19

Large N methods in QCD_4

We have encountered the large N expansion, or the $\frac{1}{N}$ expansion, in Chapter 7 in the context of two-dimensional field theory, and in particular its application by 't Hooft to solve the mesonic spectrum of QCD in two dimensions [10]. A natural question is thus what does this limit tell us about four dimensional QCD, and in particular can one also solve the mesonic spectrum of QCD in four dimensions in the large N approximation. These questions will be the topics of this chapter. We start with the rules of counting powers of N in four-dimensional QCD, and the relations between Feynman diagrams in the double line notation and Riemann surfaces. We then briefly discuss certain applications of the expansion to the mesonic physics and then follow Witten's seminal analysis of baryons in the planar approximation [222].

The large N technique was introduced by 't Hooft in [122]. Since then there have been many follow-up papers and there is a very rich literature on large N approximation including review papers and books like [223], [66], [165], [46] and [160]. In this chapter we use mainly the latter.

19.1 Large N QCD in four dimensions

Let us remind the reader the basic notations and the classical action of QCD in four dimensions. The two-dimensional ones were presented in (8),

$$S_{QCD} = \int \mathrm{d}^4x \left[-\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] + \bar{\Psi}_i(i\not{D} - m_i)\Psi_i \right], \tag{19.1}$$

where the gauge fields are spanned by $N \times N$ Hermitian matrices T^A such that $A_\mu = A_\mu^A T^A$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\frac{g}{\sqrt{N}}[A_\mu, A_\nu]$, the covariant derivative $D_\mu = \partial_\mu + i\frac{g}{\sqrt{N}}A_\mu$, the fermions Ψ are in the fundamental representation of the color group and $i = 1, \ldots, N_\mathrm{f}$ indicates the flavor degrees of freedom. The gauge coupling was chosen to be $\frac{g}{\sqrt{N}}$, to accommodate a large N approximation with g fixed. This can be shown for instance in applying the large N expansion to the β *function*. The latter, when g is used in the covariant derivatives, is given by (17.73),

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = -\left[\frac{11}{3}N - \frac{2}{3}N_\mathrm{f}\right]\frac{g^3}{16\pi^2} + \mathcal{O}(g^5), \tag{19.2}$$

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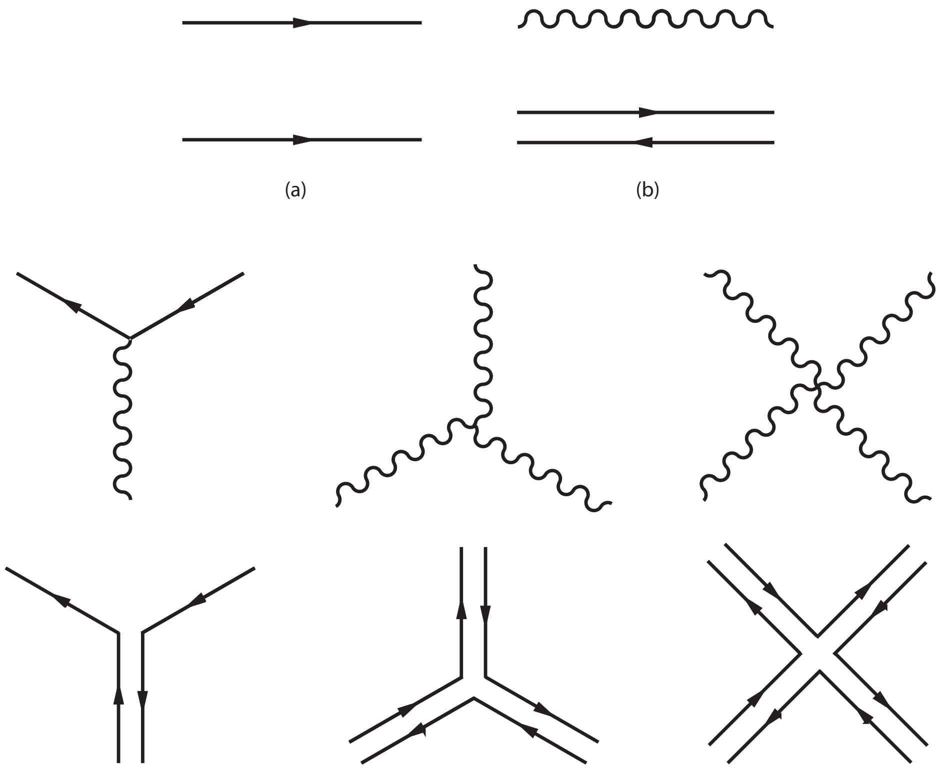

Fig. 19.1. Four-dimensional Feynman rules in the usual form and in the double line notation.

which obviously is not suitable for a large N expansion whereas if one replaces $g \to \frac{g}{\sqrt{N}}$ the β function takes the form,

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = -\left[\frac{11}{3} - \frac{2N_f}{3N}\right] \frac{g^3}{16\pi^2} + \mathcal{O}(g^5). \tag{19.3}$$

The rules of the Feynman diagrams in two-dimensional QCD (see Fig. 10.1) include the fermion propagator, the gluon propagator and the quark gluon vertex, all expressed in the double line notation. Using the light-cone in two dimensions one eliminates the three- and four-gluon vertices. In four dimensions due to the transverse directions the gluon vertices cannot be eliminated by choosing a gauge. Thus all together the four-dimensional Feynman rules are expressed in Fig. 19.1.

The figures a,b,c are identical to the two-dimensional ones (Fig. 10.1) whereas figures d and e are the three- and four-gluon vertices. The quark propagator (19.1a) is given by,

$$<\psi^a(x)\bar{\psi}^b(y)> = S(x-y)\delta^{ab}. \tag{19.4}$$

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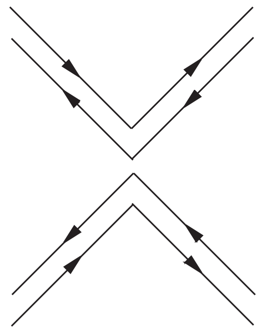

Fig. 19.2. Color flow in the double line notation associated with two traces.

