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A characterization of the Fermat quartic K3 surface by means of finite symmetries

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Abstract

We characterize the Fermat quartic K3 surface, among all K3 surfaces, by means of its finite group symmetries.

1. Introduction

The aim of this paper is to characterize the Fermat quartic surface, among all complex K3 surfaces, in terms of finite group symmetries. Our main result is Theorem 1.2.

Throughout this paper, we shall work over the complex number field $\mathbb{C}$. By a K3 surface, we mean a simply connected smooth complex surface X which admits a nowhere vanishing global holomorphic 2-form ω_X. As is well known, K3 surfaces form a 20-dimensional family and projective ones form countably many 19-dimensional families [PS71]. Among them, one of the simplest examples is the Fermat quartic surface:

$$\iota : X_4 := (x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0) \subset \mathbb{P}^3.$$

From the explicit form, we see that X_4 admits a fairly large projective transformation group, namely,

$$\tilde{F}_{384} := (\mu_4^4 : S_4)/\mu_4 = (\mu_4^4/\mu_4) : S_4.$$

Here the symbol $A : B$ means a semi-direct product(A being normal) and $\mu_I := \langle \zeta_I \rangle$ (where $\zeta_I := e^{2\pi i/I}$) is the multiplicative subgroup of order I of $\mathbb{C}^\times$. This group $\tilde{F}_{384}$ is a solvable group of order $4^3 \cdot 4! = 2^9 \cdot 3$. The action of $\tilde{F}_{384}$ is an obvious one, that is, μ_4^4 or μ_4^4/μ_4 acts on X_4 diagonally and S_4, the symmetric group of degree 4, acts as the permutation of the coordinates.

Let $\tilde{F}_{128}$ be a Sylow 2-subgroup of $\tilde{F}_{384}$. Then $\tilde{F}_{128}$ is a nilpotent group of order 2^9. We have an action of $\tilde{F}_{128}$ on X_4 which is a restriction of the action of $\tilde{F}_{384}$. We call the action

$$\iota_{384} : \tilde{F}_{384} \times X_4 \longrightarrow X_4, \quad \text{respectively} \quad \iota_{128} : \tilde{F}_{128} \times X_4 \longrightarrow X_4,$$

defined here, the standard action of $\tilde{F}_{384}$ (respectively of $\tilde{F}_{128}$) on X_4. Note that $\tilde{F}_{384}$ has exactly three Sylow 2-subgroups, corresponding to the three Sylow 2-subgroups ($\simeq D_8$) of S_4. However, they are conjugate to one another by the Sylow theorem, and their standard actions on X_4 are isomorphic to one another in the sense below. The group $\tilde{F}_{128}$ is also interesting from the point of view of Mukai's classification of symplectic K3 groups [Muk88]. In fact, it is an extension of a Sylow 2-subgroup F_{128} of the Mathieu group M_{23} by μ_4 (see also § 2).

Definition 1.1. We call a finite group G a *K3 group* (on X) if there is a faithful action of G on X, say, $\rho : G \times X \longrightarrow X$. Let G_i be a K3 group on X_i acting by $\rho_i : G_i \times X_i \longrightarrow X_i$ ($i = 1, 2$). We say that (G_i, X_i, ρ_i) are *isomorphic* if there are a group isomorphism $f : G_1 \simeq G_2$ and an isomorphism

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$\varphi : X_1 \simeq X_2$ such that the following diagram commutes.

$$\begin{tikzcd} G_2 \times X_2 \arrow[r, "\rho_2"] & X_2 \\ G_1 \times X_1 \arrow[r, "\rho_1"] \arrow[u, "f\times\varphi"] & X_1 \arrow[u, "\varphi"'] \end{tikzcd}$$

The aim of this paper is to show the following main theorem.

Theorem 1.2.

(i) *Let G be a solvable K3 group on X acting by $\rho : G \times X \longrightarrow X$. Then $|G| \leqslant 2^9 \cdot 3$. Moreover, if $|G| = 2^9 \cdot 3\,(= 1536)$, then $\mathrm{Pic}(X)^G = \mathbb{Z}H$, $(H^2) = 4$ and $(G, X, \rho) \simeq (\tilde{F}_{384}, X_4, \iota_{384})$, the standard action of $\tilde{F}_{384}$ on the Fermat quartic surface X_4.*

(ii) *Let G be a nilpotent K3 group on X acting by $\rho : G \times X \longrightarrow X$. Then $|G| \leqslant 2^9$. Moreover, if $|G| = 2^9$, then $\mathrm{Pic}(X)^G = \mathbb{Z}H$, $(H^2) = 4$ and $(G, X, \rho) \simeq (\tilde{F}_{128}, X_4, \iota_{128})$, the standard action of $\tilde{F}_{128}$ on X_4.*

The most basic class of finite groups is the class of cyclic groups of prime order. This class is extended to the following sequences of important classes of groups of rather different nature:

$$\text{(abelian groups)} \subset \text{(nilpotent groups)} \subset \text{(solvable groups)};$$
$$\text{(quasi-simple non-commutative groups)} \subset \text{(quasi-perfect groups)}.$$

Here a quasi-simple non-commutative group (respectively a quasi-perfect group) is a group which is an extension of a simple non-commutative group (respectively a perfect group) by a cyclic group (from the right).

From the point of view of these sequences, our theorem is regarded as both an analogy and a counterpart of previous work of Kondo [Kon99] for the quasi-perfect K3 group $M_{20} : \mu_4$, which is also the group of maximum order among K3 groups, and work of Zhang and the author [OZ02] for the quasi-simple non-commutative group $L_2(7) \times \mu_4$. (See also [KOZ05].) In terms of the coarse moduli space $\mathcal{M}_4$ of quasi-polarized K3 surfaces of degree 4, our theorem says that the large stabilizer subgroups $\tilde{F}_{384}$ and $\tilde{F}_{128}$ identify the point corresponding to the Fermat K3 surface (naturally polarized by ι) in $\mathcal{M}_4$. However, our theorem claims much more, because we do not assume *a priori* a degree of invariant polarization. Indeed, as in [OZ02], the determination of the degree of invariant polarization is one of the key steps in our proof (Proposition 3.3 and § 6). For this step, as in [Kon99] and [OZ02], we apply Kondo's embedding theorem [Kon98, Lemmas 5 and 6] (see also Theorem 6.3 and Corollary 6.4), which is based on fundamental work about even lattices due to Nikulin [Nik80b], especially [Nik80b, Theorem 1.12.2]. (See also [Nik80b, Remark 1.14.7 and Proposition 1.14.8] for relevant observations.) On the other hand, our theorem can also be viewed as a characterization of a 2-group $\tilde{F}_{128}$ by means of geometry. It might be worth noticing the following table in [Har99, p. 11] of the number $p(n)$ of isomorphism classes of 2-groups of order 2^n:

n	1	2	3	4	5	6	7	8	9
$p(n)$	1	2	5	14	51	267	2328	56092	10494213

Section 2 is a summary of known results, relevant to us, about K3 groups from Nikulin [Nik80a] and Mukai [Muk88]. In § 3, we reduce our main theorem to three propositions (Propositions 3.1, 3.2, and 3.3). In §§ 4, 5 and 6, we prove these three propositions.

2. Some basic properties of K3 groups after Nikulin and Mukai

A systematic study of K3 groups started by Nikulin [Nik80a, Nik80b] and further developed by Mukai [Muk88], Xiao [Xia96], Kondo [Kon98] and others. In this section, we recall basic results, relevant to us, about K3 groups from Nikulin [Nik80a] and Mukai [Muk88].

Let X be a K3 surface and G be a K3 group acting on X by $\rho : G \times X \longrightarrow X$. Then G has a natural one-dimensional representation on $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$ defined by $g^*\omega_X = \alpha(g)\omega_X$, and we have the exact sequence, called the basic sequence:

$$1 \longrightarrow G_N := \operatorname{Ker}\alpha \longrightarrow G \xrightarrow{\alpha} \mu_I \longrightarrow 1.$$

The basic sequence was first introduced by [Nik80a]. We call G_N the symplectic part and μ_I (respectively I) the transcendental part (respectively the transcendental value) of the action $\rho : G \times X \longrightarrow X$.

By the basic sequence, the study of K3 groups is divided into three parts: study of symplectic K3 groups G_N, study of transcendental values I, and study of possible extensions of symplectic parts by transcendental parts.

Example 2.1. The group $\tilde{F}_{384} = (\mu_4^4 : S_4)/\mu_4$ fits into the following exact sequence:

$$1 \longrightarrow \mu_4^4/\mu_4 \longrightarrow \tilde{F}_{384} \xrightarrow{p} S_4 \longrightarrow 1.$$

Then the group $\langle(1324), (34)\rangle \simeq D_8$ is a Sylow 2-subgroup (one of three) of S_4 and $p^{-1}(\langle(1324), (34)\rangle)$ is a 2-Sylow subgroup (one of three) of $\tilde{F}_{384}$. We fix $\tilde{F}_{128}$ as this subgroup. The basic sequences of the standard actions of $\tilde{F}_{384}$ and of $\tilde{F}_{128}$ on the Fermat K3 surface X_4 are as follows:

$$1 \longrightarrow F_{384} := (\tilde{F}_{384})_N \longrightarrow \tilde{F}_{384} \xrightarrow{\alpha} \mu_4 \longrightarrow 1,$$
$$1 \longrightarrow F_{128} := (\tilde{F}_{128})_N \longrightarrow \tilde{F}_{128} \xrightarrow{\alpha} \mu_4 \longrightarrow 1.$$

The orders of the symplectic parts F_{384} and F_{128} are $384 = 2^7 \cdot 3$ and $128 = 2^7$ respectively. Moreover, both basic sequences split: $\tilde{F}_{384} = F_{384} : \mu_4$ and $\tilde{F}_{128} = F_{128} : \mu_4$. Here the splittings are given by $\alpha(\operatorname{diag}(1, 1, 1, \zeta_4)) = \zeta_4$.

The next theorem due to Nikulin [Nik80a] is the first important result about the symplectic part.

