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SYMMETRY RESULT FOR SOME OVERDETERMINED VALUE PROBLEMS

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Abstract

The aim of this article is to prove a symmetry result for several overdetermined boundary value problems. For the two first problems, our method combines the maximum principle with the monotonicity of the mean curvature. For the others, we use essentially the compatibility condition of the Neumann problem.

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1. Introduction

We assume throughout that $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded ball which contains all the domains we use. If ω is an open subset of D, let ν be the outward normal to $\partial\omega$ and let $|\partial\omega|$ (respectively $|\omega|$) be the perimeter (respectively the volume) of ω.

Consider the following overdetermined boundary value problem:

$$\mathrm{S}(k)\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega, \\ \quad u_\Omega = 0 & \text{on } \partial\Omega, \\ -\dfrac{\partial u_\Omega}{\partial \nu} = k & \text{on } \partial\Omega. \end{cases}$$

Notice that since u_Ω vanishes on $\partial\Omega$ then $-(\partial u_\Omega/\partial\nu) = |\nabla u_\Omega|$.

In 1971, Serrin [19] proved that if Problem $\mathrm{S}(k)$ has a solution $u_\Omega \in C^2(\overline{\Omega})$ then Ω must be a ball and u_Ω is radially symmetric. The method used by Serrin combines the maximum principle together with the device of moving planes [11] to a critical position and then showing that the solution is symmetric about the limiting plane. In the same year, Weinberger [21] gave a simplified proof for this problem. His

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strategy of proof consists first of showing that $|\nabla u|^2 + (2/N)u = k^2$ in Ω and then deriving a radial symmetry from this. Concerning other methods, we refer the reader to the paper by Payne [15] and the references therein. The last problem studied in Section 5 comes from this paper. For more details about the symmetry results see [9, Introduction] and the references therein. Fragalà *et al.* [9], obtained their symmetry result by combining a maximum principle for a suitable P-function with some geometric arguments involving the mean curvature of $\partial\Omega$. However, they assumed the solution Ω to be star-shaped with respect to the origin. This assumption seems to be crucial for the proof of their main results and they cannot remove it.

The method we present here needs the use of the maximum principle together with the monotonicity of the mean curvature. Therefore, it can be extended to more general divergence operators such as the p-Laplacian for which one can use Hopf's comparison principle or the operator $-\mathrm{div}(A(|\nabla u|)\nabla u)$ for which a boundary point principle is considered [9]. The novelty of our method is the following. First, to prove the main results of Section 3, we do not ask Ω to be star-shaped with respect to the origin. Second, this method can be extended to other problems such as $\mathrm{P}(c)$, see Section 4:

$$\mathrm{P}(c)\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega,\ u_\Omega = 0 \text{ on } \partial\Omega \text{ denoted } P(\Omega), \\ -\Delta v_\Omega = u_\Omega & \text{in } \Omega,\ v_\Omega = 0 \text{ on } \partial\Omega \text{ denoted } Q(\Omega), \\ |\nabla u_\Omega||\nabla v_\Omega| = c & \text{on } \partial\Omega. \end{cases}$$

The problem $\mathrm{P}(c)$ arises from the variational problem in probability [10, 13]. Fromm and McDonald [10] related this problem to the fundamental result of Serrin. Then, using the moving plane method combined with Serrin's boundary point lemma, they showed that if this problem admits a solution Ω then it must be a ball. Huang and Miller [12] established the variational formulas for maximizing the functionals (which they considered) over C^k domains with a volume constraint and obtained the same symmetry result for their maximizers.

Section 2 contains some preliminary results which are useful for solving the shape optimization problems presented in Sections 3 and 4. Section 3 is devoted to the problem $\mathrm{S}(k)$ whereas Section 4 concerns the problem $\mathrm{P}(c)$. In Section 5, by using the compatibility condition of the Neumann problem [14], we obtain the same symmetry result for other boundary value problems for which the overdetermined condition is not constant.

2. Preliminaries

DEFINITION 2.1. Let K_1 and K_2 be two compact subsets of D. We call a Hausdorff distance of K_1 and K_2 (or briefly $d_H(K_1, K_2)$) the following positive number:

$$d_H(K_1, K_2) = \max[\rho(K_1, K_2), \rho(K_2, K_1)],$$

where $\rho(K_i, K_j) = \max_{x\in K_i} d(x, K_j)$, $i, j = 1, 2$, and $d(x, K_j) = \min_{y\in K_j} |x - y|$.

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DEFINITION 2.2. Let ω_n be a sequence of open subsets of D and let ω be an open subset of D. Let K_n and K be their complements in $\bar{D}$. We say that the sequence ω_n converges in the Hausdorff sense, to ω (or briefly $\omega_n \xrightarrow{H} \omega$) if

$$\lim_{n\to+\infty} d_H(K_n, K) = 0.$$

DEFINITION 2.3. Let $\{\omega_n, \omega\}$ be a sequence of open subsets of D. We say that the sequence ω_n converges in the compact sense to ω (or briefly $\omega_n \xrightarrow{K} \omega$) if:

- every compact subset of ω is included in ω_n for n *large enough*; and
- every compact subset of $\bar{\omega}^c$ is included in $\overline{\omega}_n^c$ for n *large enough*.

DEFINITION 2.4. Let $\{\omega_n, \omega\}$ be a sequence of open subsets of D. We say that the sequence ω_n converges in the sense of characteristic functions to ω (or briefly $\omega_n \xrightarrow{L} \omega$) if χ_{ω_n} converges to χ_ω in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $p \neq \infty$ (χ_ω is the characteristic function of ω).

DEFINITION 2.5 ([3]). Let C be a compact convex set. The bounded domain ω satisfies C-GNP if:

(1) $\omega \supset \mathrm{int}(C)$;
(2) $\partial\omega \setminus C$ is locally Lipschitz;
(3) for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected; and
(4) for every $x \in \partial\omega \setminus C$ the inward normal ray to ω (if exists) meets C.

REMARK 2.6. If Ω satisfies the C-GNP and C has a nonempty interior, then Ω is connected.