The $SU(N)$ gluon propagator in the double line notation reads,

$$<(A_\mu)^a_b(x)(A_\nu)^c_d(y)> = D_{\mu\nu}(x-y)\frac{1}{2}\left(\delta^a_d\delta^c_b - \frac{1}{N}\delta^a_b\delta^c_d\right). \qquad (19.5)$$

For a $U(N)$ gauge group the propagator does not include the term which is proportional to $\frac{1}{N}$. One can view the $SU(N)$ color indices as those of a $U(N)$ theory plus an additional "ghost" $U(1)$ gauge field that cancels the contribution of the $U(1)$ gauge field in the $U(N)$ gauge group. For many applications the distinction between the $U(N)$ and $SU(N)$ in the $1/N$ expansion is sub-leading. The three and four gluon vertices (see Fig. 19.1 d,e) emerge obviously from the $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$ term in the action. Note that it is a single trace operator and hence the four-gluon vertex is the one depicted in (19.1 e) and not in Fig. 19.2 which corresponds to the color flow of a two-trace operator.

To compute the N dependence of Feynman diagrams it is convenient to re-scale the gluon field and the quark field as follows,

$$\frac{gA_\mu}{\sqrt{N}} \to \hat{A}_\mu \quad \psi \to \sqrt{N}\hat{\psi} \qquad (19.6)$$

In terms of the re-scaled field the QCD action reads,

$$\mathcal{L} = N\left[-\frac{1}{2g^2}\mathrm{Tr}[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}] + \sum_{i=1}^{N_f}\bar{\hat{\psi}}_i(i\not{D} - m_i)\hat{\psi}_i\right]. \qquad (19.7)$$

From this Lagrangian we can read off the powers of N and $\lambda \equiv g^2 = N$ of each part of a Feynman diagram. The vertex operator scales like N, the propagator as $\frac{1}{N}$ and every color index loop gives a factor of N. If we combine the dependence on λ we find for the gluon, that the vertex behaves as $\frac{N}{\lambda}$, the propagator as $\frac{\lambda}{N}$ and the color loop as N, while for the fermions neither the propagator nor the quark-gluon vertex depend on the coupling.

Thus a connected vacuum diagram with V vertices, E propagators, namely edges and F loops, namely faces, is of order (see (7.7)),

$$N^{V-E+F}\lambda^{(E_{(G)}-V_{(G)})} = N^\chi\lambda^{(E_{(G)}-V_{(G)})}, \qquad (19.8)$$

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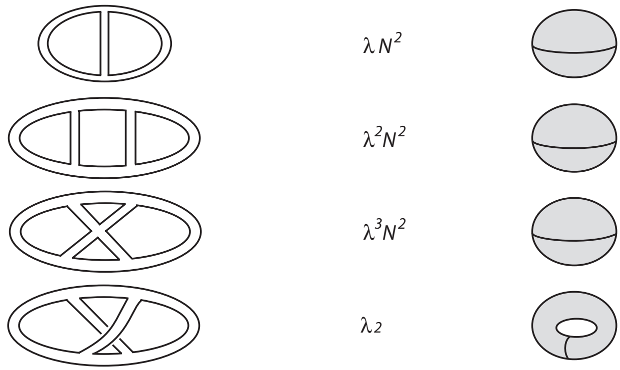

Fig. 19.3. Examples of Feynman diagrams with only gluons.

where,

$$\chi \equiv V - E + F = 2 - 2h - b, \tag{19.9}$$

is the Euler character of the surface, h is the genus, namely, the number of handles and b is the number of boundaries. $E_{(G)}$ and $V_{(G)}$ are the appropriate qualities for gluons. For instance the sphere has $\chi = 2$ since it has no handles and no boundaries, the disk has $\chi = 1$ since it has no handles and one boundary and the torus has one handle and no boundaries and hence it has $\chi = 0$. Thus the Feynman diagrams look like triangulated two-dimensional surfaces. In fact all possible gluon exchange may fill the holes of the triangulated structure forming a smooth surface with no boundaries for gluon only diagrams, and with boundaries for diagrams that include quark loops. It was conjectured that the two-dimensional surface is the world sheet of a string theory which is dual to QCD. There has been tremendous progress in this string/gauge duality in recent years following the seminal AdS/CFT duality of Maldacena [158]. This is beyond the scope of this book and we refer the reader to the relevant literature, for instance the review [10].

To further demonstrate the determination of the order of a diagram consider first the diagrams that involve only gluons which appear in Fig. 19.3. The diagram in (a) has $V = 2, E = 3, F = 3$ and thus it behaves as $N^2\lambda$. Similarly in (b) and (c) $V = 4, E = 6, F = 4$ and $V = 5, E = 8, F = 5$ so that they behave as $N^2\lambda^2$ and $N^2\lambda^3$, respectively. The three diagrams (a), (b) and (c) are all planar diagrams and have a topology of a sphere. Note however that diagram (d) which is non-planar behaves as $N^0\lambda^2$, namely of genus one. In the large N limit this last diagram is obviously suppressed.

So far we have only discussed diagrams with gluons. Quarks propagators are represented (see Fig. 19.1a) by a single line. A closed quark loop is a boundary and hence using (19.9) it contributes to the diagram a factor of $\frac{1}{N}$. Consider for example the diagram drawn in Fig. 19.4.

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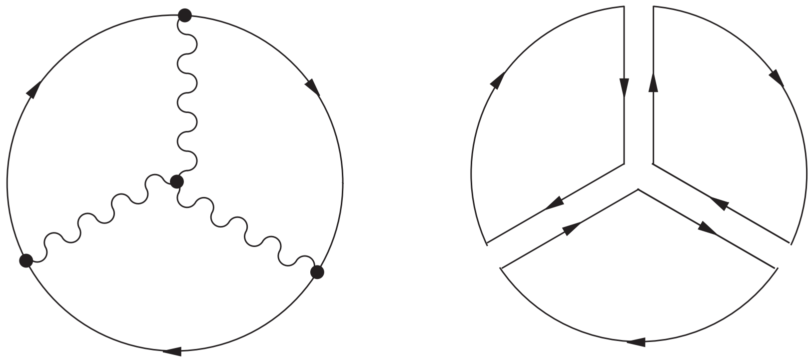

Fig. 19.4. Four-dimensional Feynman rules in the usual form and in the double line notation.

