)\times\cdots\times\mathrm{Cl}(\exp\Lambda_n) :$$

$$\prod_{j=1}^{g}[h_{2j-1},h_{2j}]\beta_n^{-1}\cdots\beta_3^{-1} = \exp(\Lambda_1 + w_2\Lambda_2)\Big\}\Big/T.$$

Since $\exp(\Lambda_1 + w_2\Lambda_2)$ is not necessarily in $\exp\mathbf{t}_+$, it is necessary to identify each component with a space

$$\Big\{ (h_1,\ldots,h_{2g},\beta_3,\ldots,\beta_n) \in G^{2g}\times \mathrm{Cl}(\exp\Lambda_3)\times\cdots\times\mathrm{Cl}(\exp\Lambda_n) :$$

$$(3.25) \qquad \prod_{j=1}^{g}[h_{2j-1},h_{2j}]\beta_n^{-1}\ldots\beta_3^{-1} = \exp w_1^{(w_2,\Lambda_1,\Lambda_2)}(\Lambda_1 + w_2\Lambda_2)\Big\}\Big/T,$$

where $w_1^{(w_2,\Lambda_1,\Lambda_2)}$ is chosen so that $\exp w_1^{(w_2,\Lambda_1,\Lambda_2)}(\Lambda_1 + w_2\Lambda_2) \in \exp\mathbf{t}_+$. This identification changes the orientation by $\epsilon(w_1^{(w_2,\Lambda_1,\Lambda_2)})$. We thus find that the exact sequence

$$(3.26) \qquad\qquad 0 \to \mathcal{T} \to \mathbf{g}^{2g}\oplus(\mathbf{t}^\perp)^{n-2} \xrightarrow{(dF,ds)} \mathbf{g} \to 0$$

used to orient the space in (3.25) is obtained from the exact sequence

$$(3.27) \qquad 0 \to \mathcal{T} \to \mathbf{g}^{2g} \oplus (\mathbf{t}^{\perp})^{n-1} \xrightarrow{(dF, ds)} \mathbf{g} \oplus \mathbf{t}^{\perp} \to 0$$

except that the orientations differ by a factor $\epsilon(w_2)\epsilon(w_1^{(w_2, \Lambda_1, \Lambda_2)})$. This completes the proof. ∎

3.7 A Differential Equation for the Volumes

Define a differential operator $\mathcal{L}$ by

$$(3.28) \qquad \mathcal{L}_{\Lambda} = \prod_{\gamma > 0} (D_{\gamma})_{\Lambda}.$$

Here, $(D_{\gamma})_{\Lambda}$ is the first order differential operator

$$(D_{\gamma})_{\Lambda} = \gamma(\partial_{\Lambda}),$$

where, if a weight $\alpha \in \Lambda^w$ is given in coordinates specified by a basis on $\mathbf{t}^*$ by $\alpha = (\alpha_1, \ldots, \alpha_l)$, we define

$$\alpha(\partial_{\Lambda}) = \sum_r \alpha_r \partial_{\lambda_r},$$

in terms of coordinates $\Lambda = (\lambda_1, \ldots, \lambda_l)$ on $\mathbf{t}$ determined by the dual basis of the basis on $\mathbf{t}^*$ which was used to determine the coordinates $(\alpha_1, \ldots, \alpha_l)$.

Lemma 3.17 *Assume $V_{g,1}^{(m)}(\Lambda)$ is T-regular. Then for any class $\eta \in H^* \big(\mathcal{M}_{g,n}(\boldsymbol{\Lambda}) \big)$ we have*

$$(3.29) \qquad \gamma(\partial_{\Lambda}) \int_{\mathcal{M}_{g,1}(\Lambda)} e^{\omega_{\Lambda}} \eta = \int_{\mathcal{M}_{g,1}(\Lambda)} c_1 \big(L_{\gamma}(\Lambda) \big) e^{\omega_{\Lambda}} \eta.$$

Proof The moduli spaces $\mathcal{M}_{g,1}(\Lambda)$ are obtained from the extended moduli spaces $\tilde{\mathcal{M}}_{g,1}$ by reduction at the coadjoint orbit $\mathcal{O}_{\Lambda}$ parametrized by Λ. Let ω_{Λ} denote the symplectic form on $\mathcal{M}_{g,1}(\Lambda)$. Then by the Duistermaat-Heckman theorem[4] we have that

$$\gamma(\partial_{\Lambda}) \omega_{\Lambda} = c_1 \big(L_{\gamma}(\Lambda) \big).$$

This immediately yields the Lemma. ∎

Theorem 3.18 *Assume $V_{g,1}^{(m)}(\Lambda)$ is T-regular. Then we have*

$$(\mathcal{L}_{\Lambda})^2 S_{g,1}(\Lambda) = C S_{g-1,1}(\Lambda)$$

where $C = \mathrm{vol}_{\omega}(T \times T)$. Here, the differential operator $\mathcal{L}_{\Lambda}$ was defined in (3.28).

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[4]This version of the Duistermaat-Heckman theorem may be proved without requiring that the 2-form ω is nondegenerate: see Section 2.3.2. It is a consequence of the role played by ω in equivariant cohomology, and requires only that Λ be *T*-regular.

