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# REPRESENTATIONS OF ORDERED GROUPS WITH COMPATIBLE TIGHT RIESZ ORDERS

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## 1. Introduction

A tight Riesz group $G$ is a partially ordered group $G$ that satisfies a strengthened form of the Riesz interpolation property. The term "tight" was introduced by Miller in (1970) and the tight interpolation property has been considered by Fuchs (1965), Miller (1973), (to appear), (preprint), Loy and Miller (1972) and Wirth (1973). If $G$ is free of elements called pseudozeros then $G$ is a non-discrete Hausdorff topological group with respect to the open interval topology $\mathscr{U}$. Moreover the closure $\bar{P}$ of the cone $P$ of the given order is the cone of an associated order on $G$. This allows an interesting interplay between the associated order, the tight Riesz order and the topology $\mathscr{U}$. Loy and Miller found of particular interest the case in which the associated partial order is a lattice order. This situation was considered in reverse by Reilly (1973) and Wirth (1973), who investigated the circumstances under which a lattice ordered group, and indeed a partially ordered group, permits the existence of a tight Riesz order for which the initial order is the associated order. These tight Riesz orders were then called compatible tight Riesz orders. In Section one we relate these ideas to the topologies defined on partially ordered groups by means of topological identities, as described by Banaschewski (1957), and show that the topologies obtained from topological identities are precisely the open interval topologies from compatible tight Riesz orders.

Miller (1973) and (preprint) has recently investigated possible representations of lattice ordered groups with compatible tight Riesz orders. If $\{A_\alpha : \alpha \in A\}$ is a set of dense totally ordered groups and $H$ is their product then two natural partial orders can be described on $H$ as follows. First there is the usual cardinal order $\preccurlyeq$ and then there is the order defined by: $f > g$ if and only if $f(\alpha) > g(\alpha)$, for all $\alpha \in A$. Then $(H, \preccurlyeq)$ is a lattice ordered group and $\leqq$ is a compatible tight Riesz order. Miller (1973) determines certain conditions under which an abelian lattice ordered group $(G, \preccurlyeq)$ with a compatible tight Riesz order $\leqq$ can be represented as a subdirect product of a product $H$ of totally ordered groups with two inherited orders as described above. Miller (to appear) then investigates two representations due to Ribenboim and Jaffard and obtains conditions under which they can be used to represent $(G, \preccurlyeq, \leqq)$. In doing so he makes use of the following type of construction. Let $\{A_\alpha : \alpha \in A\}$, $\{B_\beta : \beta \in B\}$ be totally ordered groups and let $H$ be their product. Then $H$ together with the cardinal order $\preccurlyeq$ is a lattice ordered group. If an order is defined on $H$ by: $f > g$ if and only if $f(\alpha) \geqq g(\alpha)$, for all $\alpha \in A$, and $f(\beta) > g(\beta)$, for all $\beta \in B$, then $(H, \preccurlyeq, \leqq)$ is a *hybrid product*. In this paper we show any abelian lattice ordered group with a compatible tight Riesz order can be represented as a subdirect product of a hybrid product. We also extend this result to partially ordered groups and also to non-abelian groups by using hybrid products of permutation groups of ordered sets.

The reader should note that the interpolation property assumed in the definition of a tight Riesz group in Loy and Miller (1972), Reilly (1973) and Wirth (1973) is stronger than that assumed in Miller (1973) and Miller (to appear) and in this paper. However, the compatible tight Riesz orders on a lattice ordered group that satisfy the weaker interpolation condition also satisfy the stronger condition and so, in that context, it is immaterial which condition is assumed. In the context of compatible tight Riesz orders for partially ordered groups, however, the weaker interpolation property, which we consider here, appears to be the more appropriate. One further point to notice is that other authors have usually assumed that a tight Riesz group is directed. As this assumption appears to be largely immaterial, we have not made this assumption.

## 2. Compatible tight Riesz orders on partially ordered groups

A partially ordered set $(G, \leqq)$ is said to satisfy the *tight Riesz* (1, 2) *property* if for any elements $a, b, c \in G$ with $a < b$ and $a < c$ there exists an element $d \in G$ with $a < d < b$ and $a < d < c$. The *tight Riesz* (2, 1) *property* is defined analogously and the two are clearly equivalent if $(G, \leqq)$ is a partially ordered group. The relationship between various tight Riesz orders has been considered by Cameron and Miller (to appear). As we shall be concerned almost exclusively with the tight Riesz (1, 2) property we shall refer to it as *the* tight Riesz property. A partially ordered group satisfying the tight Riesz property will be called a *tight Riesz group*. Clearly a totally ordered group is a tight Riesz group if and only if it is dense.

For the definitions and basic properties of pseudopositive elements, pseudozero elements and associated orderings the reader is referred to Loy and Miller (1972) and Miller (1973) and, unless otherwise mentioned, the notation and terminology of this paper will be that of Loy and Miller (1972) and Miller (1973).

For any partially ordered group $(G, \leqq)$ the open interval toplogy $\mathscr{U}$ is the topology on $G$ having as a subbase the set of all intervals of the form $(a, b) = \{x \in G: a < x < b\}$. From Loy and Miller (1972), in which they took tight Riesz groups to be directed and abelian, we have the following result that did not depend on either of these assumptions.

PROPOSITION 2.1. *A tight Riesz group $(G, \leqq)$ is a topological group with respect to its open interval topology $\mathscr{U}$. Morever, $\mathscr{U}$ is not discrete and the collection of all open intervals $(a, b)$ forms a base for $\mathscr{U}$.*

*Let $P$ denote the positive cone of $(G, \leqq)$. Then*

(1) *the boundary $\partial P$, of $P$, is the set of all pseudopositive elements together with* 0;

(2) *the set of pseudozeros is $\partial P \cap (-\partial P)$. This set is also equal to $\overline{\{0\}}$, the closure of* $\{0\}$.

*If $(G, \leqq)$ has no pseudozeros then*

(3) *$(G, \mathscr{U})$ is a Hausdorff topological group,*

*and*

(4) *the cone of the associated order is $\bar{P}$, the closure of $P$.*

The next result is due to Fuchs (1965a), Lemma 8.2.

LEMMA 2.2. *Let $(G, \leqq)$ be a tight Riesz group. Let $Z = \overline{\{0\}}$. Then $Z$ is a closed normal trivially ordered subgroup and $G/Z$ is a tight Riesz group without pseudozeros with respect to the naturally induced order.*

Let $\preccurlyeq$ and $\leqq$ be partial orderings on a group $G$ such that $(G, \preccurlyeq)$ is a partially ordered group and $(G, \leqq)$ is a tight Riesz group without pseudozeros. If $\preccurlyeq$ is the associated ordering for $\leqq$ then $\leqq$ is said to be a *compatible tight Riesz order* for $(G, \preccurlyeq)$. In this context we adopt the following notation:

$$\begin{aligned} G^+ &= \{g \in G: 0 \preccurlyeq g\} \\ P &= \{g \in G: 0 \leqq g\} \\ P^* &= P \backslash \{0\} \\ \partial P &= \{g \in G \backslash P^*: P^* + g \subseteq P^*\}. \end{aligned}$$

Then $G^+ = P^* \cup \partial P$. We refer to $P^*$ as the *strict cone* of the compatible tight Riesz order $\leqq$.

