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# OPERATIONS IN GROTHENDIECK RINGS AND THE SYMMETRIC GROUP

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In [1] Atiyah described how to use the complex representations of the symmetric group, $S_n$, to define and investigate operations in complex topological $K$-theory. In this paper operations for more general Grothendieck groups are described in terms of the integral representations of $S_n$ using the representations directly without passing to the dual as Atiyah did. The principal tool, which is proved in the first section, is the theorem that the direct sum of the Grothendieck groups of finite integral representations of $S_n$ form a bialgebra isomorphic to a polynomial ring with a sequence of divided powers. A consequence of this theorem is that the only operations that can be constructed from the symmetric groups will be polynomials in the symmetric powers.

The second section shows how properties of operations dealing with composition, preservation of addition, and preservation of multiplication can be respectively characterized in terms of the symmetrized outer product, the comultiplication, and another cobinary operation in the theory of the representations of the symmetric groups. Methods of verifying such properties based on the well-developed theory of the symmetric groups are described. Connections to individual parts of the definition of special λ-rings are made.

In the third section the Adams operations are defined in terms of representations and the theorems of the preceding section are applied to describe their properties.

Throughout this paper $G_R(\pi)$, $\pi$ a group, will be the abelian group generated by $R(\pi)$ modules which are finitely generated and projective as $R$ modules with the relations, $[M] = [M'] + [M'']$ if there is an exact sequence $0 \to M' \to M \to M'' \to 0$. If $M$ and $M'$ are $R(\pi)$ modules, then $M \otimes_\pi M'$ denotes the tensor product over $R(\pi)$.

## 1.

PROPOSITION 1.1. *The homomorphism from $G_Z(S_n)$ to $G_Q(S_n)$ induced from the map sending $[M]$ to $[M \otimes_Z Q]$ is an isomorphism.*

*Proof.* A theorem of Heller and Reiner [7] states that if $R$ is a Dedekind domain of characteristic zero with quotient field $K$, which is a splitting field for a finite group $\pi$, then the map given by the tensor product from $G_R(\pi)$ to $G_K(\pi)$ is an isomorphism. Since the rationals are a splitting field for $S_n$, the proposition is just a special case of the theorem.

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Let $\Gamma$ be the graded $Z$-module with $\Gamma_0 = Z$, $\Gamma_n = G_Z(S_n)$ for $n > 0$, and $\Gamma_n = 0$ for $n < 0$. For convenience $\Gamma_0$ is sometimes denoted as $G_Z(S_0)$. Multiplication in $\Gamma$ is defined by the map, $\phi: \Gamma \otimes \Gamma \to \Gamma$, which is the composite of the canonical map, $h$, from $G_Z(S_i) \otimes G_Z(S_j)$ to $G_Z(S_i \times S_j)$ and the map of $G_Z(S_i \times S_j)$ to $G_Z(S_{i+j})$ induced from the inclusion of $S_i \times S_j$ into $S_{i+j}$. Since $S_i \times S_j$ and $S_j \times S_i$ are conjugate subgroups of $S_{i+j}$, $\Gamma$ is a commutative, graded ring. This product is well-known in the theory of the characters of $S_n$, where it is called the outer product [**9**, p. 52]. A comultiplication on $\Gamma$,

$$\Delta: \Gamma \to \Gamma \otimes \Gamma,$$

is defined as the composite of the restriction map from $G_Z(S_{i+j})$ to $G_Z(S_i \times S_j)$, $S_i \times S_j$ again considered as a subgroup of $S_{i+j}$, with $h^{-1}: G_Z(S_i \times S_j) \to G_Z(S_i) \otimes G_Z(S_j)$. $h$ is an isomorphism since $G_Z(S_n)$ is isomorphic to $G_Q(S_n)$ and $Q$ is a splitting field for $S_n$.

The image of $Z$ with the trivial $S_n$ action is denoted by the sumbol $[n]$ and the image of $Z$ with the action of $S_n$ given by the sign of the permutation by $[\tilde{n}]$. $[0] = [\tilde{0}] = 1$.

