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ON THE INTEGRAL COHOMOLOGY OF THE SEVEN-CONNECTIVE COVER OF BO

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Let BO, BSO and BSpin be the classifying spaces for the infinite orthogonal, infinite special orthogonal and infinite spinor groups respectively. It is well known that their integral cohomology rings have torsion only of order 2. In this paper we present an elementary proof that for the 7-connective cover of BO, $BO\langle 8\rangle$, the integral cohomology ring $H^*(BO\langle 8\rangle;\mathbb{Z})$ too has torsion only of order 2. The method follows that of Borel and Hirzebruch and a result of Wu concerning the Steenrod reduced mod p operation for an odd prime p on the Pontrjagin classes.

1. INTRODUCTION

Recently I wished to know whether the integral cohomology of $BO\langle 8\rangle$, the 7-connective cover of BO, the classifying space for the infinite orthogonal group, has any element of order $\geqslant 4$. The purpose of this note is to answer this in the negative, more specifically, I prove

THEOREM 1.1. *The torsion elements of* $H^*(BO\langle 8\rangle;\mathbb{Z})$ *are of order* 2.

Recall that $BO\langle 8\rangle$ fibres over BSpin with fibre $K(\mathbb{Z},3)$, the Eilenberg-MacLane space of type $(\mathbb{Z},3)$. To prove Theorem 1.1 we need the structure of $H^*(BO\langle 8\rangle;\mathbb{Z}_2)$.

2. COHOMOLOGY OF $BO\langle 8\rangle$

In [4] I defined a system of generators for $H^*(BSO;\mathbb{Z}_2)$ as follows.

$$H^*(BSO;\mathbb{Z}_2)\cong \mathbb{Z}_2[v_i \mid i\geqslant 2]$$

where

$$v_i=\begin{cases} Sq^{2^r}\dots Sq^{2^2}Sq^2w_4 \text{ if } i=2^{r+1}+2,\ r\geqslant 0,\\ Sq^{2^j(2^{t+1}+1)}\dots Sq^{2^{t+1}+1}Sq^{2^t}\dots Sq^2w_4 \text{ if } i=2^{t+j+2}+2^{j+1}+1,\ t\geqslant 0,\ j\geqslant 0,\\ Sq^{2^j(2+1)}Sq^{2^{j-1}(2+1)}\dots Sq^{2+1}w_4 \text{ if } i=2^{j+2}+2^{j+1}+1,\ j\geqslant 0,\\ Sq^{2^{r-1}}\dots Sq^2Sq^1w_2 \text{ if } i=2^4+1,\ r\geqslant 0,\\ w_i \text{ otherwise.}\end{cases}$$

Then a Leray-Serre spectral sequence argument for the fibration $BO\langle 8\rangle\to$ BSpin gives:

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THEOREM 2.1. *$H^*(BO\langle 8\rangle;\mathbb{Z}_2)\cong \mathbb{Z}_2[v_i;\ \alpha(i-1)>2]$, where $\alpha(j)=$ number of 1's in the dyadic expansion of j.*

According to [4] we have

THEOREM 2.2.

$$\begin{aligned}
Sq^1 v_{2^j+2} &= v_{2^j+2+1} \text{ if } j\geqslant 2;\\
Sq^1 v_{2^j+2^k+1} &= \left(v_{2^{j-1}+2^{k-1}+1}\right)^2 \text{ for } j\geqslant k>1;\\
Sq^1 v_{2^j+2+1} &= 0,\ j\geqslant 1;\\
Sq^1 v_4 &= v_5+v_2v_3;\\
Sq^1 v_2 &= v_3.
\end{aligned}$$

3. THE SQ^1-COHOMOLOGY OF $H^*(BO\langle 8\rangle;\mathbb{Z}_2)$

Let k be a field. Let $M=\sum_{k=0}^{\infty} M_i$ be a graded vector-space of finite type over a field k, that is the dimension of M_i is finite for each i. Recall that the Poincaré Series of M, $P_k(M,t)$, is defined by

$$P_k(M,t)=\sum_{i\geqslant 0}\dim(M_i)t^i.$$

When k is the field Z_p, where p is a prime, we write $P_p(M,t)$ for $P_k(M,t)$ and for the rational field we write $P_0(M,t)$ for $P_{\mathbb{Q}}(M,t)$.

By Thomas [5] we have

THEOREM, 3.1.

$$H^*(\mathrm{BSpin};\mathbb{Q})\cong \mathbb{Q}[Q_1,\ldots,Q_i,\ldots]$$

where Q_i are generators in dimension $4i$.