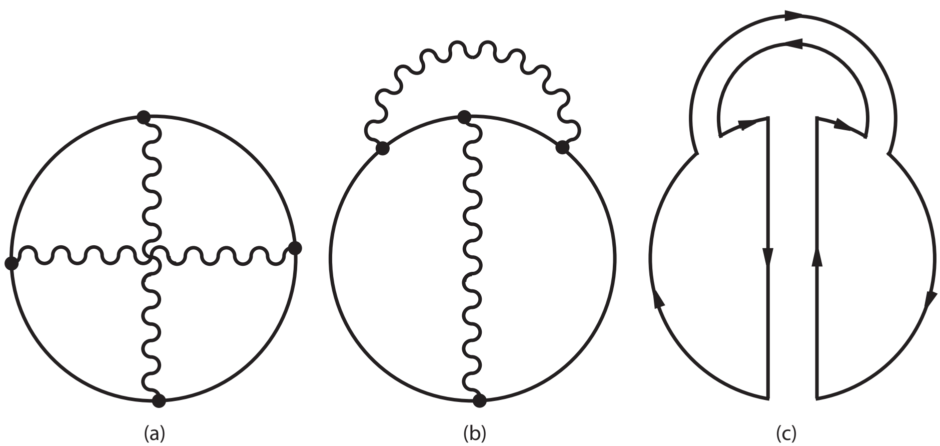

Fig. 19.5. Non-planar diagram with gluons and quarks.

It is a diagram of order N. This follows trivially from the fact that it has zero genus, $h = 0$ and one boundary $b = 1$. Alternatively we have one gluon vertex, three gluon propagators and three loops and hence $N^{1-3+3} = N$. Obviously this is also the result when one uses the unrescaled operators where each vertex contributes $\frac{1}{\sqrt{N}}$ and each index loop N, so that we get $(\frac{1}{\sqrt{N}})^4 \times N^3 = N$. Similar to the non-planar gluon diagram (19.3d), Fig. 19.5 describes a non-planar diagram that includes both gluons and quarks. This diagram scales like $\frac{1}{N}$ since there is no gluon vertex, two gluon propagators and one index loop $N^{0-2+1} = \frac{1}{N}$.

As was mentioned above the difference between the $SU(N)$ case versus the $U(N)$ can be accounted by adding a ghost $U(1)$ gauge field whose role is to cancel the extra $U(1)$ part of the $U(N)$. The $U(1)$ commutes with the $U(N)$ gauge fields and therefore does not interact with them and hence one has to incorporate only the coupling of the quark fields to the $U(1)$ gauge field. When we consider a connected diagram with gluons and $U(1)$ ghost gauge fields the contribution to the

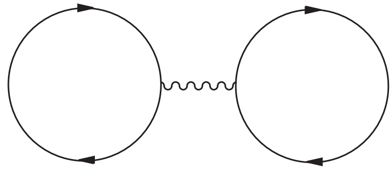

Fig. 19.6. $U(1)$ ghost propagator connecting two otherwise disconnected diagrams.

counting of orders of N due to the gluons is not affected, whereas each $U(1)$ ghost field contributes a factor of $\frac{1}{N^2}$. The latter follows from the fact that the ghost $U(1)$ propagator contributes a factor of $1/N$ and another $1/N$ factor due to the two coupling constants at the end of the propagator. This can easily be seen from the unrescaled action (19.1). For diagrams that are connected with the ghost field and otherwise disconnected as is depicted in Fig. 19.6 the situation is different. For instance that diagram is of order N^0 since it has $N \times N \times \frac{1}{N^2}$. Note however that even in this case there is a difference of order $\frac{1}{N^2}$ between the $SU(N)$ and $U(N)$ cases, (actually, in Fig. 19.6 the contribution of $SU(N)$ vanishes).

19.1.1 Counting rules for correlation functions

So far we have described the counting rules for vacuum diagrams. We now proceed to the counting rules of correlation functions of gluons and quarks which are vacuum expectation values of gauge invariant operators made out of gluon and quark fields. The latter should be color singlets, not necessarily local, that cannot be split into color singlet pieces. Thus operators like $\bar\psi\psi, Tr[F_{\mu\nu}F^{\mu\nu}], \bar\psi(y)\mathrm{e}^{-i\int_x^y A_\mu(z)dz^\mu}\psi(x)$ are allowed whereas $(\bar\psi\psi)^2$ is not. As usual the procedure to compute correlation functions of such operators is to add appropriate source terms to the action and differentiate the generating function with respect to sources that correspond to the operators. If we denote by $\hat{\mathcal{O}}_i$ a gauge invariant operator made out of the rescaled fields, we shift the Lagrangian density as follows, $\mathcal{L}_0 \to \mathcal{L}_0 + \sum_i NJ_i\mathcal{O}_i$, where $\mathcal{L}_0$ is the Lagrangian density without the sources and J_i is the source that corresponds to $\mathcal{O}_i$. Thus any correlator can be determined as follows,

$$<\hat{\mathcal{O}}_1 \ldots . \hat{\mathcal{O}}_n> = \frac{1}{iN}\frac{\partial}{\partial J_1}\cdots\frac{1}{iN}\frac{\partial}{\partial J_n}W(J). \tag{19.10}$$

In terms of N counting, for correlation functions of only gluon fields $W(J)$ is of order N^2 and hence the correlator is of order N^{2-n}. The leading order of $W(J)$ in the case where quarks fields are also involved is of order N which means that the correlator is of order N^{1-n}. Let us denote by $\hat{G}_i$ and $\hat{M}_i$ glueball and meson gauge invariant operators, respectively built from the rescaled fields $\hat{A}_\mu$ and $\hat\psi, \hat{\bar\psi}$. The leading order in N of the various correlators are summarized in the following table. The operator $\sqrt{N}\hat{M}$ is the operator that creates a meson

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Table 19.1. *The N counting for glueball and meson correlators*

Correlator	N counting
$<\hat{G}_1\hat{G}_2>_c$	N^0
$<\hat{G}_1\dots\hat{G}_{n_g}>_c$	N^{2-n_g}
$<\hat{M}_1\hat{M}_2>_c$	N^{-1}
$<\sqrt{N}\hat{M}_1\sqrt{N}\hat{M}_2>_c$	N^0
$<\sqrt{N}\hat{M}_1\sqrt{N}\hat{M}_{n_h}>_c$	$N^{1-\frac{n_h}{2}}$
$<\hat{G}_1\dots\hat{G}_{n_g}\sqrt{N}\hat{M}_1\sqrt{N}\hat{M}_{n_h}>_c$	$N^{1-n_g-\frac{n_h}{2}}$

with a unit amplitude. In particular we read from the table that the glueball meson interaction is of order $\frac{1}{\sqrt{N}}$.

19.2 Meson phenomenology

The picture that emerges from N counting is that of mesons and glueballs interacting weakly with a coupling of $\frac{1}{\sqrt{N}}$. At the tree level the singularities are poles. At one loop, namely at order $\frac{1}{N}$ the singularities are two particle cuts, at two loops three-particle cuts and so on. We now describe certain phenomena of meson physics that are accounted for by $\frac{1}{N}$ arguments and quite often cannot be explained in any other way.