Theorem 2.2 [Nik80a]. *Let $g \in G_N$. Then $\operatorname{ord} g \leqslant 8$. The fixed locus X^g is a finite set (if $g \neq 1$) and the cardinality $|X^g|$ depends only on $\operatorname{ord} g$ as in the following table:*

$\operatorname{ord}(g)$	1	2	3	4	5	6	7	8
$\lvert X^g\rvert$	X	8	6	4	4	2	3	2

Let $\Omega := \{1, 2, \ldots, 24\}$ be the set of 24 elements and $\mathcal{P}(\Omega)$ be the power set of Ω, i.e. the set of all subsets of Ω. As is classically known (see for instance [CS99, ch. 10]), $\mathcal{P}(\Omega)$ has a very remarkable subset $St(5, 8, 24)$, called the Steiner system. $St(5, 8, 24)$ is defined to be a subset of $\mathcal{P}(\Omega)$ consisting of eight-element subsets such that, for each five-element subset B of Ω, there is exactly one $A \in St(5, 8, 24)$ such that $B \subset A$. Such subsets $St(5, 8, 24)$ of $\mathcal{P}(\Omega)$ are known to be unique up to $\operatorname{Aut}\Omega = S_{24}$ and satisfy $|St(5, 8, 24)| = 759$. We fix one such $St(5, 8, 24)$. The Mathieu group M_{24} of degree 24 is then defined to be the stabilizer group of $St(5, 8, 24)$:

$$M_{24} := \{\tau \in \operatorname{Aut}(\Omega) = S_{24} \mid \tau(St(5, 8, 24)) = St(5, 8, 24)\}.$$

It is well known that M_{24} is a simple (sporadic) group of order $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ that acts 5-transitively on Ω (e.g. [CS99]). The Mathieu group M_{23} of degree 23 is the stabilizer group of one point, say $24 \in \Omega$, i.e. $M_{23} := \{\tau \in M_{24} \mid \tau(24) = 24\}$. Also M_{23} is a simple group and is of order $|M_{23}| = |M_{24}|/24 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. By definition, both M_{24} and M_{23} act naturally on Ω.

Being inspired by a curious coincidence between Nikulin's table in Theorem 2.2 and the character table of the natural action of M_{23} on Ω, Mukai [Muk88] finally obtained the following very beautiful theorem.

THEOREM 2.3 [Muk88, main theorem]. *Let K be a finite group. Then K is a symplectic K3 group on some K3 surface X if and only if K is isomorphic to a subgroup of M_{23} having at least 5-orbits on Ω (under the action induced by the action of M_{23} on Ω). Moreover, with respect to the inclusion as abstract groups, there are exactly 11 maximal such groups. The groups M_{20} and F_{384} are those of the two largest orders, which are $|M_{20}| = 960$ and $|F_{384}| = 384$.*

Later, Xiao [Xia96] and Kondo [Kon98] gave alternative proofs respectively. In the course of the proof, Xiao shows that there are exactly 80 isomorphism classes of symplectic K3 groups (as abstract groups). In our proof of the main result (Theorem 1.2), we shall also exploit an idea of Kondo's alternative proof (§ 6).

We emphasize the following consequence.

COROLLARY 2.4. *The group F_{128} is isomorphic to a Sylow 2-subgroup of M_{23}.*

Proof. By Example 2.1 and Theorem 2.3, we have $F_{128} < M_{23}$. Moreover, since $|F_{128}| = 2^7$ and $|M_{23}| = 2^7 \cdot k$ $((2, k) = 1)$, the result follows from the Sylow theorem. □

Next we recall the basic properties of the transcendental part μ_I of a K3 group on X from [Nik80a] and [MO98]. By $\varphi(I)$, we denote the Euler function of I, i.e. $\varphi(I) = |\mathrm{Gal}(\mathbb{Q}(\zeta_I)/\mathbb{Q})|$. Note that $\varphi(I)$ is even unless $I = 1, 2$. As observed in [Nik80a], X is projective if $I \geqslant 2$.

In the rest of § 2, we assume that X is projective.

Let $\mathrm{NS}(X)$ be the Néron–Severi lattice of X and $T(X)$ the transcendental lattice, i.e. the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$ with respect to the cup product:

$$T(X) := \{x \in H^2(X, \mathbb{Z}) \mid (x, \mathrm{NS}(X)) = 0\}.$$

Then, $\mathrm{NS}(X) \oplus T(X)$ is a sublattice of finite index of $H^2(X, \mathbb{Z})$ (by the projectivity of X). $T(X)$ is also the minimal primitive sublattice of $H^2(X, \mathbb{Z})$ such that the scalar extension by $\mathbb{C}$ contains the class of ω_X (by the Lefschetz $(1,1)$-theorem). Since $b_2(X) = 22$, we have $2 \leqslant \operatorname{rank} T(X) \leqslant 21$.

THEOREM 2.5 ([Nik80a], see also [MO98] for parts (ii) and (iv)).

(i) *G_N acts on $T(X)$ as identity.*

(ii) *Set $G/G_N = \langle g \bmod G_N \rangle \simeq \mu_I$. Then, there is a natural isomorphism*

$$T(X) \simeq \mathbb{Z}[\zeta_I]^{\oplus n}, \quad n = \frac{\operatorname{rank} T(X)}{\varphi(I)}$$

as $\mathbb{Z}[\zeta_I]$-modules. Here, $\mathbb{Z}[\zeta_I]$-module structure on $T(X)$ is given by $f(\zeta_I)x := f(g^*)x$.

(iii) *$\varphi(I)|\operatorname{rank} T(X)$. In particular, $\varphi(I) \leqslant 20$ and $I \leqslant 66$. Moreover, $I = 1, 2, 3, 4, 6$ if $\varphi(I) \leqslant 2$ and $I = 5, 8, 10, 12$ if $\varphi(I) = 4$.*

(iv) *$I \neq 60$. Conversely, each I such that $\varphi(I) \leqslant 20$ and $I \neq 60$ is realized as a transcendental value of some K3 group. There are exactly 40 such I. (For the explicit list, see [MO98].)*

As we reviewed above, both symplectic part and transcendental part are now well understood. However, the K3 groups, i.e. all the geometrically possible extensions of 80 symplectic parts by 40 transcendental parts, are not yet classified completely. Work on this problem is now in progress in [IOZ04].

We close § 2 by recalling the following group-theoretical nature of F_{128} from [Xia96].

Proposition 2.6.

(i) *The order structure of* F_{128} *is as follows:*

order	1	2	4	8
cardinality	1	35	76	16

(ii) *The commutator subgroup* $[F_{128}, F_{128}]$ *of* F_{128} *is isomorphic to* $C_2 \times D_8$*, where* C_n *is a cyclic group of order* n *and* D_{2n} *is a dihedral group of order* $2n$*.*

(iii) F_{128} *has a subgroup isomorphic to the binary dihedral group of order* 16*:*

$$Q_{16} := \langle a, b \mid a^8 = 1, a^4 = b^2, b^{-1}ab = a^{-1} \rangle.$$

3. Reduction of the Main Theorem to three propositions

In this section, we reduce the main theorem (Theorem 1.2) to the following three propositions 3.1, 3.2 and 3.3.

Proposition 3.1. *Let* X *be a projective K3 surface. Assume that* Q_{16} *is a symplectic K3 group on* X*. Then the following hold:*

(i) $\mathrm{NS}(X)^{Q_{16}} = \mathbb{Z}H$*, where* H *is an ample class on* X*;*

(ii) *if, in addition,* $(H^2) = 4$*, then the polarized K3 surface* (X, H) *is unique up to isomorphism. In particular,* $(X, H) \simeq (X_4, H_4)$*, where* X_4 *is the Fermat quartic K3 surface and* $H_4 := \iota^*\mathcal{O}_{\mathbb{P}^3}(1)$ *under the natural inclusion* $\iota : X_4 \subset \mathbb{P}^3$*.*

Proposition 3.2.

(i) *Let* G *be a K3 group on* X *such that* $G_N \simeq F_{384}$*. Then the transcendental value* I *of* G *is either* 1*,* 2*, or* 4*.*

(ii) *Let* G *be a solvable K3 group on* X*. Then* $|G| \leqslant 2^9 \cdot 3$*. Moreover, if* $|G| = 2^9 \cdot 3$*, then the symplectic part* G_N *is necessarily isomorphic to* F_{384} *and the transcendental part is isomorphic to* μ_4*.*

(iii) *Let* G *be a K3 group on* X *such that* $G_N \simeq F_{128}$*. Then the transcendental value* I *of* G *is either* 1*,* 2*, or* 4*. In particular,* G *is a* 2*-group and nilpotent.*

(iv) *Let* G *be a nilpotent K3 group on* X*. Then* $|G| \leqslant 2^9$*. Moreover, if* $|G| = 2^9$*, then the symplectic part* G_N *is necessarily isomorphic to* F_{128} *and the transcendental part is isomorphic to* μ_4*.*

Proposition 3.3. *Let* X *be a K3 surface. Assume that* X *admits a K3 group* G *of order* 2^9*. Then* X *is projective and* $\mathrm{NS}(X)^G = \mathbb{Z}H$*, where* H *is an ample class such that* $(H^2) = 4$*.*

We shall prove these three propositions in §§ 4, 5 and 6 respectively. In the rest of this section, we show that these propositions imply the main result (Theorem 1.2).

Proof that Propositions 3.1, 3.2 and 3.3 imply Theorem 1.2

Let Y be a K3 surface admitting a K3 group F such that $|F| = 2^9 \cdot 3$. Let G be a Sylow 2-subgroup of F. Then $|G| = 2^9$ and G is a nilpotent group. (Here we recall that any p-group is nilpotent.) Then, by Proposition 3.2(iv), $G_N \simeq F_{128}$ and $I = 4$. In particular, Y is projective by $I \geqslant 2$. Recall that Q_{16} is a subgroup of F_{128} by Proposition 2.6(iii). Then, we have embeddings: $Q_{16} < F_{128} < G < F$. Thus

$$\mathrm{NS}(Y)^F \subset \mathrm{NS}(Y)^G \subset \mathrm{NS}(Y)^{F_{128}} \subset \mathrm{NS}(Y)^{Q_{16}} = \mathbb{Z}H.$$

Here we use Proposition 3.1(i) for the last equality. Since $\mathrm{NS}(Y)^F$ contains an ample invariant class, say $\sum_{g\in F} g^*h$, h being ample on Y, we have

$$\mathrm{NS}(Y)^F = \mathrm{NS}(Y)^G = \mathrm{NS}(Y)^{F_{128}} = \mathrm{NS}(Y)^{Q_{16}} = \mathbb{Z}H.$$

Thus, $F < \mathrm{Aut}(Y,H)$. Moreover, $(H^2) = 4$ by Proposition 3.3. Hence, $\varphi : (Y,H) \simeq (X_4,H_4)$ by Proposition 3.1(ii). Then, under the isomorphism $F \simeq \varphi^{-1}\circ F\circ\varphi$, we have $((Y,H),F) \simeq ((X_4,H_4),F)$. So, we may identify $((X,H),F) = ((X_4,H_4),F)$. Under this identification, we have

$$F < \mathrm{Aut}(X_4,H_4) > \tilde{F}_{384} > F_{384}.$$

Note that $\mathrm{Aut}(X_4,H_4)$ is a finite group. This is because $\mathrm{Aut}(X_4,H_4)$ is a discrete algebraic subgroup of $\mathrm{PGL}(\mathbb{P}^3)$, whence, finite. Thus, $[\mathrm{Aut}(X_4,H_4) : F_{384}] \leqslant 4$ by Proposition 3.2(i) and Theorem 2.3. Hence $|\mathrm{Aut}(X_4,H_4)| \leqslant 2^7\cdot 3\times 4 = 2^9\cdot 3$. Since $|\tilde{F}_{384}| = |F| = 2^9\cdot 3$, we then obtain $F = \mathrm{Aut}(X_4,H_4) = \tilde{F}_{384}$. This implies the assertion (i) of the main theorem (Theorem 1.2).