Wirth (1973) gave a valuable characterization of those subsets of the cone of a partially ordered group that are the strict cones of compatible tight Riesz orders.

PROPOSITION 2.3. (Wirth (1973)). *Let $(G, \preccurlyeq)$ be a partially ordered group. A proper subset $T$ of $G^+$ is the strict cone of a compatible tight Riesz order if and only if the following conditions are satisfied*:

*$T(1)$ $T$ is a lower directed upper ideal of $(G^+, \preccurlyeq)$;*
*$T(2)$ $T = T + T$;*
*$T(3)$ $\wedge\, T = 0$, that is, $0$ is the greatest lower bound for the set $T$ in $G^+$;*
*$T(4)$ $T$ is normal.*

*If $(G, \preccurlyeq)$ is a lattice ordered group and $\leqq$ is a compatible tight Riesz order then $(G, \leqq)$ is directed.*

Before proceeding to obtain a representation theorem for partially ordered groups with compatible tight Riesz orders we relate the concept to that of topological identities. Banaschewski (1957) defined a *topological identity* in a partially ordered group $(G, \preccurlyeq)$ to be a subset $E$ of $G$ such that

$E(1)$ $E \subseteq G^+ \backslash \{0\}$;
$E(2)$ for any $e, e' \in E$ there exists a $d \in E$ with $d \preccurlyeq e, e'$;
$E(3)$ for any $e \in E$, there exists an element $d \in E$ with $d + d \preccurlyeq e$;
$E(4)$ fof any $e \in E$, $x \in G$ there exists an element $d \in E$ with $d \preccurlyeq x + e - x$;
$E(5)$ $\wedge\, E = 0$.

If $E$ is a topological identity on $(G, \preccurlyeq)$ then Banaschewski defined a topology on $G$, with respect to which $G$ is a topological group, by taking the sets of the form $B_e = \{x\colon -e \preccurlyeq x \preccurlyeq e\}$, for $e \in E$, as a subbase of neighbourhoods of 0. On account of $E(2)$ these sets also form a base of neighbourhoods for 0. We shall refer to this topology as the $E$-topology. A glance at the conditions $T(1) - T(4)$ and $E(1) - E(5)$ would suggest a close connection between compatible tight Riesz orders and topological identities. We proceed to make this precise.

For any subset $A$ of a partially ordered group $(G, \preccurlyeq)$ let

$$A^u = \{g \in G\colon g \succcurlyeq a, \text{ for some } a \in A\},$$

and for any subset $A$ of a topological space let $A^0$ denote the interior of $A$.

LEMMA 2.4. *Let $E$ be a topological identity in the partially ordered group $(G, \preccurlyeq)$. Then, with respect to the E-topology $(G^+)^0 = E^u$.*

PROOF. Let $g \in E^u$. Let $e \in E$ be such that $e \preccurlyeq g$. By $E(3)$, there exists an element $d \in E$ with $0 \prec d \prec e \preccurlyeq g$. Then $g \in [g - d,\ g + d] \subseteq G^+$. Hence $g \in (G^+)^0$.

Conversely, let $g \in (G^+)^0$. Then there exists an element $d \in E$ such that $[g - d, g + d] \subseteq G^+$. Then $0 \preccurlyeq g - d$, $d \preccurlyeq g$ and $g \in E^u$.

LEMMA 2.5. *Let $(G, \preccurlyeq)$ be a partially ordered group and $E \subseteq G^+ \backslash \{0\}$. Then the following statements are equivalent.*

(1) *$E$ is the strict cone of a compatible tight Riesz order.*
(2) *$E$ is a topological identity and $E = E^u$.*
(3) *$E$ is a topological identity and $E = (G^+)^0$, with respect to the E-topology.*

PROOF. The equivalence of (2) and (3) follows from Lemma 2.4. Let (2) hold. Then $E$ is an upper ideal of $G^+$ and by $E(2)$, $E$ is lower directed. Therefore $E$

satisfies $T(1)$. For any $e \in E$ there exists an element $d \in E$ with $d + d \preccurlyeq e$. Then $-d + e \succcurlyeq d$ and, by (2), $-d + e \in E$. Hence $e = d + (-d + e) \in E + E$. On the other hand, for $e, f \in E$, $e + f \succcurlyeq e$ and so, by (2), $e + f \in E$. Thus $E = E + E$ and $T(2)$ is satisfied.

Conditions $T(3)$ and $E(5)$ are the same and condition $E(4)$ combined with (2) clearly implies that $E$ is normal. Thus (2) implies (1). That (1) implies (2) is even simpler.

Let $E$ be a topological identity on the partially ordered group $(G, \preccurlyeq)$. It is clear that $E^u$ is also a topological identity and that the $E$-topology is equal to the $E^u$-topology. However, from Lemma 2.5, since $(E^u)^u = E^u$, $E^u$ is the strict cone of a compatible tight Riesz order $\leqq$. Let $B_e$, for $e \in E^u$, be any element of the neighbourhood base of 0 in the $E^u$-topology. From $E(3)$ there exists an element $d \in E^u$, with $0 \prec d \prec e$ and so $(-d, d) \subset [-e, e]$. Hence the $E^u$-topology is contained in the $\mathscr{U}$-topology, from the tight Riesz order determined by $E^u$. The converse containment follows similarly and we have the following result.

PROPOSITION 2.6. *Let $(G, \preccurlyeq)$ be a partially ordered group. The topologies on $(G, \preccurlyeq)$ derived from topological identities are precisely the open interval topologies on $G$ derived from compatible tight Riesz orders.*

In the remainder of this section we develop a representation theorem for isolated partially ordered groups with a compatible tight Riesz order where a partially ordered group $(G, \preccurlyeq)$ is said to be *isolated* if $na \succcurlyeq 0$ for some natural number $n$ implies that $a \succcurlyeq 0$.