Put

$$\mathcal{O}_C = \{\omega \subset D \mid \omega \text{ satisfies } C\text{-GNP}\}.$$

THEOREM 2.7. *If $\omega_n \in \mathcal{O}_C$, then there exist an open subset $\omega \subset D$ and a subsequence (again denoted by ω_n) such that* (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\omega_n \xrightarrow{K} \omega$, (iii) χ_{ω_n} *converges to* χ_ω *in $L^1(D)$ and* (iv) $\omega \in \mathcal{O}_C$. *Furthermore, the assertions* (i), (ii) *and* (iii) *are equivalent.*

Barkatou proved this theorem [3, Theorem 3.1] and the equivalence between (i), (ii) and (iii) [3, Propositions 3.4, 3.5, 3.6, 3.7 and 3.8].

PROPOSITION 2.8. *Let $\{\omega_n, \omega\} \subset \mathcal{O}_C$ such that $\omega_n \xrightarrow{H} \omega$. Let u_n and u_ω be respectively the solutions of $P(\omega_n)$ and $P(\omega)$. Then u_n converges strongly in $H^1_0(D)$ to u_ω (u_n and u_ω are extended by zero in D).*

This proposition was proven for $N = 2$ or 3 [3, Theorem 4.3].

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DEFINITION 2.9. Let C be a convex set. We say that an open subset ω has the C-SP if:

(1) $\omega \supset \operatorname{int}(C)$;
(2) $\partial\omega \setminus C$ is locally Lipschitz;
(3) for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected; and
(4) for all $x \in \partial\omega \setminus C$ $K_x \cap \omega = \emptyset$, where K_x is the closed cone defined by

$$\{y \in \mathbb{R}^N \mid (y - x) \cdot (z - x) \leq 0, \text{ for all } z \in C\}.$$

REMARK 2.10. K_x is the normal cone to the convex hull of C and $\{x\}$.

PROPOSITION 2.11 ([3, Proposition 2.3]). *ω has the C-GNP if and only if ω satisfies the C-SP.*

DEFINITION 2.12 ([8]). We say that a domain ω satisfies the ε-cone property if for all $x \in \partial\omega$ there exists a direction vector $\xi \in \mathbb{R}^N$ such that the cone $C(y, \xi, \varepsilon) \subset \omega$ for all $y \in B(x, \varepsilon) \cap \overline{\omega}$. ε denotes both the angle and the height of the cone.

Denoting by $\mathcal{O}_\varepsilon$ the class of domains which have the ε-cone property, we have the following lemma.

LEMMA 2.13 ([8]). *If $\omega_n \in \mathcal{O}_\varepsilon$, then there exist an open subset $\omega \subset D$ and a subsequence (again denoted by ω_n) such that* (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\bar{\omega_n} \xrightarrow{H} \bar{\omega}$, (iii) $\partial\omega_n \xrightarrow{H} \partial\omega$, (iv) *$\chi_{\omega_n}$ converges to χ_ω in $L^1(D)$,* (v) *$\omega \in \mathcal{O}_\varepsilon$ and* (vi) *u_{ω_n} converges strongly in $H_0^1(D)$ to u_ω (u_ω is the solution of $P(\omega)$).*

PROPOSITION 2.14 ([5, Theorem 3.5]). *Let v_n and v_ω be respectively the solutions of the Dirichlet problems $P(\omega_n, g_n)$ and $P(\omega, g)$. If g_n converges strongly in $H^{-1}(D)$ to g then v_n converges strongly in $H_0^1(D)$ to v_ω (v_n and v_ω are extended by zero in D).*

LEMMA 2.15 ([6, 17]). *Let ω_n be a sequence of open and bounded subsets of D. There exist a subsequence (again denoted by ω_n) and some open subset ω of D such that:*

(1) *ω_n converges to ω in the Hausdorff sense; and*
(2) $|\partial\omega| \leq \liminf_{n\to\infty} |\partial\omega_n|$.

2.1. Shape derivative In this subsection, we use the standard tool of the domain derivative to write down the optimality conditions. Before doing this, recall the definition of the domain derivative [20]. Suppose that ω is of class C^2. Consider a deformation field $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and set $\omega_t = \{x + tV(x) \mid x \in \omega\}$, $t > 0$. The application $\mathrm{Id} + tV$ (a perturbation of the identity) is a Lipschitz diffeomorphism for t small enough and, by definition, the derivative of J at ω in the direction V is

$$dJ(\omega, V) = \lim_{t\to 0} \frac{J(\omega_t) - J(\omega)}{t}.$$

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As the functional J depends on the domain ω through the solution of some Dirichlet problem, we need to also define the domain derivative u'_ω of u_ω:

$$u'_\omega = \lim_{t\to 0} \frac{u_{\omega_t} - u_\omega}{t}.$$

Furthermore, u'_ω is the solution of the following problem [20]:

$$\begin{cases} -\Delta u'_\omega = 0 & \text{in } \omega, \\ u'_\omega = -\dfrac{\partial u_\omega}{\partial \nu} V \cdot \nu & \text{on } \partial\omega. \end{cases} \tag{2.1}$$

Now to compute the derivative of the functionals we consider below, recall the following [20].

(1) The shape derivative of the volume is

$$\int_{\partial\omega} V \cdot \nu \, d\sigma. \tag{2.2}$$

(2) The shape derivative of the perimeter is

$$\int_{\partial\omega} (N-1) H_{\partial\omega} V \cdot \nu \, d\sigma. \tag{2.3}$$

(3) Suppose that u_ω is in $H_0^1(D)$ and ω is of class C^2.

(a) If $F(\omega) = \int_\omega u_\omega^2 \, dx$, then the Hadamard formula gives

$$dF(\omega, V) = 2 \int_\omega u_\omega u'_\omega \, dx.$$

But v_ω is in $H_0^1(D)$ and $-\triangle v_\omega = u_\omega$ in ω, so by Green's formula we obtain

$$dF(\omega, V) = 2 \int_{\partial\omega} |\nabla u_\omega||\nabla v_\omega| V \cdot \nu \, d\sigma.$$

(b) If $G(\omega) = \int_\omega |\nabla u_\omega|^2 \, dx$, by the Hadamard formula we get

$$dG(\omega, V) = \int_{\partial\omega} |\nabla u_\omega|^2 V \cdot \nu \, d\sigma.$$

Since the set ω satisfies some geometric property (the ε-cone property or the C-GNP), we ask the deformation set ω_t to satisfy the same property (for t sufficiently small). To keep the ε-cone property any direction is admissible. The aim in what follows is to prove the same thing for the C-GNP. With ω having the C-GNP, by Proposition 2.11, it satisfies the C-SP. Then

$$\text{for all } x \in \partial\omega \setminus C : K_x \cap \omega = \emptyset.$$

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For t sufficiently small, let $\omega_t = \omega + tV(\omega)$ be the deformation of ω in the direction V. Let $x_t \in \partial\omega_t$. There exists $x \in \partial\omega$ such that $x_t = x + tV(x)$. Using the definition of K_{x_t} and the equality above, we get (for t small enough and for every displacement V)

$$\text{for all } x_t \in \partial\omega_t \setminus C : K_{x_t} \cap \omega_t = \emptyset,$$

which means that ω_t satisfies the C-SP (and so the C-GNP) for every direction V when t is sufficiently small.