Next, we shall show the assertion (ii) of the main theorem. Let X be a K3 surface admitting a K3 group G such that $|G| = 2^9$. Then, by repeating the same argument as above, we can identify $((X,H),G) = ((X_4,H_4),G)$. Since $\mathrm{Aut}(X_4,H_4) = \tilde{F}_{384}$, our G is a subgroup of $\tilde{F}_{384}$. Since $|\tilde{F}_{384}| = 2^9\cdot 3$ and $|G| = 2^9$, it follows that G is one of three Sylow 2-subgroups $\tilde{F}_{128}$ of $\tilde{F}_{384}$, which are conjugate to one another in $\tilde{F}_{384}$. This implies the result.

Remark 3.4. As a byproduct, we have obtained that $\mathrm{Aut}(X_4,H_4) = \tilde{F}_{384}$. One can also derive this equality through a more direct calculation along the same lines as in [Shi88]. We also notice that $\rho(X_4) = 20$ and, by [SI77], the full automorphism group $\mathrm{Aut}(X_4)$ is an infinite group.

4. Polarized K3 surface of degree 4 with a symplectic Q_{16}-action

In this section, we shall prove Proposition 3.1.

DEFINITION 4.1. The binary dihedral group Q_{4m} of order $4m$ is defined by

$$Q_{4m} := \langle a, b \mid a^{2m} = 1, a^m = b^2, b^{-1}ab = a^{-1}\rangle.$$

The group Q_{4m} is realized as a linear subgroup of $\mathrm{GL}(2,\mathbb{C})$ as

$$\left\langle a := \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1}\end{pmatrix}, b := \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0\end{pmatrix}\right\rangle = \{a^n, a^n b \mid 0 \leqslant n \leqslant 2m-1\}.$$

Lemmas 4.2 and 4.4 (the same as Proposition 3.1(i)) explain the reason why we pay special attention to the particular group Q_{16}.

LEMMA 4.2. *Any projective representation of Q_{4m} is induced by a linear representation, i.e. for any group homomorphism $\rho : Q_{4m} \longrightarrow \mathrm{PGL}(n,\mathbb{C}) := \mathrm{GL}(n,\mathbb{C})/\mathbb{C}^\times$, there is a group homomorphism $\tilde{\rho} : Q_{4m} \longrightarrow \mathrm{GL}(n,\mathbb{C})$ such that $\rho = p\circ\tilde{\rho}$, where $p : \mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathrm{PGL}(n,\mathbb{C})$ is the quotient map.*

Proof. This should be well known, but, the proof is so direct and easy that we shall give it here. We write $[X] = X \bmod \mathbb{C}^\times$ for $X \in \mathrm{GL}(n,\mathbb{C})$. First we remark that $Q_{4m} = \langle a, b \mid a^m = b^2, b^{-1}ab = a^{-1}\rangle$. This is because $a^m = b^2$ and $b^{-1}ab = a^{-1}$ imply that $a^{-m} = b^{-1}a^m b = b^{-1}b^2 b = b^2 = a^m$, whence $a^{2m} = 1$. Let $\rho(a) = [A]$ and $\rho(b) = [B]$. Then, $[A^m] = [B^2]$ and $[B^{-1}AB] = [A^{-1}]$. That is, $A^m = \alpha B^2$ and $B^{-1}AB = \beta A^{-1}$ in $\mathrm{GL}(n,\mathbb{C})$ for some $\alpha,\beta \in \mathbb{C}^\times$. By replacing the representative A by $A/\sqrt{\beta}$, one has $B^{-1}AB = A^{-1}$. Next, by replacing the representative B by $\sqrt{\alpha}B$, one obtains $B^{-1}AB = A^{-1}$ and $A^m = B^2$. Therefore we have a group homomorphism $\tilde{\rho} : Q_{4m} \longrightarrow \mathrm{GL}(n,\mathbb{C})$ defined by $\tilde{\rho}(a) = A$ and $\tilde{\rho}(b) = B$. This $\tilde{\rho}$ satisfies $\rho = p\circ\tilde{\rho}$. □

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Remark 4.3. Consider the dihedral group $D_8 := \langle a, b \mid a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Then the map

$$\rho(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}$$

defines a projective representation $\rho : D_8 \longrightarrow \mathrm{PGL}(2, \mathbb{C})$. However, this is not induced by any linear representation $D_8 \longrightarrow \mathrm{GL}(2, \mathbb{C})$.

Lemma 4.4. *Let X be a projective K3 surface admitting a symplectic K3 group Q_{16}. Then $\operatorname{rank} T(X) = 2$ and $\mathrm{NS}(X)^{Q_{16}} = \mathbb{Z}H$ for some ample class H.*

This lemma is proved after Proposition 4.5 and Lemma 4.6.

The following very important proposition is due to Mukai [Muk88].

Proposition 4.5. *Let X be a projective K3 surface admitting a symplectic K3 group G. Then the following hold:*

(i) *we have*

$$\operatorname{rank} H^2(X, \mathbb{Z})^G = \frac{1}{|G|}\left(24 + \sum_{n=2}^{8} m(n) f(n)\right) - 2,$$

where $m(n)$ is the number of elements of order n in G and $f(n)$ is the number of fixed points in Theorem 2.2;

(ii) *we have $\operatorname{rank} H^2(X, \mathbb{Z})^G \geqslant 3$. Moreover, if $\operatorname{rank} H^2(X, \mathbb{Z})^G = 3$, then $\operatorname{rank} T(X) = 2$ and $\mathrm{NS}(X)^G = \mathbb{Z}H$ for some ample class H.*

Proof. Consider the action of G on the total cohomology group

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}).$$

Then, by the representation theory, one has

$$\operatorname{rank} H^*(X, \mathbb{Z})^G = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g^* | H^*(X, \mathbb{Z})).$$

By the Lefschetz fixed point formula, each summand satisfies

$$\operatorname{tr}(g^* | H^*(X, \mathbb{Z})) = e(X^g).$$

Here $e(*)$ is the topological Euler number of $*$. Combining these two equalities with Theorems 2.2 and 2.5, one gets the result. □

We also state the following lemma.

Lemma 4.6. *The order structure of Q_{16} is as follows:*

order	1	2	4	8
cardinality	1	1	10	4

Proof. This follows directly from the description of Q_{16}. □

Proof of Lemma 4.4. Let us return to Lemma 4.4. By Proposition 4.5 and Lemma 4.6, we calculate that

$$\operatorname{rank} H^2(X, \mathbb{Z})^{Q_{16}} = \tfrac{1}{16}(24 + 8 \cdot 1 + 4 \cdot 10 + 2 \cdot 4) - 2 = 3.$$

This completes the proof of Lemma 4.4. □

Recall that the standard action of F_{128} on X_4 is a symplectic action on the polarized K3 surface (X_4, H_4) and that $Q_{16} < F_{128}$. Now, the next proposition completes the proof of Proposition 3.1.

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Proposition 4.7. *Polarized K3 surfaces (X, H) of degree 4 which admit a symplectic K3 group Q_{16} (which keeps H invariant) are unique up to isomorphism as polarized K3 surfaces.*

Proof. Since $\operatorname{rank} \mathrm{NS}(X)^{Q_{16}} = 1$ and $H \in \mathrm{NS}(X)$ is primitive (by $(H^2) = 4$ and by the evenness of the intersection numbers), we have $\mathrm{NS}(X)^{Q_{16}} = \mathbb{Z}H$. Then $|H|$ has no fixed components. Indeed, the fixed part of $|H|$ must also be Q_{16}-invariant, while $\mathrm{NS}(X)^{Q_{16}} = \mathbb{Z}H$. Therefore, the ample linear system $|H|$ is free by [Sai74]. Note that $\dim|H| = 3$ by the Riemann–Roch formula and by $(H^2) = 4$. Then $|H|$ defines a morphism

$$\Phi := \Phi_{|H|} : X \longrightarrow \mathbb{P}^3 = |H|^*; \quad x \mapsto \{D \in |H| \bigm| D \ni x\}.$$

This Φ is either

(I) an embedding onto a (smooth) quartic surface $W = (4)$, or

(II) a finite double cover of an irreducible, reduced quadratic surface $W = (2)$.

Since H is Q_{16}-invariant in $\mathrm{Pic}(X) \simeq \mathrm{NS}(X)$, the divisor g^*D is linearly equivalent to D whenever $D \in |H|$ and $g \in Q_{16}$. Thus, the group Q_{16} induces a Φ-equivariant, projective linear action on the image W. By Lemma 4.2, this action is also induced by a linear co-action of Q_{16} on $H^0(X, \mathcal{O}_X(H)) = \bigoplus_{i=1}^4 \mathbb{C}x_i$.

In order to complete the proof, it suffices to show the two assertions, that case (II) cannot happen (Lemma 4.9) and that the image W is uniquely determined up to projective transformations of $\mathbb{P}^3$ in case (I) (Lemma 4.11).

In both assertions, we need the following classification of the complex irreducible linear representations of $Q_{16} = \langle a, b \mid a^4 = b^2, b^{-1}ab = a^{-1} \rangle$.

Lemma 4.8. *A complex irreducible linear representation of Q_{16} is isomorphic to one of the following seven representations:*

$$\rho_{1,1} : a \mapsto 1, b \mapsto 1, \quad \rho_{1,2} : a \mapsto 1, b \mapsto -1,$$
$$\rho_{1,3} : a \mapsto -1, b \mapsto 1, \quad \rho_{1,4} : a \mapsto -1, b \mapsto -1,$$
$$\rho_{2,1} : a \mapsto \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix},$$
$$\rho_{2,2} : a \mapsto \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^{-3} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix},$$
$$\rho_{2,3} : a \mapsto \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. These seven representations are clearly irreducible and well defined. Moreover, any two are inequivalent as linear representations (by looking at the trace of the matrices). Since $16 = 1^2 \cdot 4 + 2^2 \cdot 3$, these are all. ☐

Lemma 4.9. *Case (II) in the proof of Proposition 4.7 cannot happen.*

Proof. In what follows, assuming to the contrary that case (II) happens, i.e. the image W is a quadratic surface, we shall derive a contradiction.