LEMMA 2.7. *Let $(G, \preccurlyeq)$ be an isolated partially ordered group and $\leqq$ be a compatible tight Riesz order. Then $(G, \leqq)$ is isolated.*

PROOF. Let $a \in G$ and $n$ be a natural number such that $na \geqq 0$. If $na = 0$ then $na \in G^+$ and, since $(G, \preccurlyeq)$ is isolated, $a \in G^+$ and so $a = 0$. Now suppose that $na > 0$. From conditions $T(1)$ and $T(2)$ on the strict cone $T$ of the order $\leqq$ it follows that there exists an element $h > 0$ such that $nh < na$. Then

$$0 < na - nh = \sum_{k=n-1}^{0} (a + (ka - h - ka))$$
$$= \sum_{k=n-1}^{0} (a - (ka + h - ka)).$$

Since $T$ is normal, $ka + h - ka \in T$, $k = 0, \cdots, n-1$. By $T(1)$, there exists a $g \in T$ such that $g < ka + h - ka$, for $k = 0, \cdots, n-1$. Then

$$0 < na - nh < \sum_{k=n-1}^{0} (a - g) = n(a - g).$$

Hence $n(a - g) \in G^+$. Since $(G, \preccurlyeq)$ is isolated $a - g \in G^+$. Thus $0 < g \preccurlyeq a$. Therefore $0 < a$.

This result will enable us to describe the boundary of the cone of a compatible tight Riesz order on an isolated partially ordered group. We shall also refer to the following.

PROPOSITION 2.8. *Let $(G, \preccurlyeq)$ be a partially ordered group. Let $\mathscr{S}(G)$ be the set of convex subsemigroups of $G^+$ containing* 0 *and let $\mathscr{C}(G)$ denote the set of all directed convex subgroups of G. Then the mappings*

$$A \to A \cap G^+$$

*and*

$$S \to [S] = \{x - y \colon x, y \in S\}$$

*are inverse mappings of $\mathscr{C}(G)$ onto $\mathscr{S}(G)$ and $\mathscr{S}(G)$ onto $\mathscr{C}(G)$, respectively.*

PROOF. The proof is entirely analogous to that of Conrad (1967, Theorem 1.3) establishing the correspondence between convex $l$-subgroups and convex sub-semigroups containing 0 of the positive cone of an $l$-group $(G, \preccurlyeq)$.

THEOREM 2.9. *Let $(G, \preccurlyeq)$ be an isolated partially ordered group with a compatible tight Riesz order $\leqq$. Then $\partial P = G^+ \backslash P^* = \cup \{S_i \colon i \in I\}$ where $\{S_i \colon i \in I\}$ is the set of maximal subsemigroups of $G^+ \backslash P^*$. Each $S_i$ is convex with respect to the order $\preccurlyeq$ and contains* 0. *Let $M_i = [S_i]$ be the convex directed subgroup of $(G, \preccurlyeq)$ generated by $S_i$. Then $M_i$ is a maximal convex directed subgroup of $(G, \preccurlyeq)$ with respect to the condition that $M_i \cap P^* = \varnothing$. Moreover, $S_i$ is closed with respect to the open interval topology $\mathscr{U}$ induced from $\leqq$ and $\overline{M_i} \cap G^* = S_i$, where $\overline{M_i}$ is the closure of $M_i$ with respect to $\mathscr{U}$.*

PROOF. For any $a \in G^+ \backslash P^*$, $\langle a \rangle \subseteq G^+$, where $\langle a \rangle$ denotes the subsemigroup of $G$ generated by $a$. Since $(G, \leqq)$ is isolated, $\langle a \rangle \cap P^* = \varnothing$. Hence $\langle a \rangle \subseteq G^+ \backslash P^*$ and $G^+ \backslash P^*$ is the union of its maximal subsemigroups. Let $S_i$ be any of the maximal subsemigroups of $G^+ \backslash P^*$ and let $a \preccurlyeq x \preccurlyeq b$ where $a$, $b \in S_i$ and $x \in G$. Clearly $x \in G^+$ and the subsemigroup generated by $S_i$ and $x$ is contained in $G^+ \backslash P^*$. Hence, by the maximality of $S_i$, $x \in S_i$ and $S_i$ is convex. Similarly $0 \in S_i$.

That $M_i$ is a directed convex subgroup of $(G, \preccurlyeq)$ and is maximal with respect to the condition that $M_i \cap P^* = \varnothing$. follows immediately from Proposition 2.8.

Let $\bar{S}_i$ denote the closure of $S_i$ with respect to $\mathscr{U}$. Since $G^+$ is closed, being the closure of $P$ by Proposition 2.1 (4), and $P^*$ is open it follows that $G^+ \backslash P^*$ is closed. Therefore $\bar{S}_i \subseteq G^+ \backslash P^*$ and, since $\bar{S}_i$ is a subsemigroup containing $S_i$, we must have $\bar{S}_i = S_i$, by the maximality of $S_i$.

Since $M_i \cap G^+ = S_i$, $M_i \subseteq G \backslash P^*$. Hence $\overline{M_i} \subseteq G \backslash P^*$ and $S_i \subseteq \overline{M_i} \cap G^+ \subseteq G^+ \backslash P^*$. By the maximality of $S_i$, $\overline{M_i} \cap G^* = S_i$.

It would seem to be quite possible that the subgroups $M_i$ of Theorem 2.9 will be closed thereby simplifying the statement and proof. Unfortunately, the author was unable to determine whether or not this must always be so. However,

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in the event that $(G, \preccurlyeq)$ is a lattice ordered group the situation is much improved (cf. Lemma 3.2).

To avoid repetition, we shall assume throughout the remainder of this section that $(G \preccurlyeq)$ is an isolated partially ordered group with a compatible tight Riesz order $\leqq$, that $\{S_i\colon i \in I\}$ is the set of maximal subsemigroups of $G^+ \backslash P^*$ and that $M_i = [S_i]$, for each $i \in I$. We call $\partial P = \cup \{S_i\colon i \in I\}$ the *prime decomposition* of $\partial P$. The reason will become apparent in Section 3. Topological comments refer to the open interval topology $\mathscr{U}$ induced from $\leqq$.

Let $M$ be a convex subgroup of $(G, \preccurlyeq)$. Then $M$ is also convex in $(G, \leqq)$. *Hence the set of right cosets $R(M)$ of $M$ can be endowed with orders induced from $\preccurlyeq$ and $\leqq$, respectively. We denote these orders by $\preccurlyeq$ and $\leqq$, as this should* cause no confusion. If $G$ is abelian or $M$ is normal in $G$, then $G/M$ is a partially ordered group with respect to the induced orders and the natural mappings are order preserving homomorphisms.

Lemma 2.10. *Let $M$ be a convex subgroup of* $(G, \leqq)$.

(1) *$M$ is open if and only if $M \cap P^* \neq \varnothing$.*

(2) *If $M$ is not open then $R(M)$ has the tight Riesz property. If $M$ is also a normal subgroup then $G/M$ is a tight Riesz group.*

Proof. (1) is a straightforward and is just the extension to the partially ordered case of the observation $5^0$ (IV) in Miller (1973). The proof of $5^0$ (V) given by Miller (1973) will carry over verbatim to establish (2).