3. Problem S(k)

3.1. Auxiliary lemmas

LEMMA 3.1. *Let B_ρ be a solution of* S(k) *so then $\rho = Nk$.*

PROOF. Let u_ρ be the solution of $P(B_\rho)$. Using polar coordinates, u_ρ verifies

$$-u''_\rho - \frac{N-1}{r}u'_\rho = 1 \quad \text{for } r \in]0, \rho[,$$
$$u_\rho(\rho) = 0.$$

By the first equation, $(r^{N-1}u'_\rho)' = -r^{N-1}$. Since $u_\rho(\rho) = 0$, we get

$$r^{N-1}u'_\rho(r) = \rho^{N-1}u'_\rho(\rho) + \int_r^\rho s^{N-1}\, ds.$$

As $r \to 0$, $r^{N-1}u'_\rho(r) \to 0$ (otherwise we get a distributional contribution to Δu_ρ at the origin). Thus

$$-u'_\rho(\rho) = \frac{1}{\rho^{N-1}} \int_0^\rho s^{N-1}\, ds = \frac{\rho}{N}.$$

Now if B_ρ is a solution of S(k) then $-u'_\rho(\rho) = k$. Thus $\rho = Nk$. □

LEMMA 3.2. *Let Ω be a solution of* S(k). *Let $\omega \supset \Omega$ and let u_ω be the solution of $P(\omega)$. If $\partial\omega \cap \partial\Omega \neq \emptyset$ and if $|\nabla u_\omega| \leq k$ on $\partial\omega \cap \partial\Omega$ then $\omega = \Omega$.*

PROOF. Suppose by contradiction that Ω is different to ω. As $\omega \supset \Omega$, $\partial\omega \neq \partial\Omega$. But $\partial\omega \cap \partial\Omega \neq \emptyset$, so applying the maximum principle to u_ω and u_Ω and using the fact that Ω is a solution of S(k), we obtain

$$k = |\nabla u_\Omega| < |\nabla u_\omega| \leq k \quad \text{on } \partial\omega \cap \partial\Omega,$$

which gives the contradiction. □

As a consequence, we have the following corollary.

COROLLARY 3.3. *Let ω and Ω be two solutions of* S(k). *Suppose that $\Omega \subset \omega$ and $\partial\omega \cap \partial\Omega \neq \emptyset$, then $\omega = \Omega$.*

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3.2. Shape optimization problems Let Ω be a solution of $\mathrm{S}(k)$. Let B_ρ be the ball centred at the origin of radius ρ. Denote by B the greatest ball contained in Ω. Denote by $\mathcal{O}_B$ (respectively $\mathcal{O}_{B_\rho}$) the class of all domains which satisfy the B-GNP (respectively the B_ρ-GNP). Set

$$\mathcal{O}_\Omega = \{D \supset \omega \supset \Omega \mid \omega \in \mathcal{O}_B\},$$
$$\mathcal{O}^\Omega = \{\omega \subset \Omega \subset D \mid \omega \in \mathcal{O}_\varepsilon\},$$

and

$$\mathcal{O}_\rho = \{D \supset \omega \supset B_\rho \mid \omega \in \mathcal{O}_{B_\rho}\}.$$

Consider the following functionals:

$$j_1(\omega) = \frac{N}{N-1} k^2 |\partial \omega| - \frac{1}{2} \int_\omega |\nabla u_\omega|^2,$$
$$j_2(\omega) = \frac{N}{N-1} k |\partial \omega| - |\omega|,$$

and

$$j_3(\omega) = \frac{k^2}{2} |\omega| - \frac{1}{2} \int_\omega |\nabla u_\omega|^2.$$

Here u_ω is the solution of $P(\omega)$.

We then have the following propositions.

PROPOSITION 3.4. *Suppose $N \in \{2, 3\}$. There exists $\Omega_1 \in \mathcal{O}_\Omega$ which is of class C^2 such that:*

(1) $j_1(\Omega_1) = \min\{j_1(\omega) \mid \omega \in \mathcal{O}_\Omega\}$ *and u_{Ω_1} is the solution of $P(\Omega_1)$;*

(2)

$$\begin{cases} |\nabla u_{\Omega_1}| \le N k^2 H_{\partial\Omega_1} & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ |\nabla u_{\Omega_1}| = N k^2 H_{\partial\Omega_1} & \text{on } \partial\Omega_1 \setminus \partial\Omega. \end{cases} \tag{3.1}$$

PROOF. To get item (1), we use Theorem 2.7, Proposition 2.8 and item (2) of Lemma 2.15. For item (2), using the same notation as in Section 2.1, to get Ω in $(\Omega_1)_t$ (for t small enough) the admissible directions V must satisfy

$$V \cdot \nu \ge 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_1.$$

Notice that, for $\partial\Omega_1 \setminus \partial\Omega$, each V is admissible. Now since u_{Ω_1} vanishes on $\partial\Omega_1$, (2.3) and 3(b) above imply

$$d j_1(\Omega_1, V) = \int_{\partial\Omega_1} (N H_{\partial\Omega_1} k^2 - |\nabla u_{\Omega_1}|^2) V \cdot \nu \, d\sigma.$$

Since $d j_1(\Omega_1, V) \ge 0$ for each admissible direction V, according to the preceeding calculations we obtain (3.1). □

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PROPOSITION 3.5. *Suppose that $N \geq 2$. There exists $\Omega_2 \in \mathcal{O}^\Omega$ such that:*