Claim 4.10.

(a) W is non-singular.

(b) The induced action by Q_{16} on W is faithful.

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Proof. Note that a quadratic surface is normal if it is irreducible and reduced. Since Φ is a finite double covering, it is also a Galois covering. Let τ be the covering involution. Then $W = X/\tau$. Since W is a rational surface, $\tau^*\omega_X = -\omega_X$. Thus, if $P \in X^\tau$, then there is a local coordinate (x_P, y_P) at P such that $\tau^*(x_P, y_P) = (x_P, -y_P)$. Hence, W is non-singular. The kernel of the natural map $\mathrm{Aut}(X, H) \longrightarrow \mathrm{Aut}(W, \mathcal{O}_W(1))$ is a subgroup of $\langle\tau\rangle$. Since $\tau^*\omega_X = -\omega_X$, we have $Q_{16} \cap \langle\tau\rangle = \{1\}$. This means that the induced action of Q_{16} on W is faithful. $\square$

Let us return to the proof of Lemma 4.9. Let us consider the irreducible decomposition of the co-action of Q_{16} on $H^0(X, \mathcal{O}_X(H)) \simeq \mathbb{C}^4$. Note that the representations $\rho_{2,1}$ and $\rho_{2,2}$ are transformed by the outer automorphism $a \mapsto a^3$ and $b \mapsto b$ of Q_{16}. Recall also that the action must be faithful by Claim 4.10(b). Thus, we may assume without loss of generality that $\rho_{2,1}$ appears in the decomposition. Under this assumption, there are four possible decompositions: (i) $\rho_{2,1} \oplus \rho_{2,1}$, (ii) $\rho_{2,1} \oplus \rho_{2,2}$, (iii) $\rho_{2,1} \oplus \rho_{2,3}$, (iv) $\rho_{2,1} \oplus$ (two one-dimensional irreducible representations).

In cases (i) and (ii), $a^4 = \mathrm{id}$ in $\mathrm{PGL}(4, \mathbb{C})$, a contradiction to Claim 4.10(b).

Consider the case (iii). Then the action of Q_{16} on $H^0(X, \mathcal{O}_X(H))$ is given by

$$a = \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & 0 & \zeta_4^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \zeta_4 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

under a suitable basis $\langle x_i \rangle_{i=1}^4$ of $H^0(X, \mathcal{O}_X(H))$. Let us consider the defining equation $F_2(x_1, x_2, x_3, x_4) \in \mathrm{Sym}^2 H^0(X, \mathcal{O}_X(H))$ of W. Then, F_2 is both a-semi-invariant and b-semi-invariant, i.e. $a(F_2) = \sigma(a)F_2$ and $b(F_2) = \sigma(b)F_2$. This σ defines a one-dimensional representation of Q_{16}. Thus $a(F_2) = \pm F_2$ and $b(F_2) = \pm F_2$. If $a(F_2) = F_2$, then $F_2 = \alpha x_1 x_2 + \beta x_3 x_4$ by the explicit matrix form of a. Since $b(F_2) = \pm F_2$, we have then $F_2 = \alpha x_1 x_2$ or $\beta x_3 x_4$. However, this contradicts the smoothness of W. If $a(F_2) = -F_2$, then F_2 is of the form $F_2(x_3, x_4)$ and again contradicts the smoothness of W. Thus, the case (iii) cannot happen, either.

Finally consider the case (iv). In this case, the action of Q_{16} on $H^0(X, \mathcal{O}_X(H))$ is given by

$$a = \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \zeta_4 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix},$$

under a suitable basis $\langle x_i \rangle_{i=1}^4$ of $H^0(X, \mathcal{O}_X(H))$. Let us consider the defining equation F_2 of W. Then as before $a(F_2) = \pm F_2$ and $b(F_2) = \pm F_2$. If $a(F_2) = -F_2$, then $F_2 = F_2(x_3, x_4)$ and W is singular, a contradiction. Consider the case where $a(F_2) = F_2$. By the explicit form of a, we have $F_2 = \alpha x_1 x_2 + f_2(x_3, x_4)$. Since W is non-singular, we have $\alpha \neq 0$ and $f_2 \neq 0$. Since $b(x_1 x_2) = -x_1 x_2$, we have $b(f_2) = -f_2$. Thus, again by the explicit form of b, it follows that $F_2 = \alpha x_1 x_2 + \beta x_3 x_4$ for some non-zero constants α, β. After replacing x_i by their multiples and the order of x_3 and x_4 if necessary, we finally normalize the equation of W as $F_2 = x_1 x_2 + x_3 x_4$ and we have:

$$a = \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 0 & \zeta_4 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then, it follows that $W^a = W^{a^2} = W^{a^4} = \{P_i\}_{i=1}^4 =: S$, where $P_1 = [1:0:0:0]$, $P_2 = [0:1:0:0]$, $P_3 = [0:0:1:0]$ and $P_4 = [0:0:0:1]$. Since the actions of Q_{16} on X and on W are Φ-equivariant and since Φ is a finite morphism of degree 2, it follows that a^2 and a^4 act on $T := \Phi^{-1}(S)$ as identity. Thus $X^{a^2} = X^{a^4} = T$. On the other hand, $|X^{a^2}| = 4$ and $|X^{a^4}| = 8$ by Theorem 2.2, a contradiction. This completes the proof of Lemma 4.9. $\square$

LEMMA 4.11. *Assume that case (I) in the proof of Proposition 4.7 happens, i.e. that* $\Phi : X \simeq W = (4) \subset \mathbb{P}^3$*. Then* $W = (x_1^4 + x_2^4 + x_3^3 x_4 + x_3 x_4^3 = 0)$ *in suitably chosen homogeneous coordinates of* $\mathbb{P}^3$*.*

Proof. Set $W = (F_4(x_1, x_2, x_3, x_4) = 0)$. We note that Φ-equivariant action of Q_{16} on W is symplectic and faithful. As in Lemma 4.9, we consider the irreducible decomposition of the co-action of Q_{16} on $H^0(X, \mathcal{O}_X(H))$. Again as before, we may assume that $\rho_{2,1}$ appears in the decomposition. Under this assumption, there are four possible decompositions: (i) $\rho_{2,1} \oplus \rho_{2,1}$, (ii) $\rho_{2,1} \oplus \rho_{2,2}$, (iii) $\rho_{2,1} \oplus \rho_{2,3}$, (iv) $\rho_{2,1} \oplus$ (two one-dimensional irreducible representations).

As before, cases (i) and (ii) are ruled out by $a^4 = \mathrm{id}$ in $\mathrm{PGL}(4, \mathbb{C})$.

CLAIM 4.12. Case (iii) does not happen.

Proof. Assume that case (iii) happens. Then the action of Q_{16} on $H^0(X, \mathcal{O}_X(H))$ is given by

$$a = \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & 0 & \zeta_4^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \zeta_4 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

under a suitable basis $\langle x_i \rangle_{i=1}^4$ of $H^0(X, \mathcal{O}_X(H))$. Since $\det a = 1$ and $a^* \omega_W = \omega_W$, it follows that F_4 is a-invariant. Then, by the explicit form of a, the equation F_4 must be of the following form:

$$F_4 = \alpha x_1^2 x_2^2 + \beta x_1 x_2 x_3 x_4 + f_4(x_3, x_4).$$

However the point $[1 : 0 : 0 : 0]$ is then a singular point of W, a contradiction. □

In what follows, we shall consider case (iv). Note that

$$\begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \zeta_8^5 & 0 & 0 & 0 \\ 0 & \zeta_8^3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^5$$

in $\mathrm{PGL}(4, \mathbb{C})$. So, replacing a by a^5 (for instance, by using an outer automorphism of Q_{16} defined by $a \mapsto a^5$, $b \mapsto b$) if necessary, we may assume that a is either

$$a_1 := \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad a_2 := \begin{pmatrix} \zeta_8 & 0 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In each case

$$b = \begin{pmatrix} 0 & \zeta_4 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}.$$

CLAIM 4.13. We have $a \neq a_1$.

Proof. Assume that $a = a_1$. Then, as before, by $\det a_1 = 1$ and $a^* \omega_W = \omega_W$, it follows that F_4 is a-invariant. Thus F_4 must be of the following form:

$$F_4 = \alpha x_1^2 x_2^2 + \beta x_1 x_2 f_2(x_3, x_4) + f_4(x_3, x_4).$$

However, $[1 : 0 : 0 : 0]$ is then a singular point of W, a contradiction. □

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So, $a = a_2$. By $\det a_2 = -1$, we have $a(F_4) = -F_4$. By the explicit form of a_2, the equation F_4 is of the following form:

$$F_4 = \alpha x_1^4 + \beta x_2^4 + \gamma x_4^3 x_3 + \delta x_4 x_3^3 + \epsilon x_1 x_2 x_3 x_4.$$

If $\alpha = 0$, then $\beta = 0$, because F_4 is b-semi-invariant. However, $[1:0:0:0]$ is then a singular point of W, a contradiction. Thus $\alpha \neq 0$. For the same reason, we have $\beta \neq 0$. If $\gamma = 0$, then $[0:0:0:1]$ is a singular point of W. If $\delta = 0$, then $[0:0:1:0]$ is a singular point of W. Thus $\gamma \neq 0$ and $\delta \neq 0$.

If $\det b = 1$, then $b(F_4) = F_4$. Thus $\alpha = \beta$ and $\epsilon = 0$. Then, applying a suitable linear transform like $x_1 \mapsto cx_1$, $x_2 \mapsto cx_2$, $x_2 \mapsto dx_2$, $x_3 \mapsto ex_3$, one can normalize the equation of W as in Lemma 4.11.

If $\det b = -1$, then $b(F_4) = -F_4$. Thus $\alpha = -\beta$ and $\epsilon = 0$. Then, one can again normalize the equation of W as in Lemma 4.11. This completes the proof of Lemma 4.11. $\square$

We have now completed the proof of Proposition 4.7 (and therefore that of Proposition 3.1). $\square$

5. Solvable K3 groups and nilpotent K3 groups

In this section, we shall prove Proposition 3.2. Throughout this section, we denote by G a K3 group acting on X and by

$$1 \longrightarrow G_N \longrightarrow G \xrightarrow{\alpha} \mu_I \longrightarrow 1$$

the basic sequence.

The next proposition is a special case of a more general fact in [IOZ04] and is crucial for our proof.