Lemma 2.11. *For each $i \in I$, $(R(\overline{M_i}), \leqq)$ has the tight Riesz property. Moreover, the induced orders $\preccurlyeq$ and $\leqq$ on $R(\overline{M_i})$ are identical.*

Proof. Since every neighbourhood of 0 clearly contains an element of $P^*$, $\overline{M_i}$ is not open. If $0 \preccurlyeq a \preccurlyeq b$, $b \in \overline{M_i}$, then $b \in \overline{M_i} \cap G^+ = S_i$ and, since $S_i$ is convex, $a \in S_i \subseteq \overline{M_i}$. Hence $\overline{M_i}$ is convex in $(G, \preccurlyeq)$ and so also in $(G, \leqq)$. Therefore, by Lemma 2.10, $(R(\overline{M_i}), \leqq)$ has the right Riesz property.

For any $a, b \in G$, is is clear that $\overline{M_i} + b < \overline{M_i} + a$ implies that $\overline{M_i} + b \prec \overline{M_i} + a$. We wish to establish the converse implication. So let $\overline{M_i} + b \prec \overline{M_i} + a$. Clearly we may assume that $\overline{M_i} + b = \overline{M_i}$ and that $0 \prec a$. Then $\langle S_i, a\rangle \cap P^* \neq \varnothing$, where $\langle S_i, a\rangle$ is the subsemigroup of $G$ generated by $S_i$ and $a$. Therefore, for some $m_j \in S_i$, $j = 1, \cdots, n$, we have

$$x = m_1 + a + m_2 + \cdots + a + m_n \in P^*.$$

If $m = m_1 + \cdots + m_n$ then

$$x \preccurlyeq n(m + a)$$

and so $n(m + a) \in P^*$. Since $(G, \leqq)$ is isolated this implies that $m + a \in P^*$ and so

$$\overline{M_i} < \overline{M_i} + m + a = \overline{M_i} + a.$$

Thus $(R(\overline{M_i}), \preccurlyeq) = (R(\overline{M_i}), \leqq)$.

In Reilly (1973) it has been shown that if $(G, \preccurlyeq)$ is a lattice ordered group then $(R(M_i), \preccurlyeq)$ is totally ordered. This need not hold in general as the following example demonstrates.

EXAMPLE. Let $G$ denote the additive groups of two by two matrices of real numbers with the order defined as follows:

$$(0) \preccurlyeq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ if and only if either (1) } a, b \geqq 0 \text{ but } a \text{ and } b \text{ are not both equal to } 0, \text{ or (2) } a = b = 0 \text{ and } c, d \geqq 0,$$

where (0) denotes the zero two by two matrix.

Let a second order $\leqq$ be defined as follows:

$$(0) < \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ if and only if either (1) } a, b \geqq 0 \text{ but } a \text{ and } b \text{ are not both equal to } 0 \text{ or (2) } a = b = 0,\ c \geqq 0 \text{ and } d > 0.$$

Then $(G, \preccurlyeq)$ is a partially ordered group and $\leqq$ is a compatible tight Riesz order. The boundary $\partial P$ of $P$ consists of those matrices of the form

$$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \quad c \geqq 0. \tag{1}$$

Thus $\partial P$ is itself a subsemigroup of $G^+$ and the corresponding directed convex subgroup $M$ consists of all those matrices of the form (1) above where $c$ is arbitrary; $M$ is closed. If

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then $M + x$ and $M + y$ are incomparable in $R(M)$ and therefore $R(M)$ is not totally ordered.

Let $\mathscr{A} = \{(H_\alpha, \preccurlyeq): \alpha \in A\}$ and $\mathscr{I} = \{(K_i, \preccurlyeq): i \in I\}$ be two sets of partially ordered groups (where $I \neq \varnothing$). Then we shall denote by $(\mathscr{P}, \mathscr{A}, \mathscr{I}, \preccurlyeq, \leqq)$ the full direct product of the groups $(H_\alpha, \preccurlyeq)$ and $(K_i, \preccurlyeq)$

$$\mathscr{P} = \prod_{\alpha \in A} H_\alpha \times \prod_{i \in I} K_i$$

endowed with two partial orderings $\preccurlyeq$ and $\leqq$ where $\preccurlyeq$ is just the cardinal order

$$0 \preccurlyeq f \text{ if and only if } f(\alpha) \succcurlyeq 0, \text{ for all } \alpha \in A, \text{ and } f(i) \succcurlyeq 0, \text{ for all } i \in I,$$

and $\leqq$ is the order defined by

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$$0 < f \text{ if and only if } f(\alpha) \succcurlyeq 0, \text{ for all } \alpha \in A, \text{ and } f(i) \succ 0, \text{ for all } i \in I.$$

Miller (to appear) has referred to such products as *hybrid products*. We shall say that a group with two partial orderings $(H, \preccurlyeq, \leqq)$ is a *subdirect product* of the hybrid product $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ if $H$ is a subdirect product of the group $\mathscr{P}$ in the usual group theoretic sense and the orders $\preccurlyeq$, $\leqq$ on $H$ are the restrictions to $H$ of the orders $\preccurlyeq$ and $\leqq$ on $\mathscr{P}$. An *o-isomorphism* $\phi$ of a partially ordered group $(G, \preccurlyeq)$ into a partially ordered group $(H, \preccurlyeq)$ is a group monomorphism of $G$ into $H$ such that $a \preccurlyeq b$ if and only if $\phi(a) \preccurlyeq \phi(b)$. If $(G, \preccurlyeq, \leqq)$ is a group with two partial orderings, then a *realization* of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ is a monomorphism of $G$ onto a subdirect product of $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ which is an $o$-isomorphism of $(G, \preccurlyeq)$ and $(G, \leqq)$ into $(\mathscr{P}, \preccurlyeq)$ and $(\mathscr{P}, \leqq)$, respectively.

THEOREM 2.12. *Let $(G, \preccurlyeq)$ be an abelian isolated partially ordered group with a compatible tight Riesz order $\leqq$. Let $\partial P = \cup \{S_i : i \in I\}$ be the prime decomposition of $\partial P$ and $M_i = [S_i]$. Let $\theta$ be the product of the identity mapping $\iota_G$ of $G$ and the natural homomorphisms $\theta_i$ of $G$ onto each $G/\overline{M_i}$. Then $\theta$ is a realization of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \mathscr{A}$ $\mathscr{J}, \preccurlyeq, \leqq)$ where $\mathscr{A} = \{(G, \preccurlyeq)\}$ and $\mathscr{J} = \{(G/\overline{M_i}, \preccurlyeq) : i \in I\}$ where the order $\preccurlyeq$ on $G/\overline{M_i}$ is that induced from either of the orders $\preccurlyeq$ or $\leqq$ on $G$ (by Lemma 2.11, the induced orders are the same). Each $(G/\overline{M_i}, \leqq)$ is a tight Riesz group.*