THEOREM 1.2. $\Gamma$ *is isomorphic to the graded bialgebra* $Z[X_1, \ldots, X_n, \ldots]$, $n \in N$, *where* $X_n$ *is an indeterminate of degree* $n$, *and the co-multiplication*, $\Delta$, *is given by a sequence of divided powers*,

$$\Delta(X_n) = \sum_{i=0}^{n} X_i \otimes X_{n-i}, \qquad X_0 = 1.$$

*Proof.* A homomorphism $f$ from $Z[X_1, \ldots, X_n, \ldots]$ to $\Gamma$ can be defined by sending $X_n$ to $[n]$ and extending to a ring homomorphism. A basis for $\Gamma_n$, $n > 0$, is given by the irreducible representations of $S_n$ over $Q$ by Proposition 1.1. These representations are in one-to-one correspondence with the partitions of $n$ [**6**, § 28]. A theorem based on the work of Young [**9**, Theorem 2.33] states that if $\chi$ is the element of $\Gamma$ corresponding to the partition $\{n_1, \ldots, n_k\}$ of $n$, then $\chi$ is equal to the determinant of the $k$ by $k$ matrix whose $(i, j)$ entry is the element $[n_i - (i - j)]$, when $i - j \leqq n_i$, and zero elsewhere. Thus $f$ is an epimorphism. Since the rank of $\Gamma_n$ equals the rank of $Z[X_1, \ldots, X_n, \ldots]_n$, $f$ is an isomorphism.

Since

$$\Delta([n]) = \sum_{i=0}^{n} [i] \otimes [n - i]$$

and $\{[n] | n \geqq 0\}$ generates $\Gamma$ as a $Z$-algebra, to complete the proof of the theorem it suffices to show that $\Gamma$ with the comultiplication $\Delta$ is a bialgebra. A straightforward calculation using the formula for induced characters [**6**, § 38] shows that if $c$ and $d$ are respectively characters of $S_m$ and $S_n$, then the induced character on $S_{m+n}$ restricted to the subgroup $S_k \times S_{m+n-k}$ for some $k$ is the same as the sum of the characters of $S_k \times S_{m+n-k}$ induced from the restrictions of $c$ and $d$ to the subgroups $S_i \times S_{m-i}$ and $S_j \times S_{n-j}$, $i + j = k$, where $S_j \times S_{m-i} \times S_j \times S_{n-j}$ is identified with $S_i \times S_j \times S_{m-i} \times S_{n-j}$ and considered as a subgroup of $S_k \times S_{m+n-k}$ for the induction. With Proposition 1.1 and the insertion of $h$ and $h^{-1}$ in the appropriate places, this character equation implies

$$\Delta\phi = (\phi \otimes \phi)(1 \otimes T \otimes 1)(\Delta \otimes \Delta),$$

where $T$ is the twist map on $\Gamma \otimes \Gamma$.

In [**1**] Atiyah has shown that the graded algebra associated with the $Z$-dual of the bialgebra $\Gamma$, $\Gamma^*$, with the natural filtration, is a polynomial algebra. He does not discuss the algebra $\Gamma$. The $n$th indeterminate of the polynomial basis for $\Gamma^*$ is given by the sum of all elements dual to the monomials of degree $n$ of $\Gamma$.

In [**5**] Cartier has shown that $Z[X_1, \ldots, X_n, \ldots]$ with the above bialgebra structure is the bialgebra of distributions, in the sense of formal groups, of the generalized Witt ring over $Z$. He does not discuss the algebra structure of the dual. B. Pareigis and R. Morris have informed me that they have recently shown using functorial techniques that the total dual algebra is a power series in an infinite number of variables.

A different isomorphism $\tilde{f}: Z[X_1, \ldots, X_n, \ldots] \to \Gamma$ may be constructed by sending $x_n \to [\tilde{n}]$.

Let $R$ be a commutative ring (not necessarily with unit) and $\{f_i\} = (f_1, \ldots, f_i, \ldots)$ be a sequence of set theoretic functions of $R$ to $R$. $\mathfrak{D}$ is the category of all pairs $(R, \{f_i\})$. A morphism in $\mathfrak{D}$ from $(R, \{f_i\})$ to $(S, \{g_i\})$ is a ring homomorphism $F$ from $R$ to $S$ such that $Ff_i = g_iF$ for all $i$.