(1) $j_2(\Omega_2) = \min\{j_2(\omega) \mid \omega \in \mathcal{O}^\Omega\}$;

(2) *if Ω_2 is of class C^2 then*

$$\begin{cases} NkH_{\partial\Omega_2} \leq 1 & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ NkH_{\partial\Omega_2} = 1 & \text{on } \partial\Omega_2 \setminus \partial\Omega. \end{cases} \tag{3.2}$$

PROOF. The first item is obtained by using (iv) and (v) of Lemma 2.13 together with item (2) of Lemma 2.15. The continuity with respect to the domains for the Dirichlet problem $P(\Omega_2)$ is obtained by (vi) of Lemma 2.13. For the second item, on $\partial\Omega_2 \setminus \partial\Omega$, any direction V is admissible whereas V must satisfy

$$V \cdot \nu \leq 0 \quad \text{on } \partial\Omega \cap \partial\Omega_2.$$

Then, arguing as above, (2.2) and (2.3) imply (3.2). □

PROPOSITION 3.6. *Suppose that $N \in \{2, 3\}$. There exists $\Omega_3 \in \mathcal{O}_\rho$ which is of class C^2 such that:*

(1) $j_3(\Omega_3) = \min\{j_3(\omega) \mid \omega \in \mathcal{O}_\rho\}$ *and u_{Ω_3} is the solution of $P(\Omega_3)$;*

(2)

$$\begin{cases} |\nabla u_{\Omega_3}| \leq k & \text{on } \partial\Omega_3 \cap \partial B_\rho, \\ |\nabla u_{\Omega_3}| = k & \text{on } \partial\Omega_3 \setminus \partial B_\rho. \end{cases} \tag{3.3}$$

PROOF. The first item is due to Theorem 2.7 and Proposition 2.8. For the second item, the admissible directions V must satisfy $V \cdot \nu \geq 0$ on $\partial\Omega \cap \partial\Omega_3$. Then (2.2) and 3(b) imply (3.3). □

REMARK 3.7. The C^2 regularity obtained for Ω_1 and Ω_3 is due to [4, Theorem 1.4].

REMARK 3.8. The continuity-compactness result obtained by Bucur and Trebeschi [5] allows us to extend the previous propositions to other divergence operators such as $\mathrm{div}(a(x, Du))$, especially for the p-Laplacian case.

3.3. Main results Let Ω be a solution of $\mathrm{S}(k)$.

THEOREM 3.9. *Suppose that $N = 2$. Let Ω_2 be as in Proposition 3.5*, then $\Omega = \Omega_2 = B_{2k}$.

The proof of this theorem uses the following lemma.

LEMMA 3.10 ([7, Section 30.4.1]). *Suppose that $N = 2$ and let Ω be a simply connected domain which is of class $C^{2,\alpha}$. If $H_{\partial\Omega} \leq 1/\varrho$ then Ω contains a ball of radius ϱ.*

REMARK 3.11 ([7, Section 30.4.2]). The result of the previous lemma cannot be extended to $N \geq 3$.

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PROOF OF THEOREM 3.9. Item (2) of Proposition 3.5 gives $H_{\partial\Omega_2} \leq 1/Nk$ on $\partial\Omega_2$. Since $\Omega_2 \subset \Omega$ and $N = 2$, Lemma 3.10 implies that $B_{2k} \subset \Omega_2 \subset \Omega$. Without loss of generality we may assume that B_{2k} touches $\partial\Omega$ tangentially at a point x_0, so that they have the same outward normal vector ν_0 (otherwise we shift B_{2k}). So $|\nabla u_{B_{2k}}(x_0)| = k = |\nabla u_\Omega(x_0)|$. Suppose now that $B_{2k} \neq \Omega$. As $B_{2k} \subset \Omega$, $\partial B_{2k} \neq \partial\Omega$. Now as $-\Delta u_{B_{2k}} = 1 = -\Delta u_\Omega$ in B_{2k} and $u_{B_{2k}} \leq u_\Omega$ on ∂B_{2k}, the maximum principle gives $|\nabla u_{B_{2k}}(x_0)| < |\nabla u_\Omega(x_0)|$ which is absurd. It then follows that $\Omega = \Omega_2 = B_{2k}$. □

THEOREM 3.12. *Suppose that $N \in \{2, 3\}$. Let Ω_3 be as in Proposition 3.6*, then $\Omega_3 = \Omega = B_{Nk}$.

PROOF. $B_{Nk} \subset \Omega_3 \subset \Omega$ by the definition of $\mathcal{O}_\rho$. Then using the same arguments as in the proof of Theorem 3.9, we obtain the same result for Ω, Ω_3 and B_{Nk}. □

THEOREM 3.13. *Suppose that $N \in \{2, 3\}$. Let Ω_1 and Ω_2 be as in Propositions 3.4* and 3.5. If $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$ then $\Omega_1 = \Omega = \Omega_2 = B_{Nk}$.

PROOF. Since $\Omega_2 \subset \Omega \subset \Omega_1$, $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$ implies $\partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega \neq \emptyset$. Suppose by contradiction that $\Omega_1 \neq \Omega$, then $\partial\Omega_1 \neq \partial\Omega$. According to (3.1) and (3.2), the monotonicity of the mean curvature together with the maximum principle implies

$$k = |\nabla u_\Omega| < |\nabla u_{\Omega_1}| \leq Nk^2 H_{\partial\Omega_1} \leq Nk^2 H_{\partial\Omega_2} \leq k \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega,$$

which is absurd. So $\Omega_1 = \Omega$. Therefore (3.1) gives $k = |\nabla u_\Omega| \leq Nk^2 H_{\partial\Omega}$ on $\partial\Omega$. Thus $1 \leq NkH_{\partial\Omega}$ which can be combined with (3.2) and the monotonicity of the mean curvature to get $H_{\partial\Omega_2} = 1/Nk$ on $\partial\Omega_2$, that is Ω_2 is a ball with radius Nk [1]. Now Lemma 3.1 implies that Ω_2 is a solution of $\mathrm{S}(k)$ and, since $\partial\Omega \cap \partial\Omega_2 \neq \emptyset$, Corollary 3.3 gives $\Omega = \Omega_2$. □