*For each $i \in I$, let $Z_i$ denote the subgroup of $G$ such that $Z_i/\overline{M_i}$ is the group consisting of the pseudozeros of $G/\overline{M_i}$ together with $\overline{M_i}$.*

*Let $\phi$ be the product of the identity mapping $\iota_G$ on $G$ and the natural homomorphisms $\phi_i$ of $G$ onto each $G/Z_i$. Let $G/Z_i$ have the partial order inherited from either $\preccurlyeq$ or $\leqq$. Then $\phi$ is a realization of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \mathscr{A}, \mathscr{J}', \preccurlyeq, \leqq)$ where $\mathscr{J}' = \{(G/Z_i, \preccurlyeq) : i \in I\}$ and each $G/Z_i$ is a tight Riesz group without pseudozeros.*

PROOF. The presence of $\iota_G$ as a component in both $\theta$ and $\phi$ guarantees that $\theta$ and $\phi$ are one-to-one. Clearly $\theta(G)$ and $\phi(G)$ are subdirect products of the groups $\mathscr{P}$, $\mathscr{P}'$. That $(G/\overline{M_i}, \preccurlyeq)$ is a tight Riesz group follows from Lemma 2.11. Since $Z_i/\overline{M_i}$ is the group of pseudozeros (together with zero in $G/\overline{M_i}$) it follows from Lemma 2.2 that $G/Z_i$ is a tight Riesz group without pseudozeros, with respect to the order inherited from $G/\overline{M_i}$ and therefore with respect to the order inherited from either of the orders $\preccurlyeq$ and $\leqq$ on $G$. It only remains, therefore, to show that $\theta$ and $\phi$ are $o$-isomorphisms with respect to $\preccurlyeq$ and $\leqq$ .

As all the components in $\theta$ and $\phi$ are order-preserving with respect to $\preccurlyeq$, $\theta$ and $\phi$ are themselves order-preserving with respect to $\preccurlyeq$. Moreover, the presence of $\iota_G$ as a component in $\theta$ and $\phi$ ensures that $\theta(a) \preccurlyeq \theta(b)$ $(\phi(a) \preccurlyeq \phi(b))$ implies that $a \preccurlyeq b$. Thus $\theta$ and $\phi$ are $o$-isomorphisms with respect to $\preccurlyeq$.

Let $0 < a$, $a \in G$. Then $a \notin M_i$, for any $i$, and so $M_i < M_i + a$, for all $i \in I$. Since $\overline{M}_i \cap G^+ = S_i$, this implies that $\overline{M}_i < \overline{M}_i + a$, for all $i \in I$. Thus $0 < \theta(a)$. Furthermore, $\overline{M}_i < \overline{M}_i + a$ implies that $\overline{M}_i + a$ is not a pseudozero in $G/\overline{M}_i$ and, therefore, that $a \notin Z_i$. Thus $Z_i < Z_i + a$, for all $i$, and so $0 < \phi(a)$.

Conversely, $0 < \theta(a)$ implies that $0 < \iota_G(a)$ or that $a \in G^+$. Furthermore, $0 < \theta(a)$ implies that $a \notin \overline{M_i}$, for all $i$. Therefore, $a \in G^+ \setminus \cup S_i = P^*$ and $0 < a$. Similarly $0 < \phi(a)$ implies that $0 < a$ and so $\theta$ and $\phi$ are both $o$-isomorphisms.

We now give a variant of the hybrid product in order to obtain a representation for non-abelian partially ordered groups with compatible tight Riesz orders.

By an *automorphism* $\rho$ of a partially ordered set $(X, \preccurlyeq)$ is meant a permutation $\rho$ of $X$ such that $x \preccurlyeq y$ if and only if $x\rho \preccurlyeq y\rho$. (We shall write permutations on the right.)

Let $H$ be a group of automorphisms of the partially ordered set $(X, \preccurlyeq)$. Then $H$ is a partially ordered group with respect to the naturally defined order:

$$h_1 \preccurlyeq h_2 \text{ if and only if } xh_1 \preccurlyeq xh_2, \text{ for all } x \in X. \tag{2}$$

We shall write $(H, X, \preccurlyeq)$ to indicate that $(X, \preccurlyeq)$ is a partially ordered set and that $H$ is a group of automorphisms of $X$ with the inherited order defined in (2) above.

Let $\mathscr{A} = \{(H_\alpha, X_\alpha, \preccurlyeq) : \alpha \in \mathscr{A}\}$ and $\mathscr{J} = \{(K_i, Y_i, \preccurlyeq) : i \in I\}$ be two sets of groups of automorphisms of partially ordered sets. Then we shall denote by $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ the full direct product of the groups $H_\alpha$ and $K_i$

$$\mathscr{P} = \prod_{\alpha \in A} H_\alpha \times \prod_{i \in I} K_i$$

endowed with two partial orderings $\preccurlyeq$ and $\leqq$ where $\preccurlyeq$ is the cardinal order and $\leqq$ is the order defined by

$$\iota < f \text{ if and only if (1) } xf(\alpha) \succcurlyeq x, \text{ for all } \alpha \in A,\ x \in X_\alpha$$
$$\text{and (2) } yf(i) \succ y, \text{ for all } i \in A,\ y \in Y_i,$$

where $\iota$ denotes the identity of $\mathscr{P}$. Since elements of $\mathscr{P}$ are now functions whose values are permutations we treat $\mathscr{P}$ as a multiplicative group. Once again, if $(G, \preccurlyeq, \leqq)$ is a group with two partial orderings then by a *relization* $\theta$ of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ is meant a monomorphism of $G$ onto a subdirect product of $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ which is an $o$-isomorphism of $(G, \preccurlyeq)$ and $(G, \leqq)$ into $(\mathscr{P}, \preccurlyeq)$ and $(\mathscr{P}, \leqq)$, respectively.

For any group $G$ and any subgroup $H$ of $G$, the *right regular representation* of $G$ is the homomorphism $\rho: g \to \rho_g$ of $G$ into the group of permutations of $G$ where $a\rho_g = a + g$, for all $a \in G$, and the *right regular representation of G on*

$R(H)$ is the homomorphism $\sigma\colon g \to \sigma_g$ of $G$ into the group of permutations of $R(H)$ where $(H+a)\sigma_g = H + a + g$.