- The spectrum at low energies of QCD in the large N limit include infinitely many narrow glueball and meson resonances. The fact that the number of resonances is infinite follows from the need to reproduce the logarithmic running of QCD correlation functions. A meson two-point function can be written as a sum of resonances,

$$\int \mathrm{d}^4x\mathrm{e}^{iqx} <M(x)M(0)>_c = \sum \frac{Z_i}{q^2-m_i^2}, \tag{19.11}$$

  since single meson exchange dominates in the large N limit. The logarithmic dependence on q^2 of the left-hand side can be recast only provided that the sum on the right-hand side includes infinitely many terms. The resonances are narrow since their decay width goes to zero in the large N limit. This follows from the fact that the phase space factor is N independent and the coupling constant behaves like $\frac{1}{\sqrt{N}}$.
- In Chapter 17 we encountered the pion decay constant f_π. Let us check how it scales with N. Recall its definition $<0|\bar{\psi}\gamma_5 T^A\psi|\pi^b(p)>= if_\pi p^\mu\delta^{ab}$. The corresponding gauge invariant correlator is $<N\hat{M}_1\sqrt{N}\hat{M}_2>$, where the first operator $N\hat{M}_1$ corresponds to the axial current and the second $\sqrt{N}\hat{M}_2$ to the

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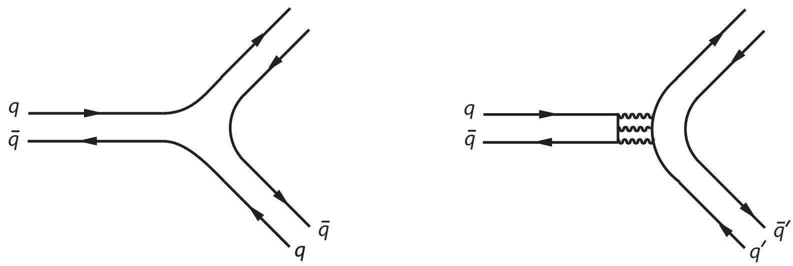

Fig. 19.7. Zweig's rule for the decay of a meson into two mesons.

pion produced from the vacuum with a unit amplitude. This correlator scales like $\sqrt{N}$ and hence,

$$f_\pi \sim \sqrt{N}. \tag{19.12}$$

- The suppression of exotic states of the form $q\bar{q}q\bar{q}$ and the fact that the meson is almost a pure $q\bar{q}$ state with little impact of the $q\bar{q}$ sea are straightforward consequences of large N. Since a quark loop, as we have seen above, is suppressed by a factor of $\frac{1}{N}$ the $q\bar{q}$ sea is irrelevant. Since in the leading order the mesons are non interacting in large N, there is no interaction that will bind two mesons into a $q\bar{q}q\bar{q}$ exotic state.
- Consider the two diagrams of Fig. 19.7.
  Using the counting rules it is obvious that the right-hand diagram is $\frac{1}{N}$ suppressed in comparison to the one on the left. Correspondingly the meson will preferably decay into two mesons of the left-hand side of the figure, what is referred to as Zweig rule conserving decay, and not to the two mesons on the right which is a Zweig rule suppressed decay. In this sense large N predicts the Zweig rule. The same mechanism is in charge of the fact that there is almost flavor singlet and octet degeneracy. In the leading order in large N the whole nonet is degenerate since the diagrams that split singlets from octets involve a $q\bar{q}$ annihilation which is order of $\frac{1}{N}$. In the large N for instance the vector mesons (ρ, w, ϕ, K^*) are degenerate.
- It is known that meson decay proceeds mainly via decay into two body states and not into states of more mesons. Large N tells us that the decay into two mesons behaves as $\frac{1}{\sqrt{N}}$, whereas a decay into four mesons is of order $\frac{1}{N^{3/2}}$.
  This can also be compared to the decay of a meson via creation of a quark anti-quark pair in the mesonic flux tube [60].
- The N counting rules tell us also that meson scattering amplitudes are given by an infinite sum of tree diagram of exchange of physical mesons. This fits nicely the so-called Regge phenomenology, where strong interactions are interpreted as an infinite sum of tree diagrams with hadron exchange.
- Another very important phenomenon is related to the axial $U(1)$, the theta term and the mass of the η'. This will be described in detail in Section 22.5, but here we present the picture in the large N.

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19.2.1 *Axial* $U(1)$ *and the mass of the* η'

Consider the full action of four-dimensional YM theory, which includes also the θ term,

$$S_{YM} = \int \mathrm{d}^4 x \left[-\frac{1}{4}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] + \frac{\theta g^2}{16\pi^2 N}\mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] \right]. \tag{19.13}$$

The normalization of the θ term is such that θ is an angular variable, namely, under the shift of $\theta \to \theta + 2\pi$ the action is shifted by $2\pi n$ where n is some integer, so that e^{iS} is unchanged. This result follows from the quantization of the θ term,

$$\int \mathrm{d}^4 x \frac{g^2}{16\pi^2 N}\mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] = \text{ integer.} \tag{19.14}$$

We would like to explore the dependence on θ of the vacuum energy in the pure YM theory and in the theory with massless quarks. In particular we would like to determine $\frac{\mathrm{d}^2 E}{\mathrm{d}\theta^2}|_{\theta=0}$.

Using the path integral formulation we find that,

$$\frac{\mathrm{d}^2 E}{\mathrm{d}\theta^2}|_{\theta=0} = \frac{1}{N^2}\left(\frac{g^2}{16\pi^2}\right)^2 \int \mathrm{d}^4 x <T(\mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}(x)]\mathrm{Tr}[[F_{\mu\nu}\tilde{F}^{\mu\nu}])(0)> . \tag{19.15}$$

Let us introduce an IR cutoff and take,

$$\begin{aligned}\frac{\mathrm{d}^2 E}{\mathrm{d}\theta^2}|_{\theta=0} &= \left(\frac{g^2}{16\pi^2 N}\right)^2 \lim_{k\to 0} U(k)\\ U(k) &= \int \mathrm{d}^4 x \mathrm{e}^{(\nu k x)} <T(\mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}](x)\mathrm{Tr}[[F_{\mu\nu}\tilde{F}^{\mu\nu}(0)])> .\end{aligned} \tag{19.16}$$