4. Problem P(c)

4.1. Auxiliary lemmas

LEMMA 4.1. *Let B_R be a solution of* P(c), *so then $R = \sqrt[4]{N^3(N+2)c}$.*

PROOF. Let u_R (respectively v_R) be the solution of $P(B_R)$ (respectively $Q(B_R)$). On the one hand, replacing ρ by R in the proof of Lemma 3.1, one obtains $-u'_R(R) = R/N$. Then a simple calculation shows that

$$u_R(r) = \frac{1}{2N}(R^2 - r^2) \quad \text{for } r \in]0, R[.$$

On the other hand, there exists a radial function v_R satisfying

$$\begin{cases} -v''_R - \dfrac{N-1}{r} v'_R = u_R & \text{for } r \in]0, R[, \\ v_R(R) = 0 \\ -v'_R(R) = \dfrac{1}{R^{N-1}} \displaystyle\int_0^R s^{N-1} u_R(s)\, ds = \dfrac{N}{N+2} \left(\dfrac{R}{N}\right)^3. \end{cases}$$

Therefore, since B_R is a solution of P(c), $R = \sqrt[4]{N^3(N+2)c}$.

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LEMMA 4.2. *Let Ω^* be a solution of* $\mathrm{P}(c)$. *Let $\omega \supset \Omega^*$ and let u_ω (respectively v_ω) be the solution of $P(\omega)$ (respectively $Q(\omega)$). If $\partial\omega \cap \partial\Omega^* \neq \emptyset$ and if $|\nabla u_\omega||\nabla v_\omega| \leq c$ on $\partial\omega \cap \partial\Omega^*$ then $\omega = \Omega^*$.*

PROOF. Suppose by contradiction that $\omega \neq \Omega^*$. Since $\omega \supset \Omega^*$, $\partial\omega \neq \partial\Omega^*$. On the one hand, Ω^* is a solution of $\mathrm{P}(c)$ so the maximum principle implies $0 \leq u_{\Omega^*} \leq u_\omega$ in Ω^* and $|\nabla u_{\Omega^*}| < |\nabla u_\omega|$ on $\partial\omega \cap \partial\Omega^*$ which is nonempty. On the other hand, one can apply the maximum principle to v_ω and v_{Ω^*} and obtain $|\nabla v_{\Omega^*}| < |\nabla v_\omega|$ on $\partial\omega \cap \partial\Omega^*$. Therefore,

$$c = |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| < |\nabla u_\omega||\nabla v_\omega| \leq c \quad \text{on } \partial\omega \cap \partial\Omega^*,$$

which gives the contradiction. □

COROLLARY 4.3. *Let ω^* and Ω^* be two solutions of* $\mathrm{P}(c)$. *Suppose that $\Omega^* \subset \omega^*$ and $\partial\omega^* \cap \partial\Omega^* \neq \emptyset$, then $\omega^* = \Omega^*$.*

4.2. Shape optimization problems Let Ω^* be a solution of $\mathrm{P}(c)$. Denote by B the greatest ball contained in Ω^*. Replacing Ω by Ω^* in $\mathcal{O}_\Omega$ (respectively in $\mathcal{O}^\Omega$)) we obtain the definition of $\mathcal{O}_{\Omega^*}$ (respectively $\mathcal{O}^{\Omega^*}$). Set

$$\mathcal{O}_R = \{D \supset \omega \supset B_R \mid \omega \in \mathcal{O}_{B_R}\}.$$

Consider the following functionals:

$$J_1(\omega) = c\frac{R}{N-1}|\partial\omega| - \frac{1}{2}\int_\omega u_\omega^2, \qquad J_2(\omega) = \frac{R}{N-1}|\partial\omega| - |\omega|,$$

and

$$J_3(\omega) = c|\omega| - \frac{1}{2}\int_\omega u_\omega^2.$$

By the Green formula,

$$J_1(\omega) = c\frac{R}{N-1}|\partial\omega| - \frac{1}{2}\int_\omega v_\omega$$

and

$$J_3(\omega) = c|\omega| - \frac{1}{2}\int_\omega v_\omega.$$

u_ω and v_ω are respectively the solutions of $P(\omega)$ and $Q(\omega)$.

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PROPOSITION 4.4. *Suppose that* $N \in \{2, 3\}$. *There exists* $\Omega_1^* \in \mathcal{O}_{\Omega^*}$ *which is of class* C^2 *such that:*

(1) $J_1(\Omega_1^*) = \min\{J_1(\omega) \mid \omega \in \mathcal{O}_{\Omega^*}\}$ *and* $u_{\Omega_1^*}$ *(respectively* $v_{\Omega_1^*}$*) is the solution of* $P(\Omega_1^*)$ *(respectively* $Q(\Omega_1^*)$*);*

(2)

$$\begin{cases} |\nabla u_{\Omega_1^*}||\nabla v_{\Omega_1^*}| \leq cRH_{\partial\Omega_1^*} & \text{on } \partial\Omega_1^* \cap \partial\Omega^*, \\ |\nabla u_{\Omega_1^*}||\nabla v_{\Omega_1^*}| = cRH_{\partial\Omega_1^*} & \text{on } \partial\Omega_1^* \setminus \partial\Omega^*. \end{cases} \tag{4.1}$$

PROOF. (1) Let u_D be the solution of the Dirichlet problem $P(D)$. By the maximum principle, $0 \leq u_\omega \leq u_D$ so $J_1(\omega) \geq -(1/2)\int_D u_D^2$ and $\inf J_1$ exists. Let Ω_n be a minimizing sequence in $\mathcal{O}_{\Omega^*}$. Since $\Omega_n \subset D$, then there exist an open set Ω_1^* and a subsequence of Ω_n (still denoted by Ω_n) such that $\Omega_n \xrightarrow{H} \Omega_1^*$. Now according to (iii) of Theorem 2.7 and Proposition 2.8, $\int_D u_n^2 \chi_{\Omega_n}$ converges to $\int_D u_{\Omega_1^*}^2 \chi_{\Omega_1^*}$ and by item (2) of Lemma 2.15 we get $J_1(\Omega_1^*) \leq \liminf_{n\to+\infty} J_1(\Omega_n)$. Then, according to (iv) of Theorem 2.7, $\Omega_1^* \in \mathcal{O}_{\Omega^*}$; therefore $J_1(\Omega_1^*) = \min_{\omega\in\mathcal{O}_{\Omega^*}} J_1(\omega)$. Now, on the one hand, Proposition 2.8 implies that $u_{\Omega_1^*}$ is the solution of $P(\Omega_1^*)$. On the other hand, Proposition 2.8 together with Proposition 2.14 implies that $v_{\Omega_1^*}$ is the solution of $Q(\Omega_1^*)$.