Proof Applying Lemma 3.17 repeatedly we find that

$$(\mathcal{L}_\Lambda)^2 S_{g,1}(\Lambda) = \int_{\mathcal{M}_{g,1}(\Lambda)} \bigcup_{\gamma > 0} \Big(c_1\big(L_\gamma(\Lambda)\big) \Big)^2 e^{\omega_\Lambda}$$

$$= \int_{\mathcal{M}_{g,1}(\Lambda)} \kappa(\mathcal{D}^2) e^{\omega_\Lambda}$$

(3.30)

$$= \int_{D_{\mathcal{D}^2} \cap \mathcal{M}_{g,1}(\Lambda)} e^{\omega_\Lambda}$$

$$= \int_{\mathcal{M}_{g-1,1}(\Lambda) \times T \times T} e^{\omega_\Lambda}$$

by Proposition 3.15. This gives the result. ∎

Remark 3.19 The value of C (which is the integral of the symplectic form over $T \times T$) has been calculated in [18, Section 10], as the determinant of the Cartan matrix of G; for $\mathrm{SU}(k)$ we have $C = k$.

Remark 3.20 For $G = \mathrm{SU}(2)$, Theorem 3.18 was proved in [21, Proposition 10].

4   A Vanishing Theorem in the Cohomology of $M(k, d)$

In this section let $G = \mathrm{SU}(k)$ and let d be coprime to k. Then

$$c = e^{2\pi i d/k} \operatorname{diag}(1, \dots, 1)$$

is a generator of $Z(G)$. Let $M(k, d)$ denote the space of representations $\rho\colon \mathbb{F}^{2g} = \pi_1(\Sigma_1^g) \to G$ for which $\rho([S_1]) = c$, where $S_1 = \prod_{j=1}^{g}[x_{2j-1}, x_{2j}]$ is the product of commutators of the generators x_j of $\mathbb{F}^{2g}$. Then $M(k, d)$ is a smooth symplectic manifold: it appears in algebraic geometry as the moduli space of semistable holomorphic vector bundles on Σ_0^g of rank k, degree d and fixed determinant. In this section we shall prove the vanishing of a subring $R(k, d)$ of $H^*\big(M(k, d)\big)$ above degree $(2g - 2)k(k - 1)$, and prove also that this estimate is sharp (in other words that there is a nonvanishing element of $R(k, d)$ in degree $(2g - 2)k(k - 1)$). These results have been obtained independently by Earl and Kirwan [7].

The material in this final section is related to that in Section 3 since the key tool being used is the formula for $S_{g,1}(\Lambda)$ (see (4.36)). The function $S_{g,1}$ is given as a Fourier series: given the formula for $S_{1,1}$ and the differential equation in Theorem 3.18 relating $S_{g,1}$ to $S_{g-1,1}$, one would be able to integrate the recurrence relation for the volume functions and obtain the formula for $S_{g,1}$.

One reason for the importance of the ring $R(k, d)$ is that it contains the *Pontrjagin ring* of $M(k, d)$ (that is, the ring generated by the Pontrjagin classes of the tangent bundle).

4.1   Fibrations and Parabolic Bundles

Let $\tilde{c} \in \mathbf{t}$ be an element of the closed fundamental alcove D_+ satisfying $\exp \tilde{c} = c$. We have

Theorem 4.1 *There is a neighbourhood U of $\tilde{c}$ in $\mathbf{t}$ such that if $\Lambda \in U$ then there is a fibration $\pi\colon \mathcal{M}_{g,1}(\Lambda) \to M(k,d)$ with fiber $\mathcal{O}_{\Lambda-\tilde{c}}$.*

Further, the induced symplectic form ω_{Λ} on $\mathcal{M}_{g,1}(\Lambda)$ satisfies

$$(4.31) \qquad\qquad \omega_{\Lambda} = \pi^* \omega_{k,d} + \tilde{\Omega}_{\Lambda-\tilde{c}}$$

where $\omega_{k,d}$ is the symplectic form on $M(k,d)$ and $\tilde{\Omega}_{\Lambda-\tilde{c}}$ restricts on each fiber of π to the standard Kirillov-Kostant symplectic form $\Omega_{\Lambda-\tilde{c}}$ on the coadjoint orbit $\mathcal{O}_{\Lambda-\tilde{c}}$.

Proof This follows by general results regarding symplectic fibrations associated to reduction at a regular value: see [14, Theorem 6.1] for a proof. (Note that $M(k,d)$ is obtained by reducing an appropriate extended moduli space at a regular value of the moment map and that the 2-form ω is nondegenerate in a neighbourhood of the preimage of this regular value under the moment map: see Proposition 5.5 of [14].) ∎

A set of generators of the rational cohomology of $M(k,d)$ is denoted

$$a_r \in H^{2r}\big(M(k,d)\big),$$
$$b_r^j \in H^{2r-1}\big(M(k,d)\big),$$
$$f_r \in H^{2r-2}\big(M(k,d)\big)$$

(for $r = 2, \ldots, k$ and $j = 1, \ldots, 2g$). The class $f_2 \in H^2\big(M(k,d)\big)$ is the cohomology class of the symplectic form $\omega_{k,d}$ on $M(k,d)$. (See [1, Section 2].)

Remark 4.2 Here the notation $\mathcal{M}_{g,1}(\Lambda)$ refers to $\mathcal{M}_{g,1}([\Lambda])$ where $[\Lambda]$ is the equivalence class in $D_+ = \mathbf{t}/W_{\mathrm{aff}}$ of the element $\Lambda \in \mathbf{t}$.

Definition 4.3 Let $R(k,d)$ denote the subring of $H^*\big(M(k,d)\big)$ generated by the $a_2, \ldots, a_k$.

The *Pontrjagin ring* of $M(k,d)$ (the ring generated by the Pontrjagin classes of $M(k,d)$) is then a subring of $R(k,d)$: see [27] for a proof of this. An explicit characterization of this ring in terms of the generators of $R(k,d)$ has been given by Earl in [6, Lemma 7].