Let $\mathfrak{C}$ be a subcategory of $\mathfrak{D}$. An operation $\alpha$ in $\mathfrak{C}$ assigns to each object $(R, \{f_i\})$ a function $\alpha(R, \{f_i\})$ from $R$ to $R$ which commutes with all the morphisms in $\mathfrak{C}$. Op $(\mathfrak{C})$ is the collection of all operations. Op $(\mathfrak{C})$ has an addition and a multiplication induced from addition and multiplication on $R$. These make Op $(\mathfrak{C})$ into a commutative ring. If $\mathfrak{C}$ includes only rings with units and morphisms preserve units, then there is a unit operation given by $1(R, \{f_i\})(r) = 1$.

PROPOSITION 1.3. *For any $\mathfrak{C}$, there is a ring homomorphism $\gamma$ from $\Gamma_+$ to* Op $(\mathfrak{C})$

*Proof.* Define $\gamma([n])((R, \{f_i\})) = f_n$ and extend to a ring homomorphism on $\Gamma_+$ using Theorem 1.2.

In the case when Op $(\mathfrak{C})$ has a unit one can extend $\gamma$ to all of $\Gamma$. In the remainder of this paper $\gamma(\xi)((R, \{f_i\}))$, $\xi \in \Gamma$, is abbreviated as $\gamma(\xi)$ when there can be no confusion. $\gamma[(0])(x)y = y = \gamma([\tilde{0}])(x)\ y$ for all $x$ and $y$ in $R$ even if $\gamma$ is only defined on $\Gamma_+$.

Let $\mathfrak{A}$ be an admissable subcategory of an abelian category with a product and $\mathscr{S}$ be some collection of short exact sequences of $\mathfrak{A}$ which are preserved by the product. Let $T_n$, $n > 0$, be functors from the direct product of the category of finitely generated $Z$-torsion free $ZS_n$ modules $\mathfrak{M}_n$ and $\mathfrak{A}$ to $\mathfrak{A}$ such that for fixed $M \in \text{ob}\ \mathfrak{M}_n$, $T_n(M,\_)$ takes exact sequences on $\mathcal{S}$ to exact sequences in $\mathcal{S}$ and for fixed $A \in \text{ob}\ \mathfrak{A}$, $T_n(\_, A)$ takes short exact sequences in $\mathfrak{M}_n$ to exact sequences in $\mathcal{S}$. $K_{\mathcal{S}}(\mathfrak{A})$ is the Grothendieck ring generated by isomorphism classes of objects in $\mathfrak{A}$ with relations $[A'] + [A''] = [A]$ when $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{S}$.

$$f_n : K_{\mathcal{S}}(\mathfrak{A}) \to K_{\mathcal{S}}(\mathfrak{A})$$

is defined by the functor from $\mathfrak{A}$ to $\mathfrak{A}$ given by $T_n(Z, \_)$. $(K_{\mathcal{S}}(\mathfrak{A}), \{f_n\})$ is an object in $\mathcal{D}$. If $M \in \text{ob}\ \mathfrak{M}_n$, then $\gamma([M])$ is defined by the functor $T_n(M, \_)$ and thus $\gamma$ is given by $T_*$. Here are examples of this situation.

$\mathfrak{A}$ is the category of finite dimensional real vector bundles over a space $X$ and $\mathcal{S}$ is all split short exact sequences. If $M \in \text{ob}\ \mathfrak{M}_n$, $M \otimes_Z R \times X$, $R$ the real numbers, is in $\mathfrak{A}$ and has an $S_n$ action. If $A \in \text{ob}\ \mathfrak{A}$, $A^{\otimes n}$ is in $\mathfrak{A}$ and has an $S_n$ action. $T_n(M, A)$ is defined to be the quotient of

$$M \otimes_Z R \times X \otimes_X A^{\otimes n}$$

formed by moding out the submodule generated by the images of the endomorphisms $1 \otimes \sigma - \sigma \otimes 1$, $\sigma \in S_n$. In this case $T_n(Z, \_)$ is given by the symmetric product of the bundle.