Proposition 5.1. *Assume that $I = 3$. Let g be an element of G such that $\alpha(g) = \zeta_3$.*

(i) *Set $\operatorname{ord} g = 3k$. Then $(k, 3) = 1$. In particular, the basic sequence splits if $I = 3$.*

(ii) *Assume that $\operatorname{ord} g = 6$. Let $P \in X^g$. Then there is a local coordinate (x, y) at P such that either $g^*(x, y) = (\zeta_6^{-1} x, \zeta_6^3 y)$ (type 1 in the notation of [IOZ04]) or $g^*(x, y) = (\zeta_6^{-5} x, \zeta_6 y)$ (type 5). Let m_1, m_5 be the numbers of points of type 1 and of type 5. Then (m_1, m_5) is either $(2, 0)$, $(4, 1)$ or $(6, 2)$.*

(iii) *Assume that $\operatorname{ord} g = 12$. Let $P \in X^g$. Then there is a local coordinate (x, y) at P such that either $g^*(x, y) = (\zeta_{12}^{-1} x, \zeta_{12}^5 y)$ (type 1), $g^*(x, y) = (\zeta_{12}^{-3} x, \zeta_{12}^7 y)$ (type 3), or $g^*(x, y) = (\zeta_{12}^{-9} x, \zeta_{12} y)$ (type 9). Let m_1, m_3 and m_9 be the numbers of points of types 1, 3 and 9, respectively. Then (m_1, m_3, m_9) is either $(1, 0, 0)$ or $(2, 1, 1)$.*

Proof. For the convenience the reader, we shall give a proof for this special case. A more general treatment will be found in [IOZ04].

Let us show part (i). If otherwise, $k = 3$ or 6 by $g^3 \in G_N$ and by Theorem 2.2. So, it suffices to show that $k \neq 3$. Assume $k = 3$. Then $\operatorname{ord} g = 9$ and $X^g \subset X^{g^3}$. Since X^{g^3} is a six-point set by Theorem 2.2, X^g is also a finite set. Let $P \in X^g$. Then, since $\operatorname{ord} g = 9$, $g^*\omega_X = \zeta_3 \omega_X$, and $P \in X^g$ is isolated, there is a local coordinate (x, y) at P such that either $g^*(x, y) = (\zeta_9^{-1} x, \zeta_9^4 y)$ (type 1), $g^*(x, y) = (\zeta_9^{-2} x, \zeta_9^5 y)$ (type 2) or $g^*(x, y) = (\zeta_9^{-7} x, \zeta_9 y)$ (type 7) holds. Let m_1, m_2 and m_7 be the numbers of fixed points of types 1, 2 and 7. Then, by the holomorphic Lefschetz fixed point formula,

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one has:

$$1+\zeta_3^{-1} = \sum_{i=0}^{2}(-1)^i \mathrm{tr}(g^*|H^i(\mathcal{O}_X))$$
$$= \frac{m_1}{(1-\zeta_9^{-1})(1-\zeta_9^4)} + \frac{m_2}{(1-\zeta_9^{-2})(1-\zeta_9^5)} + \frac{m_7}{(1-\zeta_9^{-7})(1-\zeta_9)}.$$

Note that the minimal polynomial of ζ_9 over $\mathbb{Q}$ is $x^6+x^3+1=0$. Now, a direct calculation shows that there is no solution (m_1, m_2, m_7) of the equation above even in $\mathbb{Q}$.

Let us give a proof of part (ii). In the same manner as in part (i), one obtains a list of possible local actions of g at $P \in X^g$ as described in part (ii). Then, again by the holomorphic Lefschetz fixed point formula, one has:

$$1+\zeta_3^{-1} = \frac{m_1}{(1-\zeta_6^{-1})(1-\zeta_6^3)} + \frac{m_5}{(1-\zeta_6^{-5})(1-\zeta_6)}.$$

In addition, since $X^g \subset X^{g^2}$ and X^{g^2} is an eight-point set by Theorem 2.2, one has $m_1+m_5 \leqslant 8$. Finding all the non-negative integer solutions (m_1, m_5) in this range, we obtain the result.

The proof of part (iii) is similar. □

The next lemma completes the assertions (i) and (iii) of Proposition 3.2.

Lemma 5.2.

(i) *If* $G_N \simeq F_{128}$, *then* $I = 1$, 2 *or* 4.

(ii) *If* $G_N \simeq F_{384}$, *then* $I = 1$, 2 *or* 4.

Proof. Let us show part (i) of the lemma. We may assume that X is projective. Since $Q_{16} < F_{128}$ by Proposition 2.6, one has $\operatorname{rank} T(X) = 2$ by Lemma 4.4. Thus $I = 1$, 2, 4, 3 or 6 by Theorem 2.5. If $I = 6$, then $\mu_3 < \mu_I$ and $H := \alpha^{-1}(\mu_3)$ is a K3 group such that $H_N = F_{128}$ and $I = 3$. So, it suffices to show that $I \neq 3$. Assume that $I = 3$. Then, by Proposition 5.1, $G = F_{128} : \langle g\rangle$ where $\alpha(g) = \zeta_3$ and $\mathrm{ord}(g) = 3$. Since $[F_{128}, F_{128}] \simeq C_2 \times D_8$ by Proposition 2.6 and since the commutator subgroup is a characteristic subgroup, we have a new K3 group $K := (C_2 \times D_8) : \langle g\rangle$ such that $K_N \simeq C_2 \times D_8$ and $I = 3$. Let c_g be the conjugate action of g on $C_2 \times D_8$. Since $C_2 \times D_8$ has exactly one subgroup isomorphic to $C_2 \times C_4$, we have a new K3 group $H = (C_2 \times C_4) : \langle g\rangle$ such that $H_N = C_2 \times C_4$ and $I = 3$. Since $C_2 \times C_4$ contains exactly four order-4 elements, c_g fixes one of them, say τ. Since there are then exactly two involutions σ such that $C_2 \times C_4 = \langle \sigma, \tau\rangle$, the conjugate action c_g also fixes one such σ. Hence, $H = (C_2 \times C_4) \times \langle g\rangle$. Consider the element $h = \tau g$. Then $\operatorname{ord} h = 12$ and $\alpha(h) = \zeta_3$. Let M_i be the set of type i points of X^h in Proposition 5.1(iii). Then, by Proposition 5.1(iii), one of M_i is a one-point set, say $M = \{P\}$. Since H is commutative, we have $a(P) = P$ for all $a \in H_N$. However, one would then have

$$C_2 \times C_4 = H_N < \mathrm{SL}(T_{X,P}) \simeq \mathrm{SL}(2, \mathbb{C}),$$

a contradiction to the fact that finite abelian subgroups of $\mathrm{SL}(2, \mathbb{C})$ must be cyclic.

Let us show part (ii) of the lemma. Note that $F_{384} = \langle F_{128}, \tau\rangle$ for some element τ of order 3, and the Sylow 2-subgroups of F_{384} are exactly F_{128}, $\tau^{-1}F_{128}\tau$ and $\tau^{-2}F_{128}\tau^2$. For the same reason as in part (i), it suffices to show that $I \neq 3$. Assume that $I = 3$. Then, by Proposition 5.1, $G = F_{384} : \langle g\rangle$ where $\alpha(g) = \zeta_3$ and $\mathrm{ord}(g) = 3$. Consider a Sylow 3-subgroup H of G containing τ. Since $|G| = 2^9 \cdot 3^2$, we have $|H| = 3^2$. Since $|G_N| = 2^9 \cdot 3$, there is an element $h \in H$ such that $\alpha(h) = \zeta_3$. This element h also acts by the conjugate on the set $\{F_{128}, \tau^{-1}F_{128}\tau, \tau^{-2}F_{128}\tau^2\}$ of Sylow 2-subgroups of G_N. Thus, replacing h by $h\tau^i$ if necessary, we have $h^{-1}F_{128}h = F_{128}$, and obtain a new K3 group $K = \langle F_{128}, h\rangle$. Since $\alpha(h) = \zeta_3$ and h is an element of a 3-group H, we have $\operatorname{ord} h = 3$ and $K = F_{128} : \langle h\rangle$ by Proposition 5.1(i). However, this contradicts Lemma 5.2(i). □

In the rest of this section, we prove parts (ii) and (iv) of Proposition 3.2.

Proof of Proposition 3.2(ii)

The next proposition was obtained by [Muk88] in the course of his proof of Theorem 2.3. For the notation of groups, we follow [Muk88].

Proposition 5.3 [Muk88, Proposition 5.2 and Theorem 5.5]. *Let G_N be a solvable symplectic K3 group. Then, G_N and its order (indicated by $[*]$) is one of the following:*

(I) *2-group* $[2^n, 0 \leqslant n \leqslant 7]$;

(II) *2 · 3-group* $[2^n 3, 0 \leqslant n \leqslant 7]$; *moreover, if it is nilpotent, then G_N is isomorphic to C_3, C_6 or $C_2 \times C_6$;*

(III) $9 \mid |G_N|$ *and G_N is one of* $C_3^2[9]$, $A_{3,3}, C_3 \times S_3[18]$, $S_3 \times S_3, C_3^2 : C_4, A_4 \times C_3[36]$, N_{72}, M_9, $A_{4,3}[72]$, $A_4 \times A_4[144], A_{4,4}[288]$;

(IV) $5 \mid |G_N|$ *and G_N is one of* $C_5[5], D_{10}(= C_5 : C_2)[10], C_5 : C_4[20], C_2^4 : C_5[80], C_2^4 : D_{10}[160]$;

(V) $7 \mid |G_N|$ *and G_N is one of* $C_7[7], C_7 : C_3[21]$.

Let us show Proposition 3.2(ii) dividing into the five cases (I)–(V) in Proposition 5.3. By Proposition 5.3, we may assume that $I \geqslant 2$. Then X is projective as well.

First we consider the case where G_N lies in cases (III), (IV) or (V).

Lemma 5.4.

(a) *If G_N is in case (III), then $I \leqslant 12$.*

(b) *If G_N is in case (IV), then $I \leqslant 12$.*

(c) *If G_N is in case (V), then $I \leqslant 6$.*

Proof. First we shall show part (a) of the lemma. Choose a subgroup $C_3^2 \simeq \langle \tau_1, \tau_2 \rangle < G_N$. Then, by Proposition 4.5, one has

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} \leqslant \operatorname{rank} H^2(X, \mathbb{Z})^{\langle \tau_1, \tau_2 \rangle} = \frac{24 + 6 \times 8}{9} - 2 = 6.$$

Thus $\operatorname{rank} T(X) \leqslant 5$ and we have $I \leqslant 12$ by Theorem 2.5.