We now relate powers of the symplectic form of $\mathcal{M}_{g,1}(\Lambda)$ to the fibration given by Theorem 4.1, using standard properties of the cohomology of flag manifolds (see for example [3, Section 21]:

Proposition 4.4 *If Λ is a regular element of $\mathbf{t}^*$, the coadjoint orbit $\mathcal{O}_{\Lambda}$ is diffeomorphic to the homogeneous space G/T so its cohomology is given by*

$$(4.32) \qquad\qquad H^*(\mathcal{O}_{\Lambda}) \cong \frac{S(\mathbf{t}^*)}{S(\mathbf{t}^*)^W},$$

in other words the quotient of ring of polynomials on $\mathbf{t}$ by the subring of symmetric polynomials.

Proposition 4.5 *The space $\mathcal{M}_{g,1}(\Lambda)$ is symplectomorphic to a splitting manifold for the universal bundle $\mathbb{U}\mid_{M(k,d)\times\{\mathrm{pt}\}}$ over $M(k,d)\times\{\mathrm{pt}\}\subset M(k,d)\times\Sigma_0^g$: in other words*

$$\pi^*(\mathbb{U}\mid_{M(k,d)\times\{\mathrm{pt}\}})=L_1\oplus\cdots\oplus L_k$$

where $c_1(L_j)=e_j$ for a collection of classes e_j in $H^2(\mathcal{M}_{g,1}(\Lambda))$ (for $j=1,\ldots,k$). Here, when $j=1,\ldots,k-1$, e_j restricts on the fibers of π to the generator α_j ($j=1,\ldots,k-1$) of $H^2(G/T,\mathbf{Z})\cong H^1(T,\mathbf{Z})$ corresponding to the j-th fundamental weight of $\mathrm{SU}(k)$ (an element of $\mathrm{Hom}(T,U(1))$, which is isomorphic to $H^1(T,\mathbf{Z})$) and $e_k=-(e_1+\cdots+e_{k-1})$.

Proof This follows from the algebro-geometric description of the moduli space of parabolic bundles (see for instance [25]): it is the moduli space parametrizing holomorphic bundles over Σ_0^g together with a flag in the fiber of each bundle over a basepoint ($\{\mathrm{pt}\})\in\Sigma_0^g$. The flag structure enables us naturally to split the universal bundle into a sum of holomorphic line bundles. ∎

We have the following Proposition:

Proposition 4.6 *If τ_r is the r-th elementary symmetric polynomial (for $r=2,\ldots,k$) then $\tau_r(e_1,\ldots,e_k)=\pi^*a_r$, where $a_r=c_r(\mathbb{U}\mid_{M(k,d)\times\{\mathrm{pt}\}})$.*

Proof See [3, Section 21, p. 284] for results on the properties of splitting manifolds and flag bundles. There, it is proved that for a complex vector bundle E over a complex manifold M with splitting manifold $\mathrm{Fl}(E)$, we have

$$(4.33)\qquad H^*(\mathrm{Fl}(E))=\frac{H^*(M)[e_1,\ldots,e_k]}{\prod_{i=1}^k(1+e_i)=c(E)},$$

where the $e_j\in H^2(\mathrm{Fl}(E))$ restrict (for $j=1,\ldots,k$) on the fiber $U(k)/U(1)^k\cong G/T$ (where $G=\mathrm{SU}(k)$ and T is its maximal torus) to the images under the coboundary map (in the Leray-Serre spectral sequence) of the elements $H^1_{U(1)^k}(\{\mathrm{pt}\},\mathbf{Z})=\mathrm{Hom}(U(1)^k,U(1))$ given by a basis for the weight lattice of $U(1)^k$. ∎

The following is a standard result (see for instance [2, Lemma 7.22]):

Proposition 4.7 *Let $\alpha_1,\ldots,\alpha_k$ (subject to $\sum_{j=1}^k\alpha_j=0$) be the basis for $H^2_T(\{\mathrm{pt}\})$ (the second equivariant cohomology group of a point for the maximal torus T of $\mathrm{SU}(k)$) which was introduced in Proposition 4.5. Then the standard Kirillov-Kostant symplectic form $\Omega_{\Lambda-\tilde c}$ on $\mathcal{O}_{\Lambda-\tilde c}$ is given by*

$$\Omega_{\Lambda-\tilde c}=\sum_{j=1}^k(\Lambda-\tilde c)_j\alpha_j.$$

We thus have the following result:

Proposition 4.8 *Let $\Lambda \in U$ where U is as in Theorem 4.1. Then we have*

$$(4.34) \qquad \tau_2(\partial_\Lambda)^{m_2} \cdots \tau_k(\partial_\Lambda)^{m_k} S_{g,1}(\Lambda) = \mathrm{vol}(\mathcal{O}_{\Lambda-\bar{c}}) \int_{M(k,d)} e^{\omega_{k,d}} a_2^{m_2} \cdots a_k^{m_k} \sum_{s \geq 0} \beta_s,$$

where for $s \geq n_+$

$$(4.35) \qquad \int_{\mathcal{M}_{g,1}(\Lambda)} \frac{[\tilde{\Omega}_{\Lambda-\bar{c}}]^s}{s!} \prod_{r=2}^n \pi^* a_r = \int_{\mathcal{M}_{g,1}(\Lambda)} \frac{[\tilde{\Omega}_{\Lambda-\bar{c}}]^{n_+}}{n_+!} \pi^* \beta_s \prod_{r=2}^n \pi^* a_r,$$

so that $\beta_s \in H^*(M(k,d))$ has (piecewise) polynomial dependence on $\Lambda - \tilde{c}$. Furthermore, on each region of $\mathbf{t}$ where β_s depends polynomially on $\Lambda - \tilde{c}$, it is a homogeneous polynomial of degree $s - n_+$.