Let $\mathfrak{A}$ be an admissable subcategory of an abelian category with product and $\mathcal{S}$ be a collection of short exact sequences preserved by the product. Suppose that the objects of $\mathfrak{A}$ are $Z$-modules in a way compatible with the morphisms. If $A \in \text{ob}\ \mathfrak{A}$, $S_n$ acts on $A^{\otimes n}$ and $M \otimes_{S_n} A^{\otimes n}$ is a $Z$-module. If $M \otimes_{S_n} A^{\otimes n}$, $M \in \mathfrak{M}_n$ and $A \in \mathfrak{A}$, is in $\mathfrak{A}$ and if $f \otimes g$, $f$ a morphism in $\mathfrak{M}_n$ and $g$ a morphism in $\mathfrak{A}$, is a morphism in $\mathfrak{A}$, $T_n(M, A)$ may be defined to be $M \otimes_{S_n} A^{\otimes n}$. If the exactness conditions are satisfied, $f_n$ is the symmetric product and $\gamma$ is given by $T_*$. The requirement that $M \otimes_{S_n} A^{\otimes n}$ be in $\mathfrak{A}$ is not satisfied in the case where $\mathfrak{A}$ is finitely generated torsion free $Z$-modules since $\tilde{Z} \otimes_{S_2} Z^{\otimes n} = Z/2Z$. If the objects in $\mathfrak{A}$ are vector spaces over $Q$, it suffices to have $Q(S_n/H) \otimes_{S_n} A^{\otimes n}$ in $\mathfrak{A}$, e.g., $\mathfrak{A}$ closed under coequalizers as $\sigma$ runs over all elements of $H$, $H$ any subgroup of $S_n$, since any $Q(S_n)$-module is the direct summand of a direct sum of modules of the form $Q(S_n/H)$.

**2.** $H$ is the subgroup of $S_{mn}$, which is the image of the canonical inclusion of the direct sum of $m$ copies of $S_n$. $N(H)/H$ is isomorphic to $S_m$ and $N(H) = \sum \sigma_i H$, where $\sigma_i$, $1 \leqq i \leqq m!$, is a permutation of the direct summands of $H$. $M \in \text{ob}\ \mathfrak{M}_m$ and $N \in \text{ob}\ \mathfrak{M}_n$. $N^{\otimes m}$ is both a $Z(H)$ module and a $Z(S_m)$ module, where the action of $S_m$ is given by permuting the tensor product. $M \otimes_Z (N^{\otimes m})$ is made into a $Z(N(H))$ module by defining

$$\sigma_i \tau (x \otimes y) = \sigma_i(x) \otimes (\sigma_i(\tau(y))),$$

$\tau \in H$, $x \in M$, and $y \in N^{\otimes m}$, and extending to the whole group. Induction from $N(H)$ to $S_{mn}$ yields a $Z(S_{mn})$ module. Since all the necessary relations are preserved, there is a function from $G_Z(S_m) \times G_Z(S_n)$ to $G_Z(S_{mn})$ which is denoted by $(\zeta, \xi) \mapsto \zeta \circ \xi$.

This function, in the case of complex representations, is called the symmetrized outer product and is discussed by Robinson [**9**, III, § 3.5] and Kerber [**8**, § 6]. Kerber makes use of the fact that the $N(H)$ is a faithful permutation representation of $S_m \wr S_n$, the wreath product, in his discussion of the symmetrized outer product.

*Definition* 2.1. $(R, \{f_i\})$ has the composition property if $\gamma(\zeta \circ \xi) = \gamma(\zeta) \circ \gamma(\xi)$ for all $\zeta, \xi$ in $\Gamma_+$, where the second circle denotes the composition of the functions.

The composition property corresponds to the formula for the composite of exterior powers in the theory of special $\lambda$-rings [**3**].

The examples at the end of the preceding section have the composition property. In particular $G_C(GL(N, C), \{f_i\})$ with $f_i$ given by the symmetric power of finite dimensional complex representations of $GL(N, C)$, has the composition property.