The proofs of part (b) (respectively (c)) are similar if we choose a subgroup C_5 (respectively C_7) in G_N:

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} \leqslant \operatorname{rank} H^2(X, \mathbb{Z})^{C_5} = \frac{24 + 4 \times 4}{5} - 2 = 6;$$

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} \leqslant \operatorname{rank} H^2(X, \mathbb{Z})^{C_7} = \frac{24 + 3 \times 6}{7} - 2 = 4. \qquad \square$$

Thus, when G_N is in cases (III), (IV), or (V), we see that $|G| = |G_N| \cdot I < 2^9 \cdot 3$ unless G_N is one of

$$\text{(i) } C_2^4 : D_{10}, \quad \text{(ii) } A_4 \times A_4, \quad \text{(iii) } A_{4,4}.$$

In case (i), we have $C_2^4 : C_5 \simeq H < G_N$. Here the order structure of H, which is also a subgroup (with no order-10 element) of the affine transformation group $\mathbb{F}_2^4 : \mathrm{GL}(4, \mathbb{F}_2)$, is as follows:

order	1	2	5
cardinality	1	15	64

Then, one has

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} \leqslant \operatorname{rank} H^2(X, \mathbb{Z})^{H} = \frac{24 + 8 \cdot 15 + 4 \cdot 64}{80} - 2 = 3.$$

Thus, $\operatorname{rank} T(X) = 2$ and $I \leqslant 6$. Hence $|G| < 160 \cdot 6 = 960 < 2^9 \cdot 3$.

Similarly, in case (ii), using the order structure of $G_N = A_4 \times A_4$ below

order	1	2	3	6
cardinality	1	15	80	48

one can calculate

$$\operatorname{rank} H^2(X,\mathbb{Z})^{G_N} = \frac{24 + 8\cdot 15 + 6\cdot 80 + 2\cdot 48}{144} - 2 = 3.$$

Thus, $\operatorname{rank} T(X) = 2$ and $I \leqslant 6$. Hence $|G| < 144\cdot 6 = 864 < 2^9\cdot 3$.

Note that $A_4 \times A_4 < A_{4,4}$ $(:= (S_4 \times S_4)\cap A_8)$. Then, from the calculation above, we also find that $I = 1, 2, 3, 4$ or 6 for $G_N = A_{4,4}$ (case (iii)). Note that $|A_{4,4}|\cdot 4 = 1152 < 2^9\cdot 3$, but $|A_{4,4}|\cdot 6 = 1728 > 2^9\cdot 3$. However, we can show the following lemma.

Lemma 5.5. *If $G_N \simeq A_{4,4}$, then $I \neq 3$, 6.*

Proof. As in Lemma 5.2, it suffices to show that $I \neq 3$. Assume that $I = 3$. Then, by Proposition 5.1, $G = A_{4,4} : \langle g\rangle$ where $\alpha(g) = \zeta_3$ and $\operatorname{ord}(g) = 3$. Since $[A_{4,4}, A_{4,4}] \simeq A_4 \times A_4$ and since the commutator subgroup is a characteristic subgroup, we have a new K3 group $H := (A_4 \times A_4) : \langle g\rangle$ such that $H_N \simeq A_4 \times A_4$ and $I = 3$. Note that $A_4 = C_2^2 : C_3$ so that $H_N = A_4 \times A_4 = C_2^4 : C_3^2$. Let H_3 be a Sylow 3-subgroup of H containing C_3^2. Since $|H| = 2^4\cdot 3^3$, we have $|H_3| = 3^3$. Note that H_3 acts on H_N by the conjugate, say ρ. Since C_2^4 is the normal Sylow 2-subgroup of H_N (and thus a characteristic subgroup of H_N), the conjugate action ρ makes C_2^4 stable, and we have a group homomorphism

$$\rho : H_3 \longrightarrow \operatorname{Aut}(C_2^4) \simeq \mathrm{GL}(4,\mathbb{F}_2).$$

Here $|\mathrm{GL}(4,\mathbb{F}_2)| = 2^6\cdot 3^2\cdot 5\cdot 7$. Thus, there is a non-trivial element $h \in \operatorname{Ker}\rho$. Since C_3^2 $(= H_3 \cap H_N)$ acts on C_2^4 faithfully, this h satisfies $\alpha(h) = \zeta_3$ (after replacing h by h^{-1} if necessary). Moreover, $\operatorname{ord}(h) = 3^n$ (by $h \in H_3$), and we have $\operatorname{ord}(h) = 3$ by Proposition 5.1. Thus, we obtain a new K3 group $K = C_2^4 \times \langle h\rangle$ such that $K_N = C_2^4$ and $I = 3$. Let σ be an involution in C_2^4. Then $h\sigma$ is of order 6 and satisfies $\alpha(h\sigma) = \zeta_3$. Let M_i be the set of type i fixed points of $X^{h\sigma}$ described in Proposition 5.1(ii). Then, by Proposition 5.1(ii), one of M_i is an at most two-point set, say $M = \{P, Q\}$. Since K is commutative, we have $a(\{P, Q\}) = \{P, Q\}$ for all $a \in K_N$. Then, one would have an index-2 subgroup $C_2^3 < K_N$ $(= C_2^4)$ such that $C_2^3 < \mathrm{SL}(T_{X,P}) \simeq \mathrm{SL}(2,\mathbb{C})$, a contradiction to the fact that finite abelian subgroups of $\mathrm{SL}(2,\mathbb{C})$ must be cyclic. $\square$

Next, we consider the case (I), i.e. the case where G_N is a 2-group. Set $|G_N| = 2^n$. By Theorem 2.3 and Corollary 2.4, we have $G_N < F_{128}$ (as abstract groups). In particular, $n \leqslant 7$ and if $n = 7$, then $G_N \simeq F_{128}$. So, by taking Lemma 5.2 into account, it suffices to show that $|G| < 2^9$ if $n \leqslant 6$.

Let us first consider the case where G_N has an order-8 element, say τ. In this case, we have

$$\operatorname{rank} H^2(X,\mathbb{Z})^{G_N} \leqslant \operatorname{rank} H^2(X,\mathbb{Z})^{\langle\tau\rangle} = \frac{24 + 8\times 1 + 4\times 2 + 2\times 4}{8} - 2 = 4.$$

Thus, $I \leqslant 6$ and we have $|G| \leqslant 2^6\cdot 6 < 2^9$.

Next we consider the case where G_N has no element of order 8. Then, we have the following order structure of G_N:

order	1	2	4
cardinality	1	$2k+1$	$2m$

where $k + m = 2^{n-1} - 1$. Moreover, $k \leqslant 17$ by $G_N < F_{128}$ and by Proposition 2.6(i).

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If $n = 6$, then $k + m + 1 = 2^5$ and one has

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} = \frac{24 + 8(2k+1) + 4 \cdot 2m}{2^6} - 2 = 2 + \frac{24 + 8k}{2^6} < 5, \quad \text{i.e.} \leqslant 4.$$

Here the last inequality is because $k \leqslant 17$. Hence, $\operatorname{rank} T(X) \leqslant 3$ and we have $I \leqslant 6$. Thus $|G| \leqslant 2^6 \cdot 6 < 2^9$.

If $n = 5$, then $k + m = 15$ and $k \leqslant 15$. Thus, one has

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} = \frac{24 + 8(2k+1) + 4 \cdot 2m}{2^5} - 2 = 2 + \frac{24 + 8k}{2^5} < 7, \quad \text{i.e.} \leqslant 6.$$

Hence, $\operatorname{rank} T(X) \leqslant 5$ and we have $I \leqslant 12$. Thus $|G| \leqslant 2^5 \cdot 12 < 2^9$.

Assume that $n \leqslant 4$. Then, if $|G| \geqslant 2^9$, we have $I \geqslant 2^5 = 32$. In this case, one can check that $\varphi(I) \geqslant 12$ (see, for instance, the explicit list in [MO98]). Then, $|G_N| \leqslant 2$ by the next lemma. We have then $|G| \leqslant 2 \cdot 66 < 2^9$.

Lemma 5.6. *Let G be a K3 group on X. If $\varphi(I) \geqslant 12$, then $|G_N| \leqslant 2$.*

Proof. By $\varphi(I) \geqslant 12$ and by Theorem 2.5, we have $\operatorname{rank} T(X) \geqslant 12$. Let g be a non-trivial element of G_N. Then $g^*|T(X) = \mathrm{id}$ and g fixes at least one ample class. Thus,

$$\operatorname{tr}(g^*|\mathrm{NS}(X)) \geqslant 1 + (-1) \cdot (22 - \operatorname{rank} T(X) - 1) = \operatorname{rank} T(X) - 20.$$

We also note that this inequality is strict if $\operatorname{ord} g = 3$. Combining this with the topological Lefschetz fixed point formula, one has

$$|X^g| = e(X^g) = 2 + \operatorname{tr}(g^*|\mathrm{NS}(X)) + \operatorname{tr}(g^*|\mathrm{T}(X)) \geqslant 2 \operatorname{rank} T(X) - 18 \geqslant 6.$$

Thus, g is an involution by Theorem 2.2 and by the remark above. Then, $G_N \simeq C_2^n$ for some n and one has by Proposition 4.5

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} = \frac{24 + 8(2^n - 1)}{2^n} - 2 = 6 + \frac{16}{2^n}.$$

Since $\operatorname{rank} T(X) < \operatorname{rank} H^2(X, \mathbb{Z})^{G_N}$, we have then

$$6 + \frac{16}{2^n} > 12, \quad \text{i.e. } n = 0, 1. \qquad \square$$

Finally we consider the case (II), i.e. the case where G_N is of order $2^n \cdot 3$. Then $n \leqslant 7$ and if $n = 7$, we have $G_N \simeq F_{384}$ by Theorem 2.3. The case $n = 7$ is settled by Lemma 5.2. Let H be a Sylow 2-subgroup of G_N. Then $|H| = 2^n$. By the argument in case (I) and by the fact that $H^2(X, \mathbb{Z})^{G_N} \subset H^2(X, \mathbb{Z})^H$, we have $I \leqslant 6$ if $n = 6$ and $I \leqslant 12$ if $n = 5$. Then $|G| < 2^9 \cdot 3$ for $n = 5$, 6. Assume that $n \leqslant 4$. Then, if $|G|$ $(= |G_N| \cdot I) \geqslant 2^9 \cdot 3$, then we have $I \geqslant 32$ and $\varphi(I) \geqslant 12$. Then, by Lemma 5.6, we would have $|G_N| \leqslant 2$, a contradiction. Thus $|G| < 2^9 \cdot 3$ as well. Now we have completed the proof of Proposition 3.2(ii).