Proof By Theorem 4.1 and Proposition 4.6, we have

$$\tau_2(\partial_\Lambda)^{m_2} \cdots \tau_k(\partial_\Lambda)^{m_k} \int_{\mathcal{M}_{g,1}(\Lambda)} e^{\pi^* \omega_{k,d}} e^{\sum_j (\Lambda_j - c_j) e_j} = \int_{\mathcal{M}_{g,1}(\Lambda)} e^{\pi^* \omega_{k,d}} e^{\sum_j (\Lambda_j - c_j) e_j} \prod_{r=2}^k \pi^* (a_r)^{m_r}.$$

Applying the Leray-Hirsch theorem to the fibration given in Theorem 4.1, we see that $H^*\big(\mathcal{M}_{g,1}(\Lambda)\big)$ is a free $H^*(M(k,d))$-module with basis given by a basis of the vector space $S(\mathbf{t}^*)/S(\mathbf{t}^*)^W \cong H^*(\mathcal{O}_{\Lambda-\bar{c}})$. It follows that the class[5] in $H^*\big(\mathcal{M}_{g,1}(\Lambda)\big)$ which multiplies $[\omega_{\Lambda-\bar{c}}]^n_+/n_+!$ in the term $[\tilde{\Omega}_{\Lambda-\bar{c}}]^s/s!$ in the power series expansion of $\exp \tilde{\Omega}_{\Lambda-\bar{c}}$ may be taken to be a pullback $\pi^* \beta_s$ for some $\beta_s \in H^*(M(k,d))$. We integrate over the fiber $\mathcal{O}_{\Lambda-\bar{c}}$ of π to obtain the result. ∎

4.2 Vanishing Theorems for Subrings of $H^*\big(M(k,d)\big)$

We have

Proposition 4.9 *The function $S_{g,1}$ is a piecewise polynomial function on D_+ whose degree on each region where it is a polynomial is $\leq (2g - 1)n_+$.*

Proof A formula for $S_{g,1}$ (as well as for all the other $S_{g,n}$) was rigorously established by Witten in Section 4.7 of [32, (4.114)], using the identification of the symplectic measure on the moduli space $\mathcal{M}_{g,n}$ in terms of Reidemeister-Ray-Singer torsion.[6] As a function from $\mathbf{t}$ to $\mathbf{R}$, $S_{g,1}$ is periodic under translations by the integer lattice Λ^I, or in other words it is a function on the torus $T = \mathbf{t}/\Lambda^I$. Witten's formula [32, (4.114)] for $S_{g,1}$ is given in terms of the characters of irreducible representations of G. Using the Weyl character formula (see for instance [2, Section 8.2]), Witten's formula for $S_{g,1}$ may be recast as a Fourier series: when

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[5]If ω is a closed 2-form, we use the notation $[\omega]$ to denote the corresponding class in de Rham cohomology.

[6]Witten proved the formula for $S_{g,1}(\Lambda)$ for a dense set of Λ, namely those for which some integer multiple of Λ lies in the integer lattice Λ^I: this establishes it for all Λ in the complement of the finite set of hyperplanes in T where $S_{g,1}$ or its derivatives have discontinuities, since $S_{g,1}$ is a piecewise polynomial function of Λ, as explained in the Introduction.

Λ is in the fundamental alcove D_+^o, $S_{g,1}(\Lambda)$ is given for an appropriate overall constant $k(G)$ depending only on the group G by[7]

$$(4.36) \qquad S_{g,1}(\Lambda) = k(G) \sum_{\mu \in \Lambda^w \cap \mathbf{t}_+^o} \frac{1}{D(\mu)^{2g-1}} \sum_{w \in W} \epsilon(w) e^{i\langle w\Lambda, \mu \rangle}.$$

Here, μ runs over the intersection of the fundamental Weyl chamber with the weight lattice Λ^w, and $D(\mu)$ is the dimension of the irreducible representation of G with highest weight $\mu - \rho$ (where ρ is half the sum of the positive roots): the Weyl dimension formula gives this as

$$D(\mu) = \prod_{\gamma > 0} \frac{\langle \gamma, \mu \rangle}{\langle \gamma, \rho \rangle}.$$

In [33, Section 5, (5.26)–(5.31)], Witten shows that a Fourier series of this type has the property that for any polynomial P on $\mathbf{t}$ of degree $> (2g-1)n_+$,

$$P(\partial_\Lambda) S_{g,1}(\Lambda) = 0$$

for Λ in the complement of the finite set of hyperplanes in D_+ where $S_{g,1}$ or any of its derivatives have discontinuities. It follows that on each region where $S_{g,1}$ is a polynomial, its degree is $\leq (2g-1)n_+$. ∎

Remark In fact there exist open regions in $\mathbf{t}_+$ where the degree of $S_{g,1}$ is *equal* to $(2g-1)n_+$. This will be shown in Section 4.3.

Combining Propositions 4.8 and 4.9, we obtain the following result.

Proposition 4.10 *The intersection pairing*

$$\int_{M(k,d)} e^{\omega_{k,d}} a_2^{m_2} \cdots a_k^{m_k}$$

is equal to zero if $\sum_{r \geq 2} rm_r > (2g-2)n_+$.