THEOREM 2.2. *Let $\zeta \in \Gamma_m$, $\xi \in \Gamma_n$, and $\rho \in \Gamma_{mn}$. Suppose $\gamma(\zeta) \circ \gamma(\xi) = \gamma(\rho)$ as operations in $G_C(GL(N, C)\ \{f_i\})$, $N \geqq mn$. Then $\gamma(\zeta) \circ \gamma(\xi) = \gamma(\rho)$ as operations in any object, $(R, \{f_i\})$, of $\mathfrak{D}$ that satisfies the composition property.*

*Proof.* It suffices to show that $\zeta \circ \xi = \rho$ in $\Gamma$. A theorem of Schur [**6**, § 67] states that if $V$ is an $n$-dimensional vector space over $C$ and $M$ is an irreducible complex representation of $S_m$, $n \geqq m$, then $M \otimes_{S_m} V^{\otimes m}$ is isomorphic to an irreducible $GL(n, C)$ submodule of $V^{\otimes m}$ and that the correspondence established in this way is one-to-one and onto. This theorem and Proposition 1.1 combine to prove that the map from $\Gamma_{mn}$ to $G_C(GL(N, C))$, $N \geqq mn$, given by

$$\rho \mapsto \gamma(\rho)(V)$$

is one-to-one. Since $\gamma(\zeta) \circ \gamma(\xi) = \gamma(\rho)$ in $G_C(GL(N, C))$, $\zeta \circ \xi = \rho$ in $\Gamma$.

The image of $G_C(S_{mn})$ in $G_C(GL(N, C))$ lies in the subgroup generated by the integral representations, i.e. the coordinates of the representing matrices are polynomials in the coordinates of elements of $GL(N, C)$. These representations are completely determined by their characters [**4**, V and VI, § 2], and by restriction it suffices to determine if $\gamma(\zeta) \circ \gamma(\xi) = \gamma(\rho)$ for $(G_C(\pi), \{f_i\})$, $\pi$ a cyclic group and $f_i$ the symmetric power over $C$, which is easily done since $G_C(\pi)$ is generated by one dimensional representations. Alternatively, since $(G_C(GL(N, C), \{f_i\})$ with the usual exterior power is a special $\lambda$-ring [**10**], the splitting principle [**2**] may be used to answer the question.

For any $(R, \{f_i\}) \in \text{ob}\ \mathfrak{D}$ and any $x, y \in R$, a group homomorphism $e_{x,y}: \Gamma \otimes \Gamma \to R$ is defined by

$$e_{x,y}(\zeta \otimes \xi) = \gamma(\zeta)(x)\gamma(\xi)(y).$$

*Definition* 2.3. $(R, \{f_i\})$ had the addition property if $\gamma([n])(x + y) = e_{x,y}\Delta([n])$ for all $x, y \in R$, i.e.

$$f_n(x + y) = \sum_{i=0}^{n} f_i(x)f_{n-i}(y).$$

$\nabla : \Gamma \to \Gamma \otimes \Gamma$ is the function given by the composite of the function $G_Z(S_n) \to G_Z(S_n \times S_n)$ induced from the inclusion of $S_n$ in $S_n \times S_n$ by $\sigma \mapsto (\sigma, \sigma)$ and

$$h^{-1}: G_Z(S_n \times S_n) \to G_Z(S_n) \otimes G_Z(S_n).$$

*Definition* 2.4. $(R, \{f_i\})$ has the multiplication property if $\gamma([n])(xy) = e_{x,y}\nabla([n])$.

The addition formula corresponds to the formula for $\lambda_i(x + y)$ in the definition of $\lambda$-rings, and the multiplication property corresponds to the formula for $\lambda_i(xy)$ in the theory of special $\lambda$-rings [**3**].

THEOREM 2.5. *Let* $\xi \in \Gamma$. *If* $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$, *then* $\gamma(\xi)(x + y) = \gamma(\xi)(x) + \gamma(\xi)(y)$ *for any* $(R, \{f_i\})$ *that has the addition property. If* $\nabla(\xi) = \xi \otimes \xi$, *then* $\gamma(\xi)(xy) = \gamma(\xi)(x)\gamma(\xi)(y)$ *for any* $(R, \{f_i\})$ *that has the multiplication property.*

*Proof.* Suppose $(R, \{f_i\})$ has the addition property. Since $\Gamma$ is a bialgebra under $\Delta$ it is easy to show that $\gamma(\xi)(x + y) = e_{x,y}\Delta(\xi)$ for any $\xi$ which is a monomial in the $[n]$'s. These monomials are a $Z$-basis of $\Gamma$ so $\gamma(\xi)(x + y) = e_{x,y}\Delta(\xi)$ for all $\xi \in \Gamma$. If $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$, then

$$\gamma(\xi)(x + y) = e_{x,y}\Delta(\xi) = e_{x,y}(\xi \otimes 1 + 1 \otimes \xi) = \gamma(\xi)(x) + \gamma(\xi)(y).$$