(2) Since $u_{\Omega^*} = 0$ on $\partial\Omega^*$, (2.3) and 3(a) above imply that

$$dJ_1(\Omega^*, V) = \int_{\partial\Omega_1^*} (cRH_{\partial\Omega_1^*} - |\nabla u_{\Omega_1^*}||\nabla v_{\Omega_1^*}|)V \cdot \nu \, d\sigma$$

for all admissible directions V. Thus we obtain (4.1). □

PROPOSITION 4.5. *Suppose that* $N \geq 2$. *There exists* $\Omega_2^* \in \mathcal{O}^{\Omega^*}$ *such that:*

(1) $J_2(\Omega_2^*) = \min\{J_2(\omega) \mid \omega \in \mathcal{O}^{\Omega^*}\}$, $u_{\Omega_2^*}$ *(respectively* $v_{\Omega_2^*}$*) is the solution of* $P(\Omega_2^*)$ *(respectively* $Q(\Omega_2^*)$*);*

(2) *if* Ω_2^* *is of class* C^2 *then*

$$\begin{cases} RH_{\partial\Omega_2^*} \leq 1 & \text{on } \partial\Omega_2^* \cap \partial\Omega^*, \\ RH_{\partial\Omega_2^*} = 1 & \text{on } \partial\Omega_2^* \setminus \partial\Omega^*. \end{cases} \tag{4.2}$$

PROOF. (1) The first assertion is due to Lemma 2.13 and item (2) of Lemma 2.15. Then (vi) of Lemma 2.13 together with Proposition 2.14 gives the continuity with respect to the domains for the Dirichlet problems $P(\Omega_2^*)$ and $Q(\Omega_2^*)$.

(2) Arguing as in the proof of Proposition 3.5 and using (2.2) and (2.3), we obtain (4.2). □

PROPOSITION 4.6. *Suppose that* $N \in \{2, 3\}$. *There exists* $\Omega_3^* \in \mathcal{O}_R$ *which is of class* C^2 *such that:*

(1) $J_3(\Omega_3^*) = \min\{J_3(\omega) \mid \omega \in \mathcal{O}_R\}$ *and* $u_{\Omega_3^*}$ *(respectively* $v_{\Omega_3^*}$*) is the solution of* $P(\Omega_2^*)$ *(respectively* $Q(\Omega_2^*)$*);*

(2)

$$\begin{cases} |\nabla u_{\Omega_3^*}||\nabla v_{\Omega_3^*}| \le c & \text{on } \partial\Omega_3^* \cap \partial B_R, \\ |\nabla u_{\Omega_3^*}||\nabla v_{\Omega_3^*}| = c & \text{on } \partial\Omega_3^* \setminus \partial B_R. \end{cases} \tag{4.3}$$

PROOF. (1) Theorem 2.7 and Proposition 2.8 imply the existence of the minimum Ω_3^*. Propositions 2.8 and 2.14 give the continuity with respect to the Dirichlet problems $P(\Omega_3^*)$ and $Q(\Omega_3^*)$.

(2) Arguing as in the proof of Proposition 4.4, (2.3) and 3(a) above imply (4.3). □

REMARK 4.7. The C^2 regularity obtained for Ω_1^* and Ω_3^* is due to [4, Theorem 1.4].

4.3. Main results Let Ω^* be a solution of P(c). By applying the maximum principle to $(u_{\Omega^*}; v_{\Omega^*})$ and $(u_{B_R}; v_{B_R})$, the proofs of the two first theorems are similar to those of Theorems 3.9 and 3.12.

THEOREM 4.8. *Suppose that* $N = 2$. *Let* Ω_2^* *be as in Proposition* 4.5, so $\Omega^* = \Omega_2^* = B_{2\sqrt[4]{c}}$.

THEOREM 4.9. *Suppose that* $N \in \{2, 3\}$. *Let* Ω_3^* *be as in Proposition* 4.6, so $\Omega_3^* = B_R$.

THEOREM 4.10. *Suppose that* $N \in \{2, 3\}$ *and* $R = \sqrt[4]{N^3(N+2)c}$. *Let* Ω_1^* *and* Ω_2^* *be as in Propositions* 4.4 and 4.5. If $\partial\Omega_1^* \cap \partial\Omega_2^* \neq \emptyset$ then $\Omega_1^* = \Omega^* = \Omega_2^* = B_R$.

PROOF. Since $\Omega_2^* \subset \Omega^* \subset \Omega_1^*$, $\partial\Omega_1^* \cap \partial\Omega_2^* \neq \emptyset$ implies that $\partial\Omega_1^* \cap \partial\Omega_2^* \cap \partial\Omega^* \neq \emptyset$. Suppose by contradiction that $\Omega_1^* \neq \Omega^*$, then $\partial\Omega_1^* \neq \partial\Omega^*$. Using (4.1) and (4.2), the monotonicity of the mean curvature together with the maximum principle imply that

$$\begin{aligned} c = |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| &< |\nabla u_{\Omega_1^*}||\nabla v_{\Omega_1^*}| \le cRH_{\partial\Omega_1^*} \\ &\le cRH_{\partial\Omega_2^*} \le c \quad \text{on } \partial\Omega_1^* \cap \partial\Omega_2^* \cap \partial\Omega^*, \end{aligned}$$

which is absurd. So $\Omega_1^* = \Omega^*$. Therefore (4.3) gives $c = |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| \le cRH_{\partial\Omega^*}$ on $\partial\Omega^*$. Thus $1 \le RH_{\partial\Omega^*}$ which can be combined with (4.2) and the monotonicity of the mean curvature to get $H_{\partial\Omega_2^*} = 1/R$ on $\partial\Omega_2^*$, that is Ω_2^* is a ball with radius R. Since $R = \sqrt[4]{N^3(N+2)c}$, Lemma 4.1 implies that Ω_2^* is a solution of P(c) and since $\partial\Omega^* \cap \partial\Omega_2^* \neq \emptyset$, Corollary 4.3 gives $\Omega^* = \Omega_2^*$. □

5. Other problems

This section is concerned with several overdetermined boundary value problems for which the overdetermined condition is not constant. The aim here is to prove for them the same symmetry result obtained for S(k) and P(c).