Proof Suppose $\sum_{r \geq 2} rm_r > (2g-2)n_+$. Then by Proposition 4.9 $\prod_r \tau_r(\partial_\Lambda)^{m_r} S_{g,1}(\Lambda)$ is a piecewise polynomial function of Λ of degree $< n_+$. According to Proposition 4.8, this function must be divisible by $\mathrm{vol}(\mathcal{O}_{\Lambda - \bar{c}})$, which is a piecewise polynomial function of Λ of degree equal to n_+ (by Proposition 4.7); this is only possible if

$$\prod_r \tau_r(\partial_\Lambda)^{m_r} S_{g,1}(\Lambda) = 0.$$

This implies

$$\int_{M(k,d)} e^{\omega_{k,d}} \prod_r a_r^{m_r} \sum_{s \geq 0} \beta_s = 0$$

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[7]Explicitly, $k(G) = \#Z(G) \mathrm{Vol}(G)^{2g-1} (2\pi)^{-(2g-1)\dim G + \dim T} \mathrm{Vol}(T)^{-1}$ where all volumes are with respect to the bivariant metric $\langle \cdot, \cdot \rangle$ on G given by the invariant inner product on $\mathbf{g}$ normalized so that the highest root $\gamma_{\max}$ has $\langle \gamma_{\max}, \gamma_{\max} \rangle = 2$.

and in particular

$$\int_{M(k,d)} e^{\omega_{k,d}} \prod_r a_r^{m_r} = 0. \qquad \blacksquare$$

In fact a stronger result is true:

Theorem 4.11   *The product*

$$a_2^{m_2} \cdots a_k^{m_k}$$

vanishes if $\sum_{r \geq 2} r m_r > (2g - 2)n_+$.

We shall prove Theorem 4.11 by showing that if $\sum_{r \geq 2} r m_r > (2g - 2)n_+$ then the intersection pairings of $a_2^{m_2} \cdots a_k^{m_k}$ with *all* cohomology classes of complementary degree are zero: this will imply the result since $M(k, d)$ is a smooth Kähler manifold and thus satisfies Poincaré duality. In order to show the vanishing of these pairings, we shall need some recent results on relations in the cohomology of $M(k, d)$, found by Witten [33, (5.21)] and proved in Section 10 of [18]. These relations are logically independent of the formulas for the intersection numbers between the classes a_r and the Kähler class f_2, proved in Section 9 of [18]: together the relations and these formulas determine the structure of the cohomology ring of $M(k, d)$. To state these relations we must introduce the ring homomorphism $\kappa \colon S(\mathbf{t}^*)^W \to H^*\big(M(k, d)\big)$ defined by

$$\kappa(\tau_r) = (-1)^r a_r.$$

We note that the ring of Weyl invariant polynomials on $\mathbf{t}^*$ is generated by the elementary symmetric polynomials τ_r (where $r = 2, \ldots, k$). We also introduce a variable $X \in \mathbf{t}$ so that our Weyl invariant polynomials on $\mathbf{t}$ will be specified as $\tau \colon X \in \mathbf{t} \mapsto \tau(X) \in \mathbf{R}$.

For nonnegative integers $m_2, \ldots, m_k$ we define the symmetric polynomial τ by

$$(4.37) \qquad \qquad \tau(X) = \prod_{r=2}^{k} \tau_r(-X)^{m_r}.$$

It follows that $\kappa(\tau) = \prod_{r=2}^{k} a_r^{m_r}$. We also define the invariant polynomial

$$q = \tau_2 + \sum_{r=3}^{k} \delta_r \tau_r$$

(where $\delta_3, \ldots, \delta_k$ are formal nilpotent parameters). Using the invariant inner product on $\mathbf{g}$, the map $-dq \colon \mathbf{g} \to \mathbf{g}^*$ may be regarded as a G-equivariant map $B \colon \mathbf{g} \to \mathbf{g}$ specified by $B = B^{(2)} + \sum_{r=3}^{k} \delta_r B^{(r)}$, where $B^{(r)} \colon \mathbf{g} \to \mathbf{g}$. We find that $B^{(2)} = -d\tau_2 = \mathrm{id} \colon \mathbf{g} \to \mathbf{g}$. The map $B^{-1} \colon \mathbf{g} \to \mathbf{g}$ is the inverse of B: the inverse is also G-equivariant and may be written as a formal power series in the δ_r.

The Hessian of $-q$ is denoted H: it is a function from $\mathbf{g}$ to symmetric bilinear forms on $\mathbf{g}$. If k, l index an orthonormal basis $\{\hat{v}_k\}$ of $\mathbf{g}$ then the Hessian at X is the matrix

$$H(X)_{kl} = -(\partial^2 q)_X(\hat{v}_k, \hat{v}_l).$$

Proposition 4.12 ([**18, Proposition 10.2**]) *In terms of the above notation, we have*

(4.38)
$$\int_{M(k,d)} \prod_{r=2}^{k} \kappa\big(\tau(X)\big) \exp(f_2 + \delta_3 f_3 + \cdots + \delta_k f_k)$$
$$= \int_{M(k,d)} \kappa\Big(\tau\big(B^{-1}(-X)\big) \det H\big(B^{-1}(-X)\big)^{g-1}\Big) \exp f_2.$$

Let the invariant polynomial τ on $\mathbf{t}$ be as defined in (4.37). Let s_r^j be real parameters (for $r = 2, \ldots, k$ and $j = 1, \ldots, 2g$). Define an invariant polynomial $\hat{\tau}$ on $\mathbf{g}$ (or equivalently a Weyl invariant polynomial on $\mathbf{t}$, whose argument is denoted $X \in \mathbf{t}$) by