It is easy to check that in perturbation theory $U(k)$ is of order N^2 due to the contribution of the N^2 degrees of freedom of the gluons. However, perturbatively, $\lim_{k\to 0} U(k) = 0$ since $F\tilde{F}$ is a total derivative. One concludes that perturbatively the vacuum energy is θ independent. To better understand (19.16) we rewrite $U(k)$ in terms of a sum over intermediate single particle states,

$$U(k) = \sum_{\mathrm{gb}} \frac{N^2 (a_{\mathrm{gb}})_n^2}{k^2 - (m_{\mathrm{gb}})_n^2} + \sum_{\mathrm{mes}} \frac{N (a_{\mathrm{mes}})_n^2}{k^2 - (m_{\mathrm{mes}})_n^2}, \tag{19.17}$$

where gb stands for glueball and mes for meson. $N a_{\mathrm{gb}}$ and $\sqrt{N} a_{\mathrm{mes}}$ are the amplitudes for $\mathrm{Tr}[F\tilde{F}]$ to create a glueball and meson state, respectively. This result again follows from the N counting rules,

$$<0|\mathrm{Tr}[F\tilde{F}]|\mathrm{mes}> \sim \sqrt{N} \qquad <0|\mathrm{Tr}[F\tilde{F}]|\mathrm{gb}> \sim N. \tag{19.18}$$

The fact that only single states and not multi-states are taken in the intermediate states is since the latter are suppressed in the large N. In the pure YM without quarks the first term vanishes and hence $U(0) \sim N^2$ and $\frac{\mathrm{d}^2 E}{\mathrm{d}\theta^2}|_{\theta=0} \sim 1$. In the presence of massless quarks we know that there could not be any θ dependence and thus we should be able to show that the first term is cancelled out. However, it seems that there is no way that the second term can cancel the first.

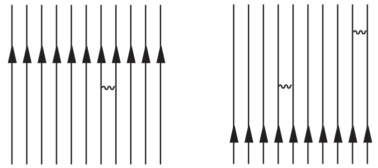

Fig. 19.8. Perturbative correction to the free propagator due to an exchange of one gluon (left) and two gluons (right).

In fact it is possible if there is one meson state with mass $m_{\text{mes}} \sim \frac{1}{\sqrt{N}}$ and if the two terms have opposite sign. This obviously can cancel the $k = 0$ term in $U(k)$ and does not cancel for non-trivial k, but this is exactly what enters (19.16). The opposite sign follows from the fact that an additional equal time commutator term has to be added to (19.15) (see the appendix of [221]). Assuming such a state with mass $m_{\text{mes}} \sim \frac{1}{\sqrt{N}}$ the form of $U(0)$ is,

$$U(0) = N\frac{a_{\eta'}^2}{M_{\eta'}^2}. \tag{19.19}$$

Using the axial anomaly equation which will be further discussed in (22.5),

$$\partial_\mu J_5^\mu = N_{\text{f}}\frac{g^2}{4\pi^2 N}\text{Tr}[F\tilde{F}], \tag{19.20}$$

we get,

$$\frac{g^2}{8\pi^2} <0|\text{Tr}[F\tilde{F}]|\eta'> = \frac{N}{2N_{\text{f}}} <0|\partial_\mu J_5^\mu|\eta'>= \frac{N}{2N_{\text{f}}} f_{\eta'} M_{\eta'}^2, \tag{19.21}$$

From this relation we get the Veneziano–Witten formula or the mass of the η',

$$M_{\eta'}^2 = \left(\frac{2N_{(\text{f})}}{f_{(\pi)}}\right)^2 \frac{\text{d}^2 E}{\text{d}\theta^2}|_{\theta=0}. \tag{19.22}$$

The picture that emerges from this discussion is that the η' is a Goldstone boson in the large N limit. It has a mass of the order of $M_{\eta'} \sim \frac{1}{\sqrt{N}}$. The dependence on η' of nonzero amplitudes can be obtained from the dependence on θ in the theory without quarks by the following replacement,

$$\theta \to \theta + \left(\frac{2N_{(\text{f})}}{f_{(\eta')}}\right)\eta'. \tag{19.23}$$

Note that $f_{(\eta')} = f_{(\pi)}$ to leading order in $\frac{1}{N}$.

19.3 Baryons in the large N expansion

Whereas we have seen that the large N expansion is very useful in discussing mesons, it may seem that it is not the case for baryons. Baryonic diagrams depend on N both via the combinatorial factors associated with the diagrams, as well as the fact that the diagrams themselves include N quarks.

The problem is clearly demonstrated when computing the perturbative correction to the free propagator of an N quarks state. The correction occurs due to an exchange of a gluon between two quarks (see Fig. 19.8). The gluon exchange

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diagram scales as $\frac{1}{N}$. However since there are $\frac{1}{2}N(N-1)$ possible pairs the net effect is of order N. In a similar manner the exchange of two gluons is of order $(\frac{1}{\sqrt{N}})^4 N^4 \sim N^2$, where the first factor comes from the four vertices and the second from the number of ways to choose the four quarks. Higher-order exchange diagrams will have higher order divergence in large N. We will now show, that in spite of this fatal obstacle, there is a large N approximation to the problem of the baryons. The idea is to divide the problem into two parts, in the first one uses diagrammatic methods to study the problem of n quark interaction, and then the effect of these forces on an N quark state. Let us first apply this approach for determining the dependence of the mass of the baryon on N in the quark model. Assuming that the mass gets contributions from the quark masses, quark kinetic energy and quark–quark potential energy, the mass of the baryon reads,

$$M_{\mathrm{B}} = N\left[m_{\mathrm{q}} + T_{\mathrm{q}} + \frac{1}{2}V_{\mathrm{q}}\right], \tag{19.24}$$

where m_{q} is the quark mass, T_{q} is the kinetic energy of the quark and V_{q} is the quark–quark potential energy. Thus we observe that the mass of the baryon scales as N. This result will be shown to hold even beyond the quark model. Again we have made use of the fact that the potential energy is combined from the N^2 combinatorial factor and the $\frac{1}{N}$ factor that comes from the vertices, or gluon propagator.

Leaving aside the quark model, we want to address first the baryonic system made out of very heavy quarks.

19.3.1 The Hartree approximation

In the baryons, the quarks are anti-symmetric in color. Thus they are symmetric in flavor, space and spin combined. Hence they act like bosons. A natural framework to analyze such a system of bosonic charged particles that are subjected to a central potential is the Hartree approximation, in which each particle moves independently of the others in a potential which is determined self consistently by the motion of all the other particles. The justification of the use of this approximation is the large N limit which renders the interactions to be weak. Therefore neglecting the fact that the particle trajectory affects the state of all the other particles and hence the potential that it feels, is justified. Also it is obvious that taking the potential created by all particles and not the one created by all the particles apart from the one we consider, is a $\frac{1}{N}$ effect. Since, as mentioned above, the particles in the non-color degrees of freedom are bosons it implies that in the ground state of the baryon all the particles will sit in the ground state of the Hartree potential.