THEOREM 2.13. *Let $(G, \preccurlyeq)$ be an isolated partially ordered group with a compatible tight Riesz order $\leqq$. Let $\partial P = \bigcup\{S_i\colon i \in I\}$ be the prime decomposition of $\partial P$ and $M_i = [S_i]$. Let $\theta$ denote the product of the right regular representation $\rho$ of $G$ and the right regular representations $\rho_i$ of $G$ on $R(\overline{M_i})$. Then $\theta$ is a realization of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ where $\mathscr{A} = \{(\rho(G), G, \preccurlyeq)\}$ and $\mathscr{J} = \{(\rho_i(G), R(\overline{M_i}), \preccurlyeq)\colon i \in I\}$ where the order $\preccurlyeq$ on $R(\overline{M_i})$ is that induced from either of the orders $\preccurlyeq$, $\leqq$ on $G$. Each $R(\overline{M_i})$ has the tight Riesz property and $(\rho_i(G), \leqq)$ is a tight Reisz group.*

PROOF. The presence of $\rho$ as a component of $\theta$ ensures that $\theta$ is one-to-one. It is also clear that $\theta(G)$ is a subdirect product of $\mathscr{P}$. That $R/(\overline{M_i})$ has the tight Riesz property follows from Lemma 2.11. Each $\rho_i$, $i \in I$, is clearly order-preserving with respect to $\preccurlyeq$ and so also is $\rho$. Hence $\theta$ is $\preccurlyeq$-preserving. If $\theta(a) \preccurlyeq \theta(b)$ then $\rho(a) \preccurlyeq \rho(b)$ and therefore $a \preccurlyeq b$. Thus $\theta$ is an $o$-isomorphism with respect to $\preccurlyeq$. Let $0 < a$ and $j \in I$. Then $a \notin \overline{M_i}$, for any $i \in I$. Since $P^*$ is a normal subset of $G$, for any $b \in G$, $-b + S_j + b = S_i$, for some $i \in I$. Therefore $-b + \overline{M_j} + b = \overline{M_i}$, for some $i \in I$. Therefore $\overline{M_i} + a \neq \overline{M_i}$ and so

$$\begin{aligned} \overline{M_j} + b + a &= b + \overline{M_i} + a \\ &\neq b + \overline{M_i} \\ &= \overline{M_j} + b. \end{aligned}$$

Hence

$$\overline{M_j} + b < \overline{M_j} + b + a = (\overline{M_j} + b)\rho_j(a),$$

for all $b \in G$, $j \in I$. Hence $\iota < \theta(a)$ and $\theta$ is $\leqq$-preserving.

Now let $a \in G$ be such that $\iota < \theta(a)$. Then $\iota_G \preccurlyeq \rho(a)$, where $\iota_G$ is the identity permutation of $G$. Hence $0 \preccurlyeq a$. Also $\overline{M_i} < \overline{M_i}\rho_i(a) = \overline{M_i} + a$, for all $i \in I$. Hence $a \notin \cup\{M_i\colon i \in I\}$. Therefore $a \in G^+ \setminus \cup\{S_i\colon i \in I\} = P^*$ and $\theta$ is an $o$-isomorphism with respect to $\leqq$.

The kernel of $\rho_i$ is $\ker \rho_i = \cap\{-x + \overline{M_i} + x \colon x \in G\}$ and is therefore a closed subgroup. As $\overline{M_i} \cap P^* = \varnothing$, $\ker \rho_i \cap P^* = \varnothing$. Hence $\ker \rho_i$ is closed but not open. That $(\rho_i(g), \leqq) = (G/\ker \rho_i, \leqq)$ is a tight Riesz group then follows from Lemma 2.10.

## 3. Representation of lattice ordered groups with compatible tight Riesz orders

Throughout this section, $(G, \preccurlyeq)$ will denote a lattice ordered group (henceforth $l$-group) with a compatible tight Riesz order $\leqq$. Our previous notational conventions for $G^+$, $P$, $P^*$, $\partial P$, $\mathscr{U}$, etc. will be retained. If $M$ is a *convex l-subgroup* of $G$ (that is, a convex sublattice as well as a subgroup) then $(R(M), \preccurlyeq)$ is a lattice. For this and other basic facts about $l$-groups the reader is referred to Conrad (1967).

LEMMA 3.1. *For a convex l-subgroup $M$ of $(G, \preccurlyeq)$ the following are equivalent*:

(1) *$a, b \in G^+ \backslash M$ implies that $a \wedge b \in G^+ \backslash M$;*

(2) *the lattice $(R(M), \preccurlyeq)$ is totally ordered.*

A convex $l$-subgroup of $(G, \preccurlyeq)$ satisfying the conditions of Lemma 3.1 is called a *prime* subgroup. There are various other characterizations of prime subgroups for which the reader is referred to Conrad (1967).

LEMMA 3.2. *Let $\partial P = \cup \{S_i : i \in I\}$ be the prime decomposition of $\partial P$ and* $M_i = [S_i]$.

(1) *Each $M_i$ is a prime subgroup of $(G, \preccurlyeq)$.*

(2) *Each $M_i$ is closed in $(G, \mathscr{U})$.*

PROOF. (1) is proved in Reilly (1973) Theorem 2.6.

It is observed by Miller (1973) $10^0$ that $\bar{M}$ is a convex $l$-subgroup of $(G, \preccurlyeq)$, for any convex $l$-subgroup $M$ of $G$. Therefore, $\overline{M_i}$ is a convex $l$-subgroup of $(G, \preccurlyeq)$ and so is directed. By Theorem 2.9, $\overline{M_i} \cap G^+ = S_i$ and so by Proposition 2.8, $\overline{M_i} = M_i$.

The reason for the terminology "prime decomposition of $\partial P$" should now be evident from Lemma 3.2.

The following characterization of minimal prime subgroup will be useful.

LEMMA 3.3. (Conrad (1967)). *A prime subgroup $M$ of $(G, \preccurlyeq)$ is a minimal prime subgroup if and only if for every $x \in M \cap G^+$ there is a $y \in G^+ \backslash M$ with $x \wedge y = 0$.*

An $l$-group $(G, \preccurlyeq)$ with compatible tight Riesz order $\leqq$ is *androgenous* if there exist $x, y \in G$ with $x > x \wedge y$ but $x \ngeqq y$. This term was introduced by Miller (to appear) where he obtained a representation of non-androgenous $l$-groups with compatible tight Riesz orders. We give various conditions equivalent to being non-androgenous below. In this connection an element $a$ of $G$ is a *weak unit* if $a \in G^+$, $a \neq 0$ and $a \wedge x = 0$ implies that $x = 0$.

For any subset $A$ of $G$ we write $A^+ = \{x \in A : 0 \preccurlyeq x\}$.

LEMMA 3.4. *The following are equivalent.*

(1) *$G$ is non-androgenous.*

(2) *For all $x, y \in G$, $x > x \wedge y$ implies that $x > y$.*

(3) *For all* $x, y \in G$, $x < x \vee y$ *implies that* $x < y$.
(4) *If* $x \wedge y$ *is neither* $x$ *nor* $y$ *then* $x, y \in x \wedge y + \partial P$.
(5) *If* $x \vee y$ *is neither* $x$ *nor* $y$ *then* $x \vee y \in (x + \partial P) \cap (y + \partial P)$.
(6) $x > 0$, $y \succ 0$ *implies that* $x \wedge y \succ 0$.
(7) $P^* \subseteq w$, *the set of weak units of* $(G, \preccurlyeq)$.
(8) $\partial P \supseteq \cup \{M^+ : M$ *is a minimal prime subgroup of* $(G, \preccurlyeq)\}$.
(9) $\cap \{M_i : i \in I\} = \{0\}$, *where* $\{M_i : i \in I\}$ *is the set of prime subgroups of* $(G, \preccurlyeq)$ *determined in Lemma* 3.2.