(4.39)
$$\hat{\tau}(-X) = -\sum_{a,b=1}^{k-1} \sum_{r,s=2}^{k} \sum_{j=1}^{g} s_r^j s_s^{j+g} (d\tau_r)_X(\hat{u}_a)(d\tau_s)_X(\hat{u}_b)\big((\partial^2 q)_X^{-1}\big)_{ab},$$

Here, $\{\hat{u}_a : a = 1, \ldots, k-1\}$ is an oriented orthonormal basis of $\mathbf{t}$, and $(\partial^2 q)_X^{-1}$ is the formal power series in the δ_r which is the inverse of the element $(\partial^2 q)_X \in \mathrm{End}(\mathbf{g})$: this power series is constructed using the identity

(4.40)
$$(1 + A)^{-1} = \sum_{r \geq 0} (-1)^r A^r$$

which is valid for elements $A \in \mathrm{End}(\mathbf{g})[[\delta_3, \ldots, \delta_k]]$ in the ring of formal power series in the variables $\delta_3, \ldots, \delta_k$. In fact $(\partial^2 q)_X$ is an element of $\mathrm{End}(\mathbf{g})$ which is equal to $1 + A$ for a matrix A of the form $\sum_{j=3}^{k} \delta_j A_j$, where the $(A_j)_X$ are elements of $\mathrm{End}(\mathbf{g})$ with polynomial dependence on $X \in \mathbf{t}$, so the construction in (4.40) applies and gives $\big((\partial^2 q)_X^{-1}\big)_{ab} = \delta_{ab} + \nu_{ab}$ where ν is a linear combination of homogeneous polynomials in X of degree ≥ 1.

Proposition 4.13 ([**18, Proposition 10.3**]) *In terms of the above notation, we have*

(4.41)
$$\int_{M(k,d)} \kappa\big(\tau(X)\big) \exp\Big(\sum_{r=2}^{k} \sum_{j=1}^{2g} s_r^j b_r^j\Big) \exp(f_2 + \delta_3 f_3 + \cdots + \delta_k f_k)$$
$$= \int_{M(k,d)} \kappa\big(\tau(X) \exp \hat{\tau}(X)\big) \exp(f_2 + \delta_3 f_3 + \cdots + \delta_k f_k).$$

Lemma 4.14 *Suppose $\tau \in S(\mathbf{g}^*)^G$ is an invariant polynomial of degree N. Then for all values of the formal parameters $\delta_3, \ldots, \delta_k$, the invariant polynomial $\tau\big(B^{-1}(-X)\big)$ is a linear combination of homogeneous invariant polynomials of degree $\geq N$ in X.*

Proof This follows because

$$B(X) = X + \delta_3 B^{(3)} + \cdots + \delta_k B^{(k)},$$

where the components of the **g**-valued maps $B^{(j)} = -d\tau_j\colon \mathbf{g} \to \mathbf{g}$ are homogeneous polynomials on **g** of degree $j - 1$. It follows that

$$B^{-1}(X) = X + \nu(X)$$

where the components of the **g**-valued map $\nu\colon \mathbf{g} \to \mathbf{g}$ are formal power series in the $\delta_3, \ldots, \delta_k$ which may be written as linear combinations of homogeneous polynomials of degree ≥ 2 in X. ∎

Lemma 4.15 *Suppose $\tau \in S(\mathbf{g}^*)^G$ is a homogeneous invariant polynomial on **g** of degree N. Then for all values of the real parameters s_r^j, the invariant polynomial $\tau(X) \exp \hat{\tau}(X)$ is a linear combination of homogeneous invariant polynomials of degree $\geq N$ in X.*

Proof This follows immediately from the Taylor series for the exponential function and the definition (4.39) of $\hat{\tau}$, which shows that it is a linear combination of homogeneous invariant polynomials on **g** of degree greater than or equal to 2. ∎

Using Propositions 4.12 and 4.13 and Lemmas 4.14 and 4.15 we thus see

Proposition 4.16 *If τ is an invariant polynomial on **g** of degree $\geq (2g - 2)n_+$ then*

$$\int_{M(k,d)} \kappa(\tau) e^{f_2 + \delta_3 f_3 + \cdots + \delta_k f_k} e^{\sum_{r=2}^{k} \sum_{j=1}^{2g} s_r^j b_r^j} = 0.$$

This implies that under these hypotheses the intersection pairing of $\kappa(\tau)$ with any class in the cohomology ring is zero. Since $M(k, d)$ satisfies Poincaré duality, it follows immediately that if $\sum_{r=2}^{k} r m_r > (2g - 2)n_+$ then $\prod_{r=2}^{k} a_r^{m_r}$ is zero. This completes the proof of Theorem 4.11.

In particular, we thus see that the Pontrjagin ring of $M(k, d)$ vanishes in degrees above $2(2g - 2)n_+ = (2g - 2)k(k - 1)$.

4.3 Sharpness of the Estimate

In this section we still let $G = \mathrm{SU}(k)$.