Suppose $(R, \{f_i\})$ has the multiplication property. It is easily verified that $\gamma(\xi)(xy) = e_{x,y}\nabla(\xi)$ for any $\xi$ which is a monomial in the $[n]$'s using the fact that the representation of a group induced from a subgroup can be constructed by first inducing to any intermediate subgroup and then to the group. As above $\gamma(\xi)(xy) = e_{x,y}\nabla(\xi)$ for all $\xi \in \Gamma$; and, if $\nabla(\xi) = \xi \otimes \xi$,

$$\gamma(\xi)(xy) = e_{x,y}(\xi \otimes \xi) = \gamma(\xi)(x)\gamma(\xi)(y).$$

It is easy to determine if $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$ or $\nabla(\xi) = \xi \otimes \xi$ using characters as is shown by the example in the next section.

## 3.

*Definition* 3.1. $\psi_n = \sum_{k=0}^{n-1}(-1)^k\chi_k$, where $\chi_k$ is the image in $\Gamma$ of the irreducible rational representation of $S_n$ which is given by the partition of $n$, $\{n - k, 1, \ldots, 1\}$. For $(R, \{f_i\}) \in \text{ob}\,\mathfrak{D}$, $\gamma(\psi_n)$ is the $n$th Adams operation.

$$\psi_n = \sum_{k=0}^{n-1} (-1)^k(n - k)[n - k][\tilde{k}]$$

as a consequence of the theorem on representing the irreducible representations in terms of $[n]$ quoted in the proof of Theorem 1.2.

$$\sum_{i=1}^{m} t_i^{\,n} = \sum_{k=0}^{n-1} (-1)^k (n-k) h_{n-k} e_k$$

where $h_{n-k}$ is the homogeneous symmetric polynomial of degree $n - k$ in the $m$ variables, $t_i$, and $e_k$ is the elementary symmetric polynomial. Substitution of $[n - k]$ for $h_{n-k}$ and $[\tilde{k}]$ for $e_k$ shows that Definition 3.1 corresponds to the original definition of Adams.

For $\xi \in \Gamma_n$, $c(\xi)$ denotes the class function of $S_n$ constructed from $\xi$.

PROPOSITION 3.2. $c(\psi_n)(\sigma) = n$ if $\sigma$ *is a cycle of length* $n$. $c(\psi_n)(\sigma) = 0$ *for all other* $\sigma$ *in* $S_n$.

*Proof.* Let $f$ be the class function on $S_n$ which is $n$ on cycles of length $n$ and zero elsewhere. In order to show that $f = c(\psi_n)$ it suffices to show that $(f, c(\chi_k)) = (-1)^k$ and $(f, c(\chi)) = 0$ for all other images, $\chi$, of irreducible representation of $S_n$. For all $\chi$, $(f, c(\chi)) = c(\chi)(\sigma)$, $\sigma$ an $n$-cycle. $c(\chi_k)(\sigma) = (-1)^k$ and $c(\chi)(\sigma) = 0$ if $\chi = \chi_k$ [**9**, Lemma 4.11].

THEOREM 3.3. *Let* $(R, \{f_i\}) \in \mathfrak{D}$.

(a) *If* $(R, \{f_i\})$ *has the composition property*, $\gamma(\psi_m) \circ \gamma(\psi_n) = \gamma(\psi_{mn})$.

(b) *If* $(R, \{f_i\})$ *has the addition property*, $\gamma(\psi_n)(x + y) = \gamma(\psi_n)(x) + \gamma(\psi_n)(y)$.

(c) *If* $(R, \{f_i\})$ *has the multiplication property*, $\gamma(\psi_n)(xy) = \gamma(\psi_n)(x)\gamma(\psi_n)(y)$.

(d) $\gamma(\psi_p) \equiv f_1^{\,p}$ mod $p$ $R$ *for all prime integers* $p$.