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THEOREM 5.1. *Let u_Ω be a solution of* P(Ω) *such that* $(x\cdot\nu)|\nabla u_\Omega|^3 = c(N+2)$ *on* $\partial\Omega$. *Let* v_Ω *be the solution of* $Q(\Omega)$. *If* (i) $|\nabla u_\Omega||\nabla v_\Omega| \le c$ *on* $\partial\Omega$ *or if* (ii) $|\nabla u_\Omega||\nabla v_\Omega| \ge c$ *on* $\partial\Omega$ *then* Ω *is a solution of* P(c). *As a consequence* Ω *is a ball of radius* $\sqrt[4]{N^3(N+2)c}$.

PROOF. The proof needs the well-known Rellich formula [18], valid for any $v \in C^1(\overline{\Omega}) \cap H^2(\Omega)$,

$$2\int_{\partial\Omega}(x\cdot\nabla v)\frac{\partial v}{\partial\nu}\,d\sigma - \int_{\partial\Omega}(x\cdot\nu)|\nabla v|^2\,d\sigma$$
$$= 2\int_{\Omega}(x\cdot\nabla v)\Delta v\,dx + (2-N)\int_{\Omega}|\nabla v|^2\,dx. \tag{5.1}$$

Replacing in (5.1) v by u_Ω and using $\nabla u_\Omega = -|\nabla u_\Omega|\nu$ on the boundary, we find that

$$\int_{\partial\Omega}(x\cdot\nu)|\nabla u_\Omega|^2\,d\sigma = -2\int_{\Omega}(x\cdot\nabla u_\Omega)\,dx + (2-N)\int_{\Omega}|\nabla u_\Omega|^2\,dx. \tag{5.2}$$

But the Green formula gives

$$\int_{\Omega}(x\cdot\nabla u_\Omega)\,dx = -N\int_{\Omega}u_\Omega = -N\int_{\Omega}|\nabla u_\Omega|^2\,dx.$$

We then obtain the identity

$$\int_{\partial\Omega}(x\cdot\nu)|\nabla u_\Omega|^2\,d\sigma = (2+N)\int_{\Omega}u_\Omega\,dx. \tag{5.3}$$

By the Compatibility Condition of the Neumann Problem (CCNP), there exists a w solution of

$$-\Delta w = u_\Omega \quad \text{in } \Omega \quad \text{and} \quad -\frac{\partial w}{\partial\nu} = \frac{1}{N+2}(x\cdot\nu)|\nabla u_\Omega|^2 \quad \text{on } \partial\Omega.$$

Put $h = v_\Omega - w$. Then $\Delta h = 0$ in Ω and

$$\frac{\partial h}{\partial\nu} = \frac{\partial v_\Omega}{\partial\nu} + \frac{1}{N+2}(x\cdot\nu)|\nabla u_\Omega|^2 \quad \text{on } \partial\Omega. \tag{5.4}$$

Multiplying (5.4) by $\partial u_\Omega/\partial\nu$ and using $-\partial u_\Omega/\partial\nu = |\nabla u_\Omega|$, we obtain

$$\frac{\partial h}{\partial\nu}\frac{\partial u_\Omega}{\partial\nu} = \frac{\partial v_\Omega}{\partial\nu}\frac{\partial u_\Omega}{\partial\nu} - \frac{1}{N+2}(x\cdot\nu)|\nabla u_\Omega|^3 \quad \text{on } \partial\Omega. \tag{5.5}$$

Now $(x\cdot\nu)|\nabla u_\Omega|^3 = c(N+2)$ on $\partial\Omega$, so (5.5) becomes

$$\frac{\partial h}{\partial\nu}\frac{\partial u_\Omega}{\partial\nu} = \frac{\partial v_\Omega}{\partial\nu}\frac{\partial u_\Omega}{\partial\nu} - c \quad \text{on } \partial\Omega. \tag{5.6}$$

Since $\partial u_\Omega/\partial\nu = -|\nabla u_\Omega| < 0$, (i) or (ii) implies that $\partial h/\partial\nu$ has a constant sign on $\partial\Omega$. But h is harmonic in Ω so by the Green formula $\int_{\partial\Omega}(\partial h/\partial\nu) = 0$. It then follows that $\partial h/\partial\nu = 0$ and so $(\partial v_\Omega/\partial\nu)(\partial u_\Omega/\partial\nu) = c$ on $\partial\Omega$, that is Ω is a solution of P(c) and so it is a ball with radius $\sqrt[4]{N^3(N+2)c}$. □

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For the following theorems we recall that u_Ω (respectively v_Ω) is the solution of $P(\Omega)$ (respectively $Q(\Omega)$).

THEOREM 5.2. *Suppose that* $|\nabla u_\Omega||\nabla v_\Omega| = c(x \cdot \nu)$ *on* $\partial\Omega$. *If* (a) $|\nabla u_\Omega| \le \sqrt[3]{c(N+2)}$ *on* $\partial\Omega$ *or if* (b) $|\nabla u_\Omega| \ge \sqrt[3]{c(N+2)}$ *on* $\partial\Omega$ *then* Ω *is a solution of* S$(\sqrt[3]{c(N+2)})$. *As a consequence* Ω *is a ball of radius* $N\sqrt[3]{c(N+2)}$.