Theorem 4.17 *The volume $S_{1,1}(\Lambda)$ of the moduli space $\mathcal{M}_{1,1}(\Lambda)$ is a piecewise polynomial function of Λ of degree greater than or equal to n_+.*

Proof This moduli space is

$$(4.42) \qquad \{(h_1, h_2) \in G^2 : [h_1, h_2] = \exp \Lambda\}/T.$$

By the same argument as in the proof of Proposition 3.15, the Poincaré dual $D_{\mathcal{D}}$ of $\bigcup_{\gamma > 0} c_1(L_\gamma)$ lifts to the following cycle in $V_{g,1}^{(1)}(\Lambda)$:

$$(4.43) \qquad \tilde{D}_{\mathcal{D}} = \{(h_1, h_2) \in G^2 : [h_1, h_2] = \exp \Lambda, h_1 \in T\}.$$

Thus if $(h_1, h_2) \in \tilde{D}_{\mathcal{D}}$ we have

$$(4.44) \qquad h_2 h_1 h_2^{-1} = h_1 \exp(-\Lambda) \in T.$$

Thus $h_2 h_1 h_2^{-1} = w h_1$ and

$$(4.45) \qquad h_1(w h_1^{-1}) = \exp \Lambda.$$

Since h_1 is assumed to be in T, and $\Lambda \in \mathbf{t}$, this has solutions $h_1 = \exp \mu$ where

$$(4.46) \qquad (1 - w)\mu = \Lambda + \xi \quad \text{for some } \xi \in \Lambda^I.$$

We shall prove

$$(4.47) \qquad \mathcal{L}(\partial_\Lambda) S_{1,1}(\Lambda) = (\pm 1)\# \Big(\mathcal{M}_{1,1}(\Lambda) \cap \mathrm{PD}\Big[\bigcup_{\gamma > 0} c_1(L_\gamma) \Big] \Big);$$

we shall do this by showing that the Poincaré dual of $\bigcup_{\gamma > 0} c_1\big(V_{1,1,\gamma}^{(1)}(\Lambda)\big)$ in $\mathcal{M}_{1,1}(\Lambda)$ consists of isolated points all of which contribute with the same sign.

If w has any eigenvalues equal to 1, the image of the linear transformation $w - 1$ of $\mathbf{t}$ is a proper vector subspace of $\mathbf{t}$. We restrict to those Λ for which $\Lambda + \xi$ does not lie in this vector subspace for any $\xi \in \Lambda^I$, so that (4.46) will have solutions only when $w - 1$ is invertible as a transformation of $\mathbf{t}$, in other words when w does not have any eigenvalues 1. Since w acts on $\mathbf{t} \subset \mathbf{R}^k$ through the action of the permutation group on k elements, this happens if and only if the cycle decomposition of w consists of exactly one cycle of length k. We see that for all such w,

$$(4.48) \qquad \epsilon(w) = (-1)^{k-1}.$$

We now consider the orientation of $\mathrm{PD}\big(\bigcup_{i<j} c_1(L_{ij}) \big)$ where we have introduced the notation $L_{ij} = V_{1,1,\gamma_{ij}}^{(1)}(\Lambda)$. This is the orientation induced from the intersection $s(M) \cap M$ in the total space of $\bigoplus_{i<j} L_{ij}$ restricted to $M = \mathcal{M}_{1,1}(\Lambda)$ [3, Proposition 12.8] (see Proposition 3.1 and Corollaries 3.2 and 3.4).

Let $F \colon G \times G \to G$ be defined by

$$F(h_1, h_2) = h_1 h_2 h_1^{-1} h_2^{-1}.$$

Let G act on $G \times G$ and on G by the left adjoint action: then F is G-equivariant. We identify the tangent space $T_h G$ with $\mathbf{g}$ by identifying $X \in \mathbf{g}$ with the corresponding left-invariant vector field whose value at h is hX.

Proposition 4.18 *In terms of the above notation, we have*

$$(dF)_{(h_1,h_2)}(X_1, X_2) = \big(\mathrm{Ad}(h_2 h_1 h_2^{-1}) - \mathrm{Ad}(h_2 h_1)\big) X_1 + \big(\mathrm{Ad}(h_2 h_1) - \mathrm{Ad}(h_2)\big) X_2$$

or equivalently

$$(4.49) \quad (dF)_{(h_1,h_2)}(X_1, X_2) = \mathrm{Ad}(h_2 h_1 h_2^{-1})\big(1 - \mathrm{Ad}(h_2)\big) X_1 + \mathrm{Ad}(h_2)\big(\mathrm{Ad}(h_1) - 1\big) X_2.$$

We shall be interested in values (h_1, h_2) satisfying (4.44) and (4.45), so that $h_2 \in N(T)$ is a lift of an element $w \in W$, and $h_1 \in T$. We decompose $\mathbf{g}$ as $\mathbf{g} = \mathbf{t} \oplus \mathbf{t}^\perp$; thus each element $X \in \mathbf{g}$ decomposes as $X = t + Y$ where $t \in \mathbf{t}$ and $Y \in \mathbf{t}^\perp$. We choose an orientation on $\mathbf{t}$ by choosing an ordered basis: the orientation on $\mathbf{t}^\perp$ is obtained by identifying it with the $\mathbf{C}$-linear span of the positive roots. At the point (h_1, h_2) the section s of the vector bundle given by the sum of the L_{ij} is given by projecting h_1 to $\bigoplus_{i<j}(h_1)_{ij}$.