Let us take the Hamiltonian of the system to be,

$$H = \frac{1}{2m}\sum_a |\vec{p}_a|^2 + \frac{1}{2N}\sum_{a\neq b} V^2(\vec{r}_a, \vec{r}_b) + \frac{1}{6N^2}\sum_{a\neq b\neq c} V^3(\vec{r}_a, \vec{r}_b, \vec{r}_c) + \ldots, \tag{19.25}$$

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where we have suppressed the flavor and spin degrees of freedom, V^n stands for the n body interaction and is independent of N and its strength is of order N^{1-n}, as explained above. In fact since the number of clusters of n quarks is of order N^n, each term in the Hamiltonian is proportional to N.

Next one takes for the ground state wave function a product of the wave functions of each of the particles, namely,

$$\psi(\vec{r}_a, \ldots \vec{r}_N) = \prod_a \phi(\vec{r}_a) \tag{19.26}$$

where the particle wave functions are determined by a variational method. The expectation value of the Hamiltonian,

$$\begin{aligned} <\psi|H|\psi> = N \Bigg[\frac{1}{2m} \int \mathrm{d}^3\vec{r} |\nabla\phi|^2 + \frac{1}{2} \int \mathrm{d}^3\vec{r}_1 \mathrm{d}^3\vec{r}_2 V^2(\vec{r}_1, \vec{r}_1) |\phi(r_1)\phi(r_2)|^2 + \\ + \frac{1}{6} \int \mathrm{d}^3\vec{r}_1 \mathrm{d}^3\vec{r}_2 \mathrm{d}^3\vec{r}_3 V^3(\vec{r}_1, \vec{r}_2, \vec{r}_3) |\phi|(r_1)\phi(r_2)\phi(r_3)|^2 + \ldots \Bigg], \end{aligned} \tag{19.27}$$

has to be minimized with respect to $\phi(r)$ subjected to the constraint that,

$$\int \mathrm{d}^3\vec{r} |\phi|^2 = 1. \tag{19.28}$$

The minimization translates to,

$$\left[-\frac{\nabla^2}{2m} + V(\vec{r}) \right] \phi = \epsilon\phi, \tag{19.29}$$

where ϵ is the Lagrange multiplier associated with the constraint, and the Hartree potential is,

$$V = \left[\int \mathrm{d}^3\vec{r}_1 V^2(\vec{r}, \vec{r}_1) |\phi(r_1)|^2 + \frac{1}{2} \int \mathrm{d}^3\vec{r}_1 \mathrm{d}^3\vec{r}_2 V^3(\vec{r}, \vec{r}_1, \vec{r}_2) |\phi(r_1)\phi(r_2)|^2 + \ldots \right]. \tag{19.30}$$

We will now treat first the case of heavy quarks and subsequently will address, in a less rigorous manner, the case of light quarks.

19.3.2 Baryons made out of heavy quarks

A non relativistic Schrodinger equation is an adequate framework to deal with very heavy baryons. In this setup for short distances the quark–quark potential is an attractive Coulomb potential so that the Hamiltonian takes the form,

$$H = Nm_\mathrm{q} - \sum_{i=1}^{N} \frac{\partial_i^2}{2m_\mathrm{q}} - \frac{g^2}{N} \sum_{i<j} \frac{1}{|x_i - x_j|}. \tag{19.31}$$

The system is effectively that of N bosons with a Coulomb interaction with a strength of $\frac{1}{N}$.

In spite of the fact that the interaction potential behaves like $\frac{1}{N}$, we cannot treat this term as a perturbation since each quark interacts with N other quarks and hence the total quark–quark interaction of each quark is of order one. This situation calls for a Hartree approximation where, as explained above, the quark is exposed to an average effective potential. The fluctuations of the effective potential are negligible and hence we can consider a background c-number potential.

For heavy quarks where the potential is taken to be a Coulomb potential we find,

$$<\psi|H - N\epsilon|\psi>= N[M + \tfrac{1}{2m}\int \mathrm{d}^3\vec{r}|\nabla\phi|^2 - \tfrac{g^2}{2}\int \mathrm{d}^3\vec{r}\int \mathrm{d}^3\vec{r}'\tfrac{|\phi(r)|^2|\phi(r')|^2}{|r-r'|} \\ -\epsilon\int \mathrm{d}^3\vec{r}|\phi(r)|^2]. \tag{19.32}$$

The main point here is that each of the terms is proportional to N and hence the result of the minimization is N independent. The variation with respect to ϕ^* results in the following Schrodinger equation,

$$-\frac{\nabla^2\phi}{2m} - g^2\phi\int \mathrm{d}^3\vec{r}'\frac{|\phi(r')|^2}{|r-r'|} = \epsilon\phi. \tag{19.33}$$

One can convert this integro-differential equation into a fourth-order differential equation

$$-\frac{1}{2m}\nabla^2\left(\frac{\nabla^2\phi}{\phi}\right) + 4\pi g^2|\phi|^2 = 0. \tag{19.34}$$

For radial solutions, for instance, the ground state of this equation takes the form,

$$-\frac{1}{2m}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\mathrm{dd}r\right]\left(\frac{1}{\phi}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\mathrm{dd}r\right]\phi\right) + 4\pi g^2|\phi|^2 = 0, \tag{19.35}$$

which is derived by dividing (19.33) by ϕ and acting with ∇^2.

Even without solving this equation, it is clear that the mass of the baryon is linear with N and that the charge distribution of the baryon which implies in particular its size is N independent.

19.3.3 Baryons made out of light quarks

Up to this point we have used a non relativistic Hartree approximation which is valid only for heavy quarks. However, phenomenologically, one is more interested in baryons made out of light quarks and in scattering processes that involve relativistic quarks. We will show now that even for the light quark baryons, a Hartree-like approximation, namely that each quark moves independently of the others in a potential which is determined self consistently by the motion of all the other particles, is still justified. Moreover, it will be argued that just as for the

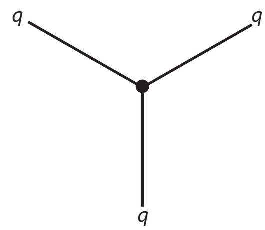

Fig. 19.9. Baryon stringy configuration for $N = 3$.

heavy quark baryons, also for the baryons made out of light quarks, the mass is linear with N, whereas the size and the shape of the baryon are N independent.