Proof of Proposition 3.2(iv)

Since G is nilpotent, G_N is also nilpotent. The previous argument for the solvable case already settled the case when G_N is in case (I) of Proposition 5.3. If a nilpotent group G_N is in case (II) of Proposition 5.3, then $|G_N| \leqslant 12$ and

$$\operatorname{rank} H^2(X, \mathbb{Z})^{G_N} \leqslant \operatorname{rank} H^2(X, \mathbb{Z})^{C_3} = \frac{24 + 6 \cdot 2}{3} - 2 = 10.$$

Thus, $\operatorname{rank} T(X) \leqslant 9$ and $\varphi(I) \leqslant 8$. This implies $I \leqslant 30$ (see e.g. an explicit list in [MO98]). We have then $|G| \leqslant 12 \cdot 30 < 2^9$. If G_N is in cases (III), (IV) or (V) of Proposition 5.3, then

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G_N is either C_5, C_7 or C_3^2. (Recall that a nilpotent group must be the direct product of its Sylow subgroups.) Thus $|G_N| \leqslant 9$. Hence by Lemma 5.4, we have $|G| \leqslant 9{\cdot}12 < 2^9$ as well. Now we are done.

6. Invariant polarization of a maximal nilpotent K3 group

In this section, we shall prove Proposition 3.3 along similar lines to [Nik80b, Remark 1.14.7], [Kon99] and [OZ02]. In each approach, the orthogonal complement of the invariant lattice [Nik80a] plays a crucial role.

First, we recall some basic facts on the Niemeier lattices needed in our arguments. As in [OZ02], our main reference concerning Niemeier lattices and their relations with Mathieu groups is [CS99, chs. 10, 11, 16, 18].

Definition 6.1. The even negative definite unimodular lattices of rank 24 are called *Niemeier lattices*. There are exactly 24 isomorphism classes of the Niemeier lattices and each isomorphism class is uniquely determined by its root lattice R, i.e. the sublattice generated by all the roots, the elements x with $x^2 = -2$. Except for the Leech lattice, which contains no root, the other 23 lattices are the over-lattices of their root lattices.

We denote the Niemeier lattice N and its root lattice R by $N(R)$. Among the 24 Niemeier lattices, the most relevant one for us is $N(A_1^{\oplus 24})$. Two other Niemeier lattices $N(A_2^{\oplus 12})$ and $N(A_3^{\oplus 8})$ will also appear in our argument.

Let $N = N(R)$ be a non-Leech Niemeier lattice. Denote by $\mathrm{O}(N)$ (respectively by $\mathrm{O}(R)$) the group of isometries of N (respectively of R) and by $\mathrm{W}(N) = \mathrm{W}(R)$ the Weyl group generated by the reflections given by the roots of N. Here $\mathrm{O}(N) < \mathrm{O}(R)$ and $\mathrm{W}(N)$ is a normal subgroup of both $\mathrm{O}(N)$ and $\mathrm{O}(R)$. The invariant hyperplanes of the reflections divide $N \otimes \mathbb{R}$ into finitely many chambers. Each chamber is a fundamental domain of the action of $\mathrm{W}(R)$. Fix a basis $\mathcal{R} := \{r_i\}_{i=1}^{24}$ of R consisting of simple roots. The quotient group $\mathrm{S}(N) := \mathrm{O}(N)/\mathrm{W}(R)$ is then identified with a subgroup of the full symmetry group $\mathrm{S}(R) := \mathrm{O}(R)/\mathrm{W}(R)$ of the distinguished chamber $\mathcal{C} := \{x \in N \otimes \mathbb{R} \mid (x, r) > 0, r \in \mathcal{R}\}$, or a bit more concretely, $\mathrm{S}(N)$ and $\mathrm{S}(R)$ are subgroups of a larger group S_{24} as:

$$\mathrm{S}(N) = \{g \in \mathrm{S}(R) \mid g(N/R) = N/R\} < \mathrm{S}(R) = \mathrm{Aut}_{\mathrm{graph}}(\mathcal{R}) < \mathrm{Aut}_{\mathrm{set}}(\mathcal{R}) = S_{24},$$

where the action of $\mathrm{S}(R)$ on N/R ($\subset R^*/R$) is induced by the natural action on R^*/R. Here and hereafter, we denote by M^* the dual lattice of a non-degenerate lattice M and regard M naturally as a submodule of finite index of M^*.

The groups $\mathrm{S}(N)$ are explicitly calculated in [CS99, chs. 18, 16]. (See also [Kon98].) We need the following proposition.

Proposition 6.2 [CS99, chs. 18, 16]. *Let N be a non-Leech Niemeier lattice. Then, we have:*

(1) $\mathrm{S}(N) = M_{24}$ *if* $N = N(A_1^{\oplus 24})$;

(2) $\mathrm{S}(N) = C_2.M_{12}$ *if* $N = N(A_2^{\oplus 12})$;

(3) $\mathrm{S}(N) = C_2 : (C_2^{\oplus 3} : L_3(2))$ *if* $N = N(A_3^{\oplus 8})$; *and*

(4) *for other* N, $\mathrm{S}(N)$ *is a subgroup of either* $C_2.S_6$ *or* $C_3.S_6$.

Let us add a few remarks about the groups in Proposition 6.2(i)–(iii).

In case (i), i.e. the case where $N = N(R)$ and $R = A_1^{\oplus 24}$, we observe that

$$\mathcal{C}_{24} := N/R \simeq \mathbb{F}_2^{\oplus 12} \subset R^*/R = \bigoplus_{i=1}^{24} \mathbb{F}_2 \overline{r}_i \simeq \mathbb{F}_2^{\oplus 24}.$$

Here $\overline{r}_i := r_i/2 \bmod \mathbb{Z} r_i$. We note that $\mathcal{R} = \{r_i\}_{i=1}^{24}$ forms a Dynkin diagram of type $A_1^{\oplus 24}$.

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Let $\mathcal{P}(\mathcal{R})$ be the power set of $\mathcal{R}$. Then, we can identify $\mathcal{P}(\mathcal{R})$ with R^*/R by the following bijective correspondence:

$$\iota : \mathcal{P}(\mathcal{R}) \ni A \leftrightarrow \overline{r}_A := \frac{1}{2}\sum_{r_j \in A} r_j \ (\mathrm{mod}\, R) \in R^*/R = (A_1^{\oplus 24})^*/A_1^{\oplus 24}.$$

In what follows, we freely identify these two sets, and we define $|x|$ ($x \in R^*/R$) to be the cardinality of $\iota^{-1}(x)$.

Then, under the identification by ι, it is well known that $\emptyset, \mathcal{R} \in \mathcal{C}_{24}$ and that if $A \in \mathcal{C}_{24}$ ($A \neq \mathcal{R}$, $\emptyset$) then $|A|$ is either 8, 12, or 16. We call $A \in \mathcal{C}_{24}$ an Octad (respectively a Dodecad) if $|A| = 8$ (respectively 12). Note that $B \in \mathcal{C}_{24}$ with $|B| = 16$ is of the form $\mathcal{R} - A$ for some Octad A. It is also well known that the set of Octads forms a Steiner system $St(5, 8, 24)$ of $\mathcal{R}$ and generates $\mathcal{C}_{24}$ as an $\mathbb{F}_2$-linear space. In this case, the embeddings $\mathrm{S}(N) < \mathrm{S}(R) < S_{24}$ explained above coincide with the natural inclusions $M_{24} < S_{24} = S_{24}$ for $N = N(A_1^{\oplus 24})$.

In the second case, the Mathieu group $M_{12} = \mathrm{S}(N)/C_2$ acts naturally on the set of 12 connected components of the Dynkin diagram $A_2^{\oplus 12}$ and C_2 interchanges the two vertices of all the components. We also note that $|M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$.

In the third case, we identify (non-canonically) the set of eight connected components of the Dynkin diagram $A_3^{\oplus 8}$ with the three-dimensional linear space $\mathbb{F}_2^{\oplus 3}$ over $\mathbb{F}_2$ by letting one connected component be 0. The group $C_2 : (C_2^{\oplus 3} : L_3(2))$ is the semi-direct product, where C_2 interchanges the two edges of all the components, $C_2^{\oplus 3}$ is the group of parallel transformations of the affine space $\mathbb{F}_2^{\oplus 3}$ and $L_3(2)$ ($\simeq L_2(7)$) is the linear transformation group of $\mathbb{F}_2^{\oplus 3}$.

As in [Kon99] and [OZ02], the next embedding theorem due to Kondo [Kon98] is an important ingredient in our proof.

Theorem 6.3 [Kon98, Lemmas 5 and 6]. *Let K be a symplectic K3 group on X. Set $L := H^2(X, \mathbb{Z})$, $L^K := \{x \in L \,|\, h(x) = x \,(\forall h \in K)\}$ and $L_K := \{y \in L \,|\, (y, x) = 0 \,(\forall x \in L^K)\}$. Then, the following hold:*

- (i) *there is a non-Leech Niemeier lattice N such that $L_K \subset N$. Moreover, the faithful action of K on L_K extends to an action on N so that $L_K \simeq N_K$ and that N^K contains a root, say r^0. Here the sublattices N^K and N_K of N are defined in the same way as L^K and L_K of L;*
- (ii) *take $\mathcal{R}$ so that $r^0 \in \mathcal{R}$. Then, the group action of K on N preserves the distinguished Weyl chamber $\mathcal{C}$ with respect to $\mathcal{R}$, and the naturally induced homomorphism $K \to \mathrm{S}(N)$ is injective.*

Corollary 6.4 [Kon98]. *Under the notation of Theorem 6.3, one has:*

- (i) $\mathrm{rank}\, N^K = \mathrm{rank}\, L^K + 2$;
- (ii) $(L^K)^*/L^K \simeq (N^K)^*/N^K$, *in particular,* $|\det N^K| = |\det L^K|$.

Proof. The assertion (i) follows from $\mathrm{rank}\, N^K = 24 - \mathrm{rank}\, N_K$, $\mathrm{rank}\, L^K = 22 - \mathrm{rank}\, L_K$, and $N_K \simeq L_K$. Since L and N are unimodular and since the embeddings $L^K \subset L$ and $N^K \subset N$ are primitive, we have natural isomorphisms $(L^K)^*/L^K \simeq (L_K)^*/L_K$ and $(N^K)^*/N^K \simeq (N_K)^*/N_K$ by [Nik80b, Proposition 1.6.1]. Now the result follows from $L_K \simeq N_K$. For the last equality, we may just note that $|\det M| = |M^*/M|$ for a non-degenerate lattice M. $\square$

We are now ready to prove Proposition 3.3. Let G be a K3 group on X such that $|G| = 2^9$. We denote by $K := G_N$ the symplectic part and by I the transcendental value. By Proposition 3.2, $K = F_{128}$ (as abstract groups) and $I = 4$. In particular, X is projective. We have also $\mathrm{rank}\, L^K = 3$ by $K = F_{128} > Q_{16}$ and by Proposition 3.1(i). Thus, $\mathrm{rank}\, T(X) = 2$ and $\mathrm{NS}(X)^G = \mathrm{NS}(X)^K = \mathbb{Z}H$ for some ample class H. As in Theorem 6.3, we set $L := H^2(X, \mathbb{Z})$. We shall fix these notations until the end of this section.