PROOF. If $|\nabla u_\Omega||\nabla v_\Omega| = c(x \cdot \nu)$ on $\partial\Omega$ then (5.5) becomes

$$\frac{\partial h}{\partial \nu}\frac{\partial u_\Omega}{\partial \nu} = (x \cdot \nu)\left[c - \frac{1}{N+2}|\nabla u_\Omega|^3\right] \quad \text{on } \partial\Omega. \tag{5.7}$$

Since $x \cdot \nu > 0$, (5.7) with (a) or with (b) implies that $\partial h/\partial \nu = 0$ on $\partial\Omega$ and so

$$|\nabla u_\Omega| = \sqrt[3]{c(N+2)} \quad \text{on } \partial\Omega.$$

Therefore Ω is a solution of S$(\sqrt[3]{c(N+2)})$, that is Ω is a ball with radius $N\sqrt[3]{c(N+2)}$. $\square$

THEOREM 5.3. *Suppose that* (H1) $|\nabla u_\Omega||\nabla v_\Omega| = (c^3/(N+2))r^4(\partial r/\partial \nu)$ *on* $\partial\Omega$. *If* (1) $|\nabla u_\Omega| \le cr$ *on* $\partial\Omega$ *or if* (2) $|\nabla u_\Omega| \ge cr$ *on* $\partial\Omega$ *then* $|\nabla u_\Omega| = cr$ *on* $\partial\Omega$. *As a consequence* Ω *is a ball.*

PROOF. Let $r = |x|$, then $\triangle r^2 = 2N$. An integration by parts gives

$$-2N\int_\Omega u_\Omega(x)\,dx = \int_\Omega \nabla(r^2)\cdot\nabla u_\Omega = 2\int_\Omega r\frac{\partial u_\Omega}{\partial r}. \tag{5.8}$$

But $\triangle(r(\partial u_\Omega/\partial r)) = -2$, so by the Green formula we obtain

$$\int_\Omega 2u_\Omega - r\frac{\partial u_\Omega}{\partial r} = \int_\Omega -u_\Omega\triangle\left(r\frac{\partial u_\Omega}{\partial r}\right) + r\frac{\partial u_\Omega}{\partial \nu}\triangle u_\Omega = \int_{\partial\Omega} r\frac{\partial u_\Omega}{\partial r}\frac{\partial u_\Omega}{\partial \nu}\,d\sigma.$$

By (5.8) we get the identity

$$(N+2)\int_\Omega u_\Omega = \int_{\partial\Omega} r\frac{\partial r}{\partial \nu}|\nabla u_\Omega|^2\,d\sigma. \tag{5.9}$$

Now, as in the proof of Theorem 5.1, the CCNP implies the existence of a function w solution of the following Neumann problem:

$$-\triangle w = u_\Omega \quad \text{in } \Omega \quad \text{and} \quad -\frac{\partial w}{\partial \nu} = \frac{1}{N+2}r\frac{\partial r}{\partial \nu}|\nabla u_\Omega|^2 \quad \text{on } \partial\Omega.$$

As above, if $h = v_\Omega - w$ then h is harmonic in Ω and

$$\frac{\partial h}{\partial \nu}\frac{\partial u_\Omega}{\partial \nu} = \frac{\partial v_\Omega}{\partial \nu}\frac{\partial u_\Omega}{\partial \nu} - \frac{1}{N+2}r\frac{\partial r}{\partial \nu}|\nabla u_\Omega|^3 \quad \text{on } \partial\Omega. \tag{5.10}$$

or again using (H1),

$$\frac{\partial h}{\partial \nu}\frac{\partial u_\Omega}{\partial \nu} = \frac{1}{N+2} r \frac{\partial r}{\partial \nu}[(cr)^3 - |\nabla u_\Omega|^3] \quad \text{on } \partial\Omega. \tag{5.11}$$

Now arguing as above, (1) (or (2)) allows us to get $\partial h/\partial \nu = 0$ on $\partial\Omega$. It then follows that $|\nabla u_\Omega| = cr$ on $\partial\Omega$ and so Ω is some ball [2]. □

THEOREM 5.4. *Suppose* (H2) $(N+2)(x\cdot\nu)^2|\nabla u_\Omega||\nabla v_\Omega| = (C_0 r^2 + C_1)^3$ *on* $\partial\Omega$. *If* (3) $(x\cdot\nu)|\nabla u_\Omega| \le C_0 r^2 + C_1$ *or if* (4) $(x\cdot\nu)|\nabla u_\Omega| \ge C_0 r^2 + C_1$ *on* $\partial\Omega$ *then* $(x\cdot\nu)|\nabla u_\Omega| = C_0 r^2 + C_1$ *on* $\partial\Omega$. *As a consequence* Ω *is a ball if* $2(NC_0 - 1)$ *is not a negative integer while it is an ellipsoid if* $C_0 = 0$.

PROOF. Applying (H2) to (5.5), we obtain

$$(N+2)(x.\nu)^2\frac{\partial h}{\partial \nu}\frac{\partial u_\Omega}{\partial \nu} = (C_0 r^2 + C_1)^3 - (x\cdot\nu)^3|\nabla u_\Omega|^3 \quad \text{on } \partial\Omega. \tag{5.12}$$

Arguing as above, (3) (or (4)) implies that $\partial h/\partial \nu = 0$ on $\partial\Omega$. Then

$$(x\cdot\nu)|\nabla u_\Omega| = C_0 r^2 + C_1 \quad \text{on } \partial\Omega,$$

which gives the conclusion [15]. □

REMARK 5.5. Suppose that u_Ω is the solution of $P(\Omega)$. If $|\nabla u_\Omega| = cr$ on $\partial\Omega$ then one can prove that Ω is a ball. In fact, replacing $|\nabla u_\Omega|$ by cr in (5.9), one can obtain

$$c^2(N+2)\int_\Omega r^2 = \frac{c^2}{4}\int_\Omega \triangle(r^4) = \frac{c^2}{4}\int_{\partial\Omega}\frac{\partial(r^4)}{\partial\nu} = c^2\int_{\partial\Omega} r^3\frac{\partial r}{\partial \nu} = (N+2)\int_\Omega u_\Omega.$$

So

$$\int_\Omega u_\Omega = c^2\int_\Omega r^2.$$

Put $u = u_\Omega$ and $\phi = u_i u_i - c^2 r^2$. A simple calculation [2] shows that $\triangle\phi \ge 0$ in Ω and $\int_\Omega \phi = \int_\Omega u_\Omega - c^2\int_\Omega r^2 = 0$. Then the maximum principle gives $\phi \equiv 0$ in Ω. One can derive that u is radially symmetric and Ω is a ball.

REMARK 5.6. Because of the use of the compatibility condition of the Neumann problem, Theorems 5.1, 5.2, 5.3 and 5.4 can be extended to the divergence operator $\mathrm{div}(a(x)Du(x))$ [14].

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