A major difference between the case of heavy quarks versus that of light ones is that in the latter case one has to introduce on top of the two-body interaction also a three-body interaction and, in general, n-body interactions. In addition one has to use a relativistic analog of the Hartree approximation. In two dimensions one can solve the relativistic Hartree approximation. Unfortunately, the four-dimensional analog is not known. Let us first discuss a non relativistic Hartree approximation with n-body interactions and then argue about the relativistic analog. The Hamiltonian for the case with any n-body interaction takes the form,

$$H = \frac{1}{2m}\sum_a |\vec{p}_a|^2 + \frac{1}{2N}\sum_{a\neq b} V^2(\vec{r}_a, \vec{r}_b) + \frac{1}{6N^2}\sum_{a\neq b\neq c} V^3(\vec{r}_a, \vec{r}_b, \vec{r}_c) + \dots, \tag{19.36}$$

where we have suppressed the flavor and spin degrees of freedom, V^n stands for the n-body interaction and is independent of N. The strength of V^n is of order N^{1-n} since breaking the n quark line costs a factor of N^{-n} and since the baryon is in a totally antisymmetric representation, each quark line carries a different color index.

We now substitute this Hamiltonian into $<\psi|H|\psi>$ and use a variational method as above. Since for each V^n term there are N^n ways to choose a set of n quarks, here again the expectation value of the Hamiltonian is linear in N.

Next we have to introduce a four-dimensional relativistic Hartree approximation. In two dimensions in the large N limit the Hartree approximation is exact. The generalization to four dimensions, however, is not known and hence one can make only the qualitative statement that even in this case the mass is linear in N and the size and shape are independent of N.

The Hartree approximation of light quarks moving in an effective potential can be also related to a string model of the baryon. In this model, the N quarks are attached to a common junction as can be seen in Fig. 19.9 for the case of $N = 3$.[1] In the large N approximation the junction can be regarded as a heavy object and its motion can be ignored. The interaction of the quarks with the fixed junction can be thought of as an interaction with an effective Hartree potential.

[1] The modern picture of the latter is that of a wrapped D brane.

19.3.4 Baryonic excited states

Low-lying excitations of the baryons are described by wave functions where out of N single particle states, a number $n_k \ll N$ of states are placed in the kth excited state of the Hamiltonian. The corresponding mass of the excited baryon is $M = M_0 + \sum_{n_k} n_k \epsilon_k$ where ϵ_k is the energy of the kth excited state.

Highly excited states have a finite fraction of single particle excited states. Denote by p this fraction, namely there are $(1-p)N$ particles in the ground state and pN ones in excited states which we take to be ϕ_1 so that the baryon wave function is,

$$\psi(\vec{r}_a, \dots \vec{r}_N) = \sum (-1)^P \prod_a^{pN} \phi_1(\vec{r}_a) \prod_a^{(1-p)N} \phi_0(\vec{r}_a). \tag{19.37}$$

Inserting this ansatz into the expectation value of the Hamiltonian one gets a set of two coupled nonlinear equations for ϕ_0 and ϕ_1. This structure can obviously be generalized to states with higher single particle excited states.

Another approach to studying excited states is to apply a time-dependent Hartree approximation. It is easy to check that starting with the Hartree ansatz for the wave function but now with single particle wave functions that are also time dependent, one finds instead of (19.33) the following time-dependent Schrodinger equation,

$$-\frac{\nabla^2 \phi}{2m} - g^2 \phi \int \mathrm{d}^3 \vec{r}' \frac{|\phi(r')|^2}{|r - r'|} = i\partial_t \phi(\vec{r}, t). \tag{19.38}$$

This equation is solved by $\phi(t, \vec{r}) = e^{-i\epsilon t} \phi(\vec{r})$ where $\phi(\vec{r})$ is a solution of the time-independent equation. By Galilean boosting along, for instance the x direction, a static baryon solution, we find the solution,

$$\phi(\vec{r}, t) = \phi(x - vt) \mathrm{e}^{i(Mvx - \epsilon t - \frac{1}{2} M v^2 t)}, \tag{19.39}$$

which is a baryon travelling with a constant velocity. This is an additional solution to the time-dependent equation. In fact starting with any function $\phi(\vec{r}, 0)$ and substituting it into (19.38) a new solution will be generated. These solutions will generically be excited states, but not in energy eigenstates, since they will not have a harmonic form.

To generate excited baryon solutions which are in eigenstates of the energy, we make use of the DHN procedure discussed in Section 5.5.1 in the context of two-dimensional field theories. The idea is to look for solutions which are periodic in time and to quantize them by requiring that the action during a period will obey,

$$\int_0^T \mathrm{d}t <\psi| H - i\partial_t |\psi> = 2\pi n, \tag{19.40}$$

where n is some integer number. Recall that this condition follows from the fact that the solutions are invariant under time translations, so from $\psi(t)$ we can also generate a solution $\psi(t - t_0)$ for any t_0 and also any linear combinations of them, and in particular a harmonic varying solution $\int_0^T \mathrm{d}t_0 \mathrm{e}^{-it_0 E} \psi(t - t_0)$.

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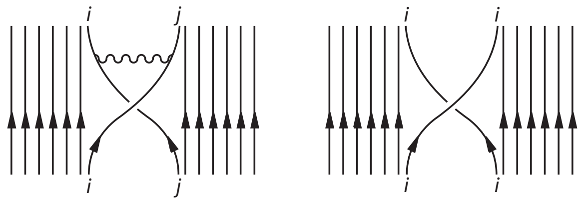

Fig. 19.10. Baryon–baryon scattering. Exchange of a quark on the right, while on the left, such an exchange plus an exchange of a gluon.

In analogy to the discussion in Section 5.5.1 here as well one can introduce a non-abelian flavor group, namely construct baryons made out of several flavors. In this case one introduces into the Hartree wave function a separate single particle wave function for each flavor.

For very heavy quarks one can neglect the spin-dependent forces, and hence anticipate that the baryons are spherically symmetric. However, for less heavy quarks this is no longer the case. For a baryon made out of a single flavor, in the ground state all the spins are aligned and hence the total spin is $\frac{1}{2}N$. Due to the fact that the total spin is very large, the effect of the coupling of this large spin to the orbit is significant and hence the ground state will no longer be spherically symmetric. If one takes the large N analog of the baryon to be composed of $\frac{N+1}{2}$ quarks of one flavor and $\frac{N+1}{2} - 1$ of the other flavor, then the net spin will be $\frac{1}{2}$ since the spin–spin interaction will align the spin of the different flavors in an antiparallel way. Unlike the one flavor case where the spin is $\frac{N}{2}$, the spin $\frac{1}{2}$ will be too small to affect the spherically symmetric ground state via spin orbit interaction, and hence for that case it will remain symmetric.