The tangent space is oriented using the map

$$(ds, dF)\colon \mathbf{g} \oplus \mathbf{g} \to \mathbf{t}^\perp \oplus \mathbf{g},$$

or equivalently

$$(ds, dF)\colon (\mathbf{t}^\perp \oplus \mathbf{t}) \oplus (\mathbf{t}^\perp \oplus \mathbf{t}) \to (\mathbf{t}^\perp \oplus \mathbf{g}).$$

It is given by

(4.50)
$$\big((t_1, Y_1), (t_2, Y_2)\big)$$
$$\mapsto \Big(Y_1, \mathrm{Ad}(h_2 h_1 h_2^{-1})\big(1 - \mathrm{Ad}(h_2)\big)(t_1, Y_1) + \mathrm{Ad}(h_2)\big(\mathrm{Ad}(h_1) - 1\big)(t_2, Y_2)\Big)$$
$$= \Big(Y_1, (1 - w)t_1 + \mathrm{Ad}(h_2 h_1 h_2^{-1})\big(1 - \mathrm{Ad}(h_2)\big)Y_1 + \mathrm{Ad}(h_2)\big(\mathrm{Ad}(h_1) - 1\big)Y_2\Big),$$

since $h_1 \in T$. We assume that $\mathrm{Ad}(\exp \Lambda)$ does not have eigenvalues 1 on $\mathbf{t}^\perp$: then by (4.45), $\mathrm{Ad}(h_1)$ also does not have eigenvalues 1 on $\mathbf{t}^\perp$. Thus

$$\mathrm{Ker}(ds, dF) \cong \big\{\big((0,0),(t_2,0)\big)\big\},$$

and it is clear from (4.50) that (ds, dF) is surjective for all (h_1, h_2) satisfying (4.44) and (4.45).

We make use of the following elementary results:

Lemma 4.19 *The adjoint action of $h \in N(T)$ takes $\mathbf{t}$ to $\mathbf{t}$ and $\mathbf{t}^\perp$ to $\mathbf{t}^\perp$. It changes the orientation of $\mathbf{t}^\perp$ by $\epsilon(w)$ if w is the element of W corresponding to h.*

Lemma 4.20 *The adjoint action of T takes $\mathbf{t}^\perp$ to $\mathbf{t}^\perp$ and preserves the orientation of $\mathbf{t}^\perp$.*

Proposition 4.21 *Let $G = \mathrm{SU}(k)$. Assume that $\mathrm{Ad}\exp(\Lambda)$ does not have any eigenvalues equal to 1 on $\mathbf{t}^\perp$, and that for all $\xi \in \Lambda^I$ and $w \in W$, $\Lambda + \xi$ does not lie in $\mathrm{Im}(w - 1)$. Then each component of $\mathrm{PD}\big(\bigcup_{i<j} c_1(L_{ij})\big)$ in $\mathcal{M}_{1,1}(\Lambda)$ is a point with orientation $(-1)^{k-1}$. In particular all such points have the same orientation.*

Proof According to (4.50), the orientation of the component $\big\{\big(h_1, \mathrm{Ad}(y)h_2\big) : y \in T\big\}$ of the solutions of (4.44) and (4.45) in $V_{1,1}^{(1)}(\Lambda)$ (which maps to a single point in $\mathcal{M}_{1,1}(\Lambda)$ under the quotient map) is thus given by the sign of $\det A_1 \det A_2$ where A_1 is the real matrix consisting of blocks (where each block is a square matrix of size $2n_+$) given by

$$A_1 = \begin{pmatrix} 1 & 0 \\ \mathrm{Ad}(h_2 h_1 h_2^{-1})\big(1 - \mathrm{Ad}(h_2)\big) & \mathrm{Ad}(h_2)\big(\mathrm{Ad}(h_1) - 1\big) \end{pmatrix}$$

and A_2 is the transformation of $\mathbf{t}$ given by $t \mapsto (1 - w)t$. The determinant of A_2 for Weyl group elements w with no eigenvalues equal to 1 is easily computed to be 2.

The determinant of A_1 is equal to the determinant of the transformation of $\mathbf{t}^\perp$ given by

$$\text{Ad}(h_2)\big(\text{Ad}(h_1) - 1\big).$$

For $h_1 \in T$, the real determinant of $\text{Ad}(h_1) - 1$ is easily seen to be positive (it is a product over the positive roots, where each root gives a factor of the form $\big(2 - \tilde{\gamma}(h_1) - \tilde{\gamma}(h_1)^{-1}\big)$, where $\tilde{\gamma}(h_1)$ is in $U(1)$). The determinant of $\text{Ad}(h_2)$ is $\epsilon(w)$: by the calculation in (4.48), this is $(-1)^{k-1}$ and in particular it is independent of (h_1, h_2). ∎

Proposition 4.21 tell us that all the points in the Poincaré dual $D_{\mathcal{D}}$ of $\prod_{i<j} c_1(L_{ij})$ in $\mathcal{M}_{1,1}(\Lambda)$ have the same orientation. This completes the proof of (4.47), which in turn completes the proof of Theorem 4.17.

Theorem 4.22   *There is a nonvanishing element $\beta \in H^{2n_+(2g-2)}\big(M(k,d)\big)$ which is of the form $\prod_{r=2}^{k} a_r^{m_r}$.*

Proof We can explicitly exhibit such an element. Let $\tilde{\beta} = \kappa(\mathcal{D}^{2(g-1)})$. Notice that by (4.33), $\tilde{\beta} = \pi^* \beta$ for some $\beta \in H^{2n_+(2g-2)}\big(M(k,d)\big)$. Theorem 3.18 shows that

$$\int_{\mathcal{M}_{g,1}(\Lambda)} \pi^*(\beta) e^{\omega_\Lambda} = S_{1,1}(\Lambda),$$

so Theorem 4.17 shows that β must be nonzero. ∎

Remark 4.23   Theorem 4.22 provides a counterexample to a conjecture of Ne'eman [26], which is that the Pontrjagin ring vanishes above dimension $2gk^2 - 4g(k-1) + 2$. The proof of Theorem 4.22 exhibits a nonvanishing element β of degree $(2g - 2)k(k - 1)$, and we easily see using [6, Lemma 7] that this element is in the Pontrjagin ring. The degree of this element exceeds Ne'eman's conjectured bound when g is chosen sufficiently large.

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