*Proof.* Suppose $(R, \{f_i\})$ has the composition property. By Theorem 2.2 it suffices to show $\gamma(\psi_m) \circ \gamma(\psi_n) = \gamma(\psi_{mn})$ for $(G_C(GL(N, C), \{f_i\})$, which is a special case of a result of Swan [**10**].

Suppose $(R, \{f_i\})$ has the addition property. $c(\psi_n)$ restricted to $S_i \times S_{n-i}$, $1 \leqq i < n$, is zero, so $\Delta(\psi_n) = \psi_n \otimes 1 + 1 \otimes \psi_n$. By Theorem 2.5, $\gamma(\psi_n)(x + y) = \gamma(\psi_n)(x) + \gamma(\psi_n)(y)$.

Suppose $(R, \{f_i\})$ has the multiplication property. Let $\sigma = (\sigma_1, \sigma_2) \in S_n \times S_n$. The formula for induced characters shows that $c(h\nabla(\psi_n))(\sigma) = n^2$ if both $\sigma_1$ and $\sigma_2$ are $n$-cycles and $c(h\nabla(\psi_n))(\sigma) = 0$ otherwise. Thus $\nabla(\psi_n) = \psi_n \otimes \psi_n$ and by Theorem 2.5, $\gamma(\psi_n)(xy) = \gamma(\psi_n)(x)\gamma(\psi_n)(y)$.

Let $p$ be a prime. $\theta_p$ is the image in $\Gamma$ of the representation of $S_p$ induced from the trivial representation of a cyclic subgroup of order $p$ of $S_p$. $c(\theta_p)(\sigma) = p - 1$ if $\sigma$ is a $p$-cycle, $c(\theta_p)(e) = (p - 1)!$, and $c(\theta_p)(\sigma) = 0$ if $\sigma$ is any other element of $S_p$. A character computation shows that $(p - 1)\psi_p = p\theta_p - [1]^p$ in $\Gamma$. Thus

$$f_1^{\,p} = \gamma([1]^p) \equiv \gamma(\psi_p) \bmod pR.$$

The verification that $\gamma(\psi_m) \circ \gamma(\psi_n) = \gamma(\psi_{mn})$ in the case $G_C(GL(N, C))$ can be done without reference to the work of Swan. By the remark at the end of

Theorem 2.2 it suffices to consider the case $G_C(\pi)$, $\pi$ a cyclic group. Using the formula

$$\psi_n = \sum_{k=0}^{n-1} (-1)^k (n-k)[n-k][\tilde{k}]$$

and the fact that $\gamma([\tilde{k}])$ is the $k$th exterior power, it is easy to see that for the image, $x$, of a one-dimensional representations of $\pi$, $\gamma(\psi_n)(x) = x^n$. Since the images of the one-dimensional representations span $G_C(\pi)$ and the hypothesis of part $b$ of Theorem 3.3 is valid, $\gamma(\psi_m)\gamma(\psi_n) = \gamma(\psi_{mn})$ for $G_C(\pi)$.

## References


1. M. F. Atiyah, *Power operations in k-theory*, Quart. J. Math. Oxford Ser. *17* (1966), 165–93.
2. M. F. Atiyah and D. O. Tall, *Group representations, λ-rings and the J-homomorphism*, Topology *8* (1969), 253–97.
3. P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des Intersections et Théorème de Riemann-Roch* (SGA6), Lecture Notes in Math. *225* (Springer, Berlin, 1971).
4. H. Boerner, *Representations of groups* (North-Holland Publishing Co., Amsterdam, 1963).
5. P. Cartier, *Groupes formels associés aux anneaux de Witt généralisés*, C. R. Acad. Sci. Paris, Sér. A-B *265* (1967), 49–42.
6. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, (Interscience Publ., New York, 1962).
7. A. Heller and I. Reiner, *Grothendieck groups of orders in semi-simple algebras*, Trans. Amer. Math. Soc. *112* (1964), 344–55.
8. A. Kerber, *Representations of permutation groups. I*, Lecture Notes in Mathematics *240* (Springer, Berlin, 1971).
9. G. de B. Robinson, *Representation theory of the symmetric group* (University of Toronto Press, Toronto 1961).
10. R. Swan, *A splitting principle in algebraic K-theory*, Representation theory of finite groups and related topics, Proc. Sym. in Pure Math., *XXI* (1971), 155–160.


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