PROOF. The equivalence of (1)–(7) is given in Miller (1973).

Let (7) hold, let $M$ be a minimal prime subgroup of $(G, \preccurlyeq)$ and let $a$ be any non-zero element of $M^+$. Then by Lemma 3.3, there is a non-zero element $b$ of $G^+$ with $a \wedge b = 0$. Hence $a \notin w$ and so $a \notin P^*$. Thus $P^* \cap M^+ = \varnothing$, $M^+ \subseteq \partial P$ and (8) holds.

Let (8) hold and suppose that $A = \cap \{M_i : i \in I\} \neq \{0\}$. Then $A^+ \neq \{0\}$. So let $a$ be any non-zero element of $A^+$ and let $x$ be any element of $P^*$ such that $a$ and $x$ are incomparable with respect to $\preccurlyeq$. (Such elements must exist, since $a \preccurlyeq P^*$ implies that $a = 0$, by $T(3)$.) Let $a'$, $x'$ be such that

$$a = a \wedge x + a'$$

$$x = a \wedge x + x'.$$

Then $a'$, $x'$ are non-zero and $a' \wedge x' = 0$. Since the intersection of all minimal prime subgroups is $\{0\}$, there exists a prime subgroup $M$ such that $a' \notin M$; then $x' \in M$ and, by (8), $x' \in \partial P = \cup \{S_i : i \in I\}$. But $a \wedge x \in S_i$ for all $i \in I$ and so, for some $j \in I$, $x \in S_j \subseteq G \backslash P^*$, a contradiction. Hence (9) holds.

Finally, let (9) hold. Let $x \in P^* \backslash w$. Then, for some non-zero element $a \in G^+ \backslash \{0\}$, $x \wedge a = 0$. Since each $M_i$ is a prime subgroup and $x \notin \cup M_i$, we must have $a \in \cap M_i = \{0\}$, a contradiction.

In relation to (7) in Lemma 3.4, it is interesting to note the following.

LEMMA 3.5. *The following statements are equivalent.*

(1) $P^* = w$.
(2) $\partial P = \cup \{M^+ : M$ *is a minimal prime*$\}$.

PROOF. Suppose that (1) holds and $A = \cup \{M^+ : M$ is a minimal prime$\}$. From Lemma 3.4, (7) and (8) we have that $A \subseteq \cup \{M_i^+ : i \in I\} = B$, say. For any $m \in B \backslash A$, $m \notin P^*$ and so there exists, by (1), an element $a \in G^+$ with $a \neq 0$ and $a \wedge m = 0$. Hence $m \in \cup \{M^+ : M$ a minimal prime$\}$, a contradiction since $m \notin A$. Thus (2) holds.

If (2) holds, then, from Lemma 3.4, we have that $P^* \subseteq w$. Let $a \in w$. By Lemma 3.3, $a \notin M$, for any minimal prime $M$. Hence, by (2), $a \notin \partial P$ and so $a \in P^*$.

In order to refine the representation theorems of Section 1, we wish to find subgroups $M$ which relate to the $\preccurlyeq$ and $\leqq$ structure and for which $R(M)$ is totally ordered with respect to the induced orders.

LEMMA 3.6. *For all* $i \in I$, $(R(M_i), \preccurlyeq) = (R(M_i), \leqq)$ *and* $(R(M_i), \preccurlyeq)$ *is a dense totally ordered set.*

PROOF. We know that $(R(M_i), \preccurlyeq) = (R(M_i), \leqq)$ from Lemma 2.11. Since $M_i$ is a prime subgroup of $(G, \preccurlyeq)$ by Lemma 3.2, we know that $(R(M_i), \preccurlyeq)$ is totally ordered by Lemma 3.1. Since $(R(M_i), \preccurlyeq)$ has the tight Riesz property by Lemma 2.11, it must be dense.

An alternative source of prime subgroups that relate to the two orders is given in the following.

LEMMA 3.7. *If* $M$ *is a prime subgroup of* $(G, \preccurlyeq)$ *such that* $M \cap P^* \neq \emptyset$ *then* $(R(M), \preccurlyeq) = (R(M), \leqq)$ *is totally ordered.*

PROOF. It is shown in Theorem 3.3 of Reilly (1973) that $(R(M), \leqq)$ is totally ordered. So also is $(R(M), \preccurlyeq)$ totally ordered. Since $\preccurlyeq$ is a refinement of $\leqq$ and both are totally ordered, we must have equality.

We now have a sufficient supply of suitable prime subgroups of $(G, \preccurlyeq)$ to approach the representation theorems.

Let $\mathscr{A} = \{H_\alpha : \alpha \in A\}$ and $\mathscr{J} = \{K_i : i \in I\}$ be two sets of totally ordered groups where $I \neq \emptyset$ (but possibly $A = \emptyset$). Suppose further that each $K_i$ is dense. Let $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ be the hybrid product of the totally ordered groups $H_\alpha$, $K_i$ described in Section 2. Then it is observed by Miller (to appear), 5$^0$ that $(\mathscr{P}, \preccurlyeq)$ is an $l$-group and that $\leqq$ is a compatible tight Riesz order.

If $(G, \preccurlyeq)$ is an $l$-group with a compatible tight Riesz order $\leqq$ then by an *l-realization* $\theta$ of $(G, \preccurlyeq, \leqq)$ as a subdirect product of $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ is meant a realization $\theta$ which is also a lattice homomorphism in the sense that $\theta(a \vee b) = \theta(a) \vee \theta(b)$ and $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$, for all $a, b \in G$. In this connection we observe that an $l$-monomorphism of an $l$-group $(G, \preccurlyeq)$ into an $l$-group $(H, \preccurlyeq)$ is necessarily an $o$-isomorphism of $(G, \preccurlyeq)$ into $(H, \preccurlyeq)$.

We note that an $l$-group is always isolated (Conrad (1967)).