19.4 Scattering processes

In Section 19.2 it was shown that in the leading order of the large N there is no meson–meson scattering. The same applies also for meson–glueball and glueball–glueball scattering. Let us now address the question of baryon–baryon and baryon–meson scattering.

Baryon–baryon scattering is dominated by an interchange of one quark between two baryons. Whether the process involves only an interchange or also in addition an exchange of a gluon, as is shown in Fig. 19.10, the amplitude is of order N. In the case of no gluon exchange, there is a choice of the interchanging quark in one of the baryon, which goes like N. Once a quark in one baryon is chosen it can be interchanged only with a quark in the second baryon that carries exactly the same color index, hence there is no additional N dependence. Thus altogether the amplitude is order N. Note also that the diagram (19.10) comes with a factor of (-1). The amplitude for an interchange that is accompanied with an exchange of a gluon is also of order N, which follows from the fact that there is a factor of N from choosing the quark in the first baryon,

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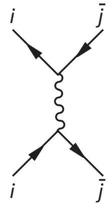

Fig. 19.11. Annihilation of a quark coming from the baryon and anti-quark coming from the anti-baryon.

another factor of N from the other baryon and a factor of $\frac{1}{N}$ from the quark gluon vertices.

This fact that the amplitude is order N is behind why there is a smooth large N limit to the baryon–baryon scattering. Recall that the mass of the baryon and hence also the non-relativistic kinetic energy of the baryon are linear in N. Thus the total Hamiltonian can be written as $H = N\hat{H}$ where $\hat{H}$ is N independent. The eigenvectors of $\hat{H}$ and hence of the scattering process are N independent.

Quantitatively one addresses the question of baryon–baryon scattering using a non relativistic time-dependent Schrodinger equation for a system of $2N$ quarks. Due to the exclusion principle the total wave function should be a product of two orthonormal space and spin wave functions $\phi_i(x,t)$ where $i = 1, 2$ in the following way,

$$\psi(x_1, \ldots, x_{2N}, t) = \sum_P (-1)^P \prod_{i=1}^{N} \phi_1(x_i, t) \prod_{j=1}^{N} \phi_2(x_j, t). \tag{19.41}$$

Using again the time-dependent variational principle, we find that in the case where all the spins of the quarks are parallel so we can ignore them,

$$\begin{aligned} i\partial_t \phi_1(x,t) = \frac{\nabla^2}{2M}\phi_1(x,t) - g^2\phi_1(x,t) \int \frac{\mathrm{d}y \phi_1^* \phi_1(y,t)}{|x-y|} \\ - g^2\phi_1(x,t) \int \frac{\mathrm{d}y \phi_2^* \phi_1(y,t)}{|x-y|}, \end{aligned} \tag{19.42}$$

and another equation where $\phi_1 \leftrightarrow \phi_2$. Apart from the last term this equation is identical to the one describing a single baryon (19.38), hence the last term obviously describes the interaction between the two baryons. To describe baryon–baryon scattering we start with inital conditions where the wave functions ϕ are localized at two far away regions of space, but heading for a collision. When the two wave functions overlap the interaction term is important and determines the scattering via (19.42).

The baryon anti-baryon scattering is dominated by an annihilation of a quark coming from the baryon and an anti-quark from the anti-baryon. The amplitude of this process is of order N since choosing one quark is order N, choosing an anti-quark is order N and the coupling is order $\frac{1}{N}$ (see Fig. 19.11). Again this is like the scaling of the kinetic term and hence there is a smooth limit. The

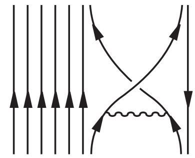

Fig. 19.12. Exchange of a quark and a gluon in meson–baryon scattering.

variational procedure now involves a wave function composed of N quark and N anti-quark wave functions, namely,

$$\psi(\vec{r}_a, \ldots, \vec{r}_N) = \prod_a \phi(\vec{r}_a) \prod_b \bar{\phi}(\vec{\bar{r}}_b), \tag{19.43}$$

where $\bar{\phi}(\vec{\bar{r}}_b)$ is the wave function of a single anti-quark. The minimization now yields a pair of coupled equations for ϕ and $\bar{\phi}$.

The meson–baryon scattering is described in a diagram like Fig. 19.12. The diagram is of order N^0 since there is a factor of $\frac{1}{N}$ from the coupling and N from the number of ways to choose the quark from the baryon. Recall that the baryon kinetic energy is order N and that of the meson is order one. Hence the interaction term is negligible from the point of view of the baryon and it does not feel the meson but the meson motion is affected by the interaction and thus there is a meson baryon non-trivial scattering. Denoting again the wave function of a quark of the baryon as $\phi(x, t)$ and that of the meson as $\phi_{\mathrm{M}}(x_{\mathrm{M}}, y_{\mathrm{M}}, t)$, the trial many body wave function reads,

$$\psi(\vec{r}_a, \ldots, \vec{r}_N, x_{\mathrm{M}}, y_{\mathrm{M}}, t) = \prod_a \phi(\vec{r}_a, t)\phi_{\mathrm{M}}(x_{\mathrm{M}}, y_{\mathrm{M}}, t). \tag{19.44}$$

Again we substitute this wave function into the variational principle $\int \mathrm{d}t <\psi|H - i\partial_t|\psi>$. The solution for ϕ of the corresponding equations is identical to the solution of the baryon and hence indeed the baryon is not affected by the presence of the meson. On the other hand the equation for ϕ_M is affected by the presence of ϕ. The equation will be that of a free meson plus two additional terms describing the interaction that take the form,

$$H_{\mathrm{int}} = -g^2\phi(x)\left[\int \frac{\mathrm{d}z\phi^*(z,t)\phi_{\mathrm{M}}(z, y_{\mathrm{M}}, t)}{|x_{\mathrm{M}} - z|} + \int \frac{\mathrm{d}z\phi^*(z,t)\phi_{\mathrm{M}}(z, y_{\mathrm{M}}, t)}{|z - y_{\mathrm{M}}|}\right]. \tag{19.45}$$

Thus the interaction term and hence the whole equation is linear in ϕ_{M}.

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