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It remains to show $(H^2) = 4$. This will be completed in Lemma 6.11.

Let us first determine the Niemeier lattice N for our K.

Lemma 6.5. *The Niemeier lattice N in Theorem 6.3 for K is $N(A_1^{\oplus 24})$.*

Proof. By Theorem 6.3(ii), $|\mathrm{S}(N)|$ must be divided by $|K| = 2^7$. Thus, N is either $N(A_1^{\oplus 24})$, $N(A_2^{\oplus 12})$ or $N(A_3^{\oplus 8})$ by Proposition 6.2. Suppose that the second case occurs. Since K fixes at least one element in $\mathcal{R}$ by Theorem 6.3(i), we have $K < M_{12}$. However, this is impossible, because $|K| = 2^7$ but $|M_{12}| = 2^6 \cdot k$ $((2,k) = 1)$. Suppose that the third case occurs. Again for the same reason as above, we have $K < C_2.L_3(2)$. However, this is impossible, because $|K| = 2^7$ but $|C_2.L_3(2)| = 2^4 \cdot k'$ $((2,k') = 1)$. Now we are done. $\square$

From now on we set $N := N(A_1^{\oplus 24})$, $R := A_1^{\oplus 24}$ and take $\mathcal{R} = \{r_i\}_{i=1}^{24}$ as in Theorem 6.3(ii). By Proposition 6.2, Theorem 6.3(ii) and Lemma 6.5, we have

$$K < M_{24} < S_{24} = \mathrm{Aut}_{\mathrm{graph}}(\mathcal{R}) = \mathrm{Aut}_{\mathrm{set}}(\mathcal{R}).$$

Lemma 6.6. *The orbit decomposition type of K on $\mathcal{R}$ is* $[1, 1, 2, 4, 16]$.

Proof. Note that $\operatorname{rank} R^K = \operatorname{rank} N^K = 5$ by $\operatorname{rank} L^K = 3$ and by Corollary 6.4(i). Thus $\mathcal{R}$ is divided into exactly five K-orbits. Since K is a 2-group and K fixes at least one element by Proposition 6.2(i), the orbit decomposition type is of the form $[1, 2^b, 2^c, 2^d, 2^e]$. We may assume that $0 \leqslant b \leqslant c \leqslant d \leqslant e$. In addition, $1 + 2^b + 2^c + 2^d + 2^e = |\mathcal{R}| = 24$. It is now easy to see that $(b, c, d, e) = (0, 1, 2, 4)$. $\square$

By Lemma 6.6, after renumbering of the elements of $\mathcal{R}$, we have

$$R^K = \langle s_1, s_2, s_3, s_4, s_5 \rangle$$

where

$$s_1 = r_1, \quad s_2 = r_2, \quad s_3 = r_3 + r_4, \quad s_4 = r_5 + \cdots + r_8, \quad s_5 = r_9 + \cdots + r_{24}.$$

Lemma 6.7. *We have*

$$N^K = \left\langle s_1, s_2, s_3, \frac{s_1 + s_2 + s_3 + s_4}{2}, \frac{s_5}{2} \right\rangle.$$

In particular, $(N^K)^*/N^K \simeq \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/8$.

Proof. Since $R^K \subset N^K \subset (R^*)^K = \langle s_1/2, s_2/2, s_3/2, s_4/2, s_5/2 \rangle$, the lattice N^K is generated by R^K and by the (representatives of) K-invariant elements of $\mathcal{C}_{24}$. Let us find out all such elements in $\mathcal{C}_{24}$. In what follows, we freely identify $\mathcal{C}_{24}$ with a subset of $\mathcal{P}(\mathcal{R})$ by ι, as explained after Proposition 6.2. By the shape of the orbit decomposition (Lemma 6.6), there is no K-invariant Dodecad. Moreover, for the same reason, if there is a K-invariant Octad, then it must be

$$(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8)/2 = (s_1 + s_2 + s_3 + s_4)/2,$$
$$\text{i.e. } \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8\}.$$

Let us show that this is indeed an Octad, i.e. an element of $\mathcal{C}_{24}$. Recall that the set of Octads of $\mathcal{C}_{24}$ forms a Steiner system $St(5, 8, 24)$ of $\mathcal{R}$. Then, there is an Octad $A \in \mathcal{C}_{24}$ containing a five-element set $\{r_1, r_5, r_6, r_7, r_8\}$. Note that

$$K(\{r_1, r_5, r_6, r_7, r_8\}) = \{r_1, r_5, r_6, r_7, r_8\}$$

by Lemma 6.6 and by $s_1, s_4 \in R^K$. Then, by the Steiner property, we have $A = g(A)$ for all $g \in K$. Thus, this A is a K-invariant Octad. So $(s_1 + s_2 + s_3 + s_4)/2$, which is the only possible candidate A, is indeed a K-invariant Octad.

Since the length-16 element of $\mathcal{C}_{24}$ is the complement of an Octad, it follows that $\{r_9, r_{10}, \dots, r_{16}\}$ is the unique length-16, K-invariant element of $\mathcal{C}_{24}$. Hence

$$N^K = \left\langle s_1, s_2, s_3, s_4, s_5 \frac{s_1 + s_2 + s_3 + s_4}{2}, \frac{s_5}{2} \right\rangle,$$

that is,

$$N^K = \left\langle s_1, s_2, s_3, \frac{s_1 + s_2 + s_3 + s_4}{2}, \frac{s_5}{2} \right\rangle.$$

The intersection matrix of N^K with respect to this basis is

$$-\begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 1 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix},$$

and the elementary divisors of this matrix are $(1, 1, 4, 8, 8)$. This implies the result. □

LEMMA 6.8.

(i) $(L^K)^*/L^K \simeq \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/8$. *In particular*, $|\det L^K| = 2^8$.

(ii) *If* $x \in L^K$, *then* $(x^2) \equiv 0 \bmod 4$.

Proof. The assertion (i) follows from Lemma 6.7 and Corollary 6.4(ii). By rank $L^K = 3$ and by (i), we can choose an integral basis $\langle f_1, f_2, f_3 \rangle$ of $(L^K)^*$ so that $\langle 4f_1, 8f_2, 8f_3 \rangle$ forms an integral basis of L^K. For $x = x_1 \cdot 4f_1 + x_2 \cdot 8f_2 + x_3 \cdot 8f_3$ ($x_i \in \mathbb{Z}$), one has

$$(x^2) = 4x_1(x, f_1) + 8x_2(x, f_2) + 8x_3(x, f_3) \in 4\mathbb{Z}.$$

This implies the second assertion. □

LEMMA 6.9. *With respect to a suitable integral basis* $\langle v_1, v_2 \rangle$ *of* $T(X)$, *the intersection matrix of* $T(X)$ *becomes of the following form:*

$$\begin{pmatrix} 4m & 0 \\ 0 & 4m \end{pmatrix} \text{ for some } m \in \mathbb{Z}.$$

Proof. By Theorem 2.5, we have an isomorphism $T(X) \simeq \mathbb{Z}[\sqrt{-1}]$ as $\mathbb{Z}[\sqrt{-1}]$-modules. Since $\sqrt{-1}$ acts on the integral basis $\langle e_1 := 1, e_2 := \sqrt{-1} \rangle$ of $\mathbb{Z}[\sqrt{-1}]$ as $e_1 \mapsto e_2$, $e_2 \mapsto -e_1$, the group $G/K = \langle g \bmod K \rangle \simeq \mu_4$ acts on the corresponding integral basis $\langle v_1, v_2 \rangle$ of $T(X)$ by $g^*(v_1) = v_2$ and $g^*(v_2) = -v_1$. Thus $(v_1, v_2) = (g^*(v_1), g^*(v_2)) = (v_2, -v_1)$, and $(v_1, v_2) = 0$. Similarly, $(v_1, v_1) = (g^*(v_1), g^*(v_1)) = (v_2, v_2)$. The result now follows from these two equalities and Lemma 6.8(ii). □

LEMMA 6.10. *Set* $l := [L^K : \mathbb{Z}H \oplus T(X)]$. *Then* $l = 1$ *or* 2. *Moreover, if* $l = 2$, *then*

$$L^K = \mathbb{Z}\left\langle \frac{H + v_1 + v_2}{2}, v_1, v_2 \right\rangle.$$

Here $\langle v_1, v_2 \rangle$ *is an integral basis of* $T(X)$ *as in Lemma 6.9.*

Proof. The proof is identical to [Kon99, p. 1248] and [OZ02, p. 177]. □

The next lemma completes the proof of Proposition 3.3.

LEMMA 6.11. *We have* $(H^2) = 4$.

Proof. By Lemma 6.8(ii), we can write $(H^2) = 4n$ for some positive integer n. We need to show that $n = 1$. Let m be a positive integer in Lemma 6.9.

First consider the case where $l = 2$ (here l is the index defined in Lemma 6.10). In this case, we have by Lemma 6.8(i)

$$4 \cdot 2^8 = l^2 \cdot \det L^K = (H^2) \cdot \det T(X) = 4n \cdot 16m^2.$$

Thus $nm^2 = 16$. Moreover, by $(H + v_1 + v_2)/2 \in L^K$ and by Lemma 6.8(ii), we have

$$n + 2m = ((H + v_1 + v_2)/2)^2) \equiv 0 \bmod 4.$$

Thus $(m, n) = (2, 4)$ and the intersection matrix of L^K (with respect to the basis in Lemma 6.10) becomes

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 0 \\ 4 & 0 & 8 \end{pmatrix}.$$

However, the elementary divisors of this matrix are $(4, 4, 16)$, a contradiction to Lemma 6.8(i).

Next consider the case where $l = 1$. In this case, we have

$$2^8 = \det L^{G_N} = (H^2) \cdot \det T(X) = 4n \cdot 16m^2.$$

Thus $nm^2 = 4$ and (m, n) is either $(1, 4)$ or $(2, 1)$. Assume that $(m, n) = (1, 4)$. Then, the intersection matrix of $L^K = \mathbb{Z}H \oplus T(X)$ (with respect to the basis $\langle H, v_1, v_2\rangle$) becomes

$$\begin{pmatrix} 16 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

However, the elementary divisors of this matrix are $(4, 4, 16)$, a contradiction to Lemma 6.8(i). Thus $(m, n) = (2, 1)$ and we are done. □

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