THEOREM 3.8. *Let* $(G, \preccurlyeq)$ *be an abelian l-group with a compatible tight Riesz order* $\leqq$. *Let* $\{N_\alpha : \alpha \in A\}$ *be the set of prime subgroups* $N_\alpha$ *of* $(G, \preccurlyeq)$ *such that* $N_\alpha \cap P^* \neq \emptyset$. *Let* $\partial P = \cup \{S_i : i \in I\}$ *be the prime decomposition of* $\partial P$ *and* $M_i = [S_i]$. *Let* $\mathscr{A} = \{(G/N_\alpha, \preccurlyeq) : \alpha \in A\}$ *and* $\mathscr{J} = \{(G/M_i, \preccurlyeq) : i \in I\}$. *Then the product* $\theta$ *of the natural homomorphisms* $\theta_\alpha : G \to G/N_\alpha$ *and* $\theta_i : G \to G/M_i$ *is an l-realization of* $(G, \preccurlyeq, \leqq)$ *as a subdirect product of the hybrid product* $(\mathscr{J}, \mathscr{A}, \mathscr{P}, \preccurlyeq, \leqq)$ *of the abelian totally ordered groups* $(G/N_\alpha, \preccurlyeq)$ *and the dense abelian totally ordered groups* $(G/M_i, \preccurlyeq)$.

*The product $\phi$ of the natural homomorphisms $\theta_i: G \to G/M_i$ is an l-realization of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \varnothing, \mathscr{J}, \preccurlyeq, \leqq)$ of the abelian totally ordered groups $G/M_i$ if and only if $G$ is non-androgenous.*

PROOF. We have from Lemmas 3.6, 3.7 that each $G/N_\alpha$, $G/M_i$ is a totally ordered abelian group and from Lemma 3.6 that $G/M_i$ is dense.

Since each $\theta_\alpha$ is an $l$-homomorphism, the kernel, $\ker\theta$, of $\theta$ is a convex $l$-subgroup of $(G, \preccurlyeq)$. Let $g \in \ker\theta$ and $g \succcurlyeq 0$. Then $g \in \cap \{N_\alpha : \alpha \in A\}$ and $g \in \cap \{M_i : i \in I\} = B$, say. Let $x \in P^*$ and let $x'$, $g'$ be such that

$$x = x \wedge g + x'$$

$$g = x \wedge g + g'.$$

Then $x' \wedge g' = 0$. If $x' = 0$, then $x = x \wedge g \preccurlyeq g$ which would imply that $g \in P^*$. But $g \in B^+ \subseteq \partial P$. Hence $x' \neq 0$. If $g' \neq 0$ then there exists a prime subgroup $M$ of $G$ with $g' \notin M$, $x' \in M$. (Take $M$ to be any maximal convex $l$-subgroup of $(G, \preccurlyeq)$ with respect to not containing $g'$. Then (Conrad (1967)) $M$ is prime and therefore $x' \in M$.) Since $x \notin \partial P$ and $x \wedge g \in B$ we must have $x' \notin \partial P$. Hence $x' \in P^*$ and $M \cap P^* \neq \varnothing$. Thus $M = N_\alpha$ for some $\alpha \in A$. Since $g \succcurlyeq g'$ we would then have $g \notin N_\alpha$, a contradiction. Therefore $g' = 0$ and $g = x \wedge g \preccurlyeq x$ for all $x \in P^*$. By $T(3)$ $g = 0$. Hence $\ker\theta = \{0\}$ and $\theta$ is an $l$-monomorphism.

If $x \in G$ and $0 < x$, then $x \notin M_i$, for any $i$ and so $0 < \theta_i(x)$, for all $i \in I$, and $0 \preccurlyeq \theta_\alpha(x)$, for all $\alpha \in A$. In other words, $0 < \theta(x)$. Conversely, let $0 < \theta(x)$. Then $0 \prec \theta(x)$ and $x \in G^+$. Since $0 \prec \theta_i(x)$, for all $i \in I$, we have $x \notin M_i$, for all $i$. Therefore, $x \in P^*$ and $\theta$ is an $o$-isomorphism with respect to $\leqq$. Thus the assertions regarding $\theta$ are proved.

Clearly $\ker\phi = \cap \{M_i : i \in I\}$ and so, by Lemma 3.4, $\phi$ is an $l$-monomorphism if and only if $G$ is non-androgenous. The result is then clear.

Corresponding to Theorem 2.13, we have the following result.

THEOREM 3.9. *Let $(G, \preccurlyeq)$ be an l-group with a compatible tight Riesz order $\leqq$. Let $\{N_\alpha : \alpha \in A\}$ be the set of prime subgroups $N_\alpha$ of $(G, \preccurlyeq)$ for which $N_\alpha \cap P^* \neq \varnothing$. Let $\partial P = \cup \{S_i : i \in I\}$ be the prime decomposition of $\partial P$ and $M_i = [S_i]$. Let $\theta$ be the product of all the right regular representations $\rho_\alpha$, $\rho_i$ of $G$ on the $R(N_\alpha)$ and $R(M_i)$, respectively. Then $\theta$ is an l-realization of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \mathscr{A}, \mathscr{J}, \preccurlyeq, \leqq)$ where $\mathscr{A} = \{(\rho_\alpha(G), R(N_\alpha), \preccurlyeq) : \alpha \in A\}$, $\mathscr{J} = \{(\rho_i(G), R(M_i), \preccurlyeq) : i \in I\}$ and each $\rho_\alpha(G)$ $(\rho_i(G))$ is an l-subgroup of the l-group of all automorphisms of the totally ordered sets $R(N_\alpha)$ $(R(M_i))$. Each $R(M_i)$ is dense.*

*Let $\phi$ denote the product of the representations $\rho_i$, $i \in I$. Then $\phi$ is an l-realization of $(G, \preccurlyeq, \leqq)$ as a subdirect product of the hybrid product $(\mathscr{P}, \varnothing, \mathscr{J}, \preccurlyeq, \leqq)$ if and only if $G$ is non-androgenous.*

PROOF. Since each component of $\theta$ and $\phi$ is an $l$-homomorphism it is clear that $\theta$ and $\phi$ are $l$-homomorphisms (with respect to $\preccurlyeq$). Since $\ker\theta \subseteq \{\cap\{N_\alpha : \alpha \in A\}\} \cap \{\cap\{M_i : i \in I\}\}$ it follows as in Theorem 3.8 that $\ker\theta = \{0\}$ and that if $G$ is non-androgenous, $\ker\phi = \{0\}$. Thus $\theta$ is an $l$-monomorphism. Since, in fact, $\{M_i : i \in I\}$ is a normal family of subgroups, that is, for all $i \in I$, $a \in G$, $a + M_i - a = M_j$, for some $j \in I$, the kernel of $\phi$ is precisely $\cap\{M_i : i \in I\}$. Hence $\phi$ is an $l$-monomorphism if and only if $G$ is non-androgenous. It follows, as in Theorem 2.13, that $\theta$ ($\phi$) is an $o$-isomorphism with respect to $\leqq$ (provided, in the case of $\phi$, that $G$ is non-androgenous).

Each $(R(M_i), \preccurlyeq)$ is dense by Lemma 3.6, and each $\rho_i(G)$ is a tight Riesz group as in Theorem 2.13.

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