Since the fibre of $BO\langle 8\rangle \to \mathrm{BSpin}$ is $K(\mathbb{Z},3)$ and since $H^*(K(\mathbb{Z},3);\mathbb{Q})\cong H^*(S^3;\mathbb{Q})$, a Leray-Serre spectral squence argument for rational cohomology gives us

THEOREM 3.2.

$$H^*(BO\langle 8\rangle;\mathbb{Q})\cong \mathbb{Q}[Q_2,\ldots,Q_j,\ldots].$$

Hence

COROLLARY 3.3.

$$P_0(H^*(BO\langle 8\rangle;\mathbb{Q}),t)=\prod_{j\geqslant 2}\left(1-t^{4j}\right)^{-1}=U(t).$$

In order to commute the Sq^1 cohomology of $H^*(BO\langle 8\rangle;\mathbb{Z}_2)$ we shall need the following

THEOREM 3.4. (Borel-Hirzebruch [1])

$$P_2(H(H^*(BSO;\mathbb{Z}_2),Sq^1),t) = P_0(H^*(BSO;\mathbb{Q}),t) = \prod_{j\geqslant 1}\left(1-t^{4j}\right)^{-1} = Q(t).$$

Now we shall rename the v_i for $\alpha(i-1)\leqslant 2$. Define R_0 to be v_2. For integers $i\geqslant j$ define $R(i,j)$ to be $v_{2^i+2^j+1}$. For Example, $R(0,0)=v_3$, $R(1,0)=v_4$, $R(1,1)=v_5$ and $R(i,i)=v_{2^{i+1}+1}i\geqslant 0$. As in [4] we define $L(k,j)$, $k\geqslant j\geqslant 0$ to be the polynomial subalgebra over $\mathbb{Z}_2$ generated by $R(r,s)$, $k>r\geqslant s\geqslant 0$ or $k=r$ and $j\geqslant s$ together with R_0. Let the ideal generated by $L(k,j)$ be denoted by $\overline{L}(k,j)$.

Define $M(k,j)$, $k\geqslant j\geqslant 0$, by

$$M(k,j) = H^*(BSO;\mathbb{Z}_2)/\overline{L}(k,j).$$

Define $M(0)=H^*(BSO;\mathbb{Z}_2)/(v_2)$. By Theorem 2.2 $\mathrm{Sq}^1(L(k,j))\subseteq L(k,j)$ for $k\geqslant j\geqslant 1$. Since $\mathrm{Sq}^1\colon H^*(BSO;\mathbb{Z}_2)\to H^*(BSO;\mathbb{Z}_2)$ is a derivation and $\mathrm{Sq}^1\mathrm{Sq}^1=0$, the cohomology $H(H^*(BSO;\mathbb{Z}_2),\mathrm{Sq}^1) = \mathrm{Ker}\,\mathrm{Sq}^1/\mathrm{Im}\,\mathrm{Sq}^1$ is a graded vector space over $\mathbb{Z}_2$. By abuse of notation we shall write d for

$$\mathrm{Sq}^1|_{L(k,j)}\ k\geqslant j\geqslant 1 \text{ or } \mathrm{Sq}^1\colon M(k,j)\to M(k,j).$$

Then $H^*(BO\langle 8\rangle;\mathbb{Z}_2)=\varinjlim M(k,j)$. Now

$$\left(H^*(BSO;\mathbb{Z}_2),\ \mathrm{Sq}^1\right) = (\mathbb{Z}_2[v_2,v_3,v_4,v_5],d)\otimes(\dot{M}(1,1),d).$$

Thus $P_2\left(H^*(BSO;\mathbb{Z}_2),\mathrm{Sq}^1\right) = \left(1-t^4\right)^{-1}\left(1-t^8\right)^{-1}P_2(M(1,1),d)$ where we denote $P_2\left(H(H^*(BSO;\mathbb{Z}_2),\ \mathrm{Sq}^1),\ t\right)$, $P_2(H(M(1,1),\ d),\ t)$ by $P_2\left(H^*(BSO;\mathbb{Z}_2),\ \mathrm{Sq}^1\right)$ and $P_2(M(1,1),\ d)$, respectively.

Now for (j,k), $j\geqslant k>1$, it follows from Theorem 2.2 that we have the following exact sequence of chain complexes

$$0\to \sum^{2^j+2^k+1} M(j,k-1)\xrightarrow{\cdot R(j,k)} M(j,k-1)\to M(j,k)\to 0 \tag{3.5}$$

where $\left(\sum^{2^j+2^k+1} M(j,k-1)\right)_r = (M(j,k-1))_{r-2^j-2^k-1}$ and $\cdot R(j,k)$ means multiplication by $R(j,k)$. This short exact sequence induces the long exact sequence

$$\cdots\to H^i\left(\sum^{2^j+2^k+1} M(j,k-1)\right)\to H^i(M(j,k-1))\to H^i(M(j,k)) \to H^{i+1}\left(\sum^{2^j+2^k+1} M(j,k-1)\right)\to\cdots$$

which is equivalent to

$$(3.6)\qquad \ldots \to H^{i-2^j-2^k-1}(M(j,k-1)) \to H^i(M(j,k-1)) \to H^i(M(j,k)) \to H^{i-2^j-2^k}(M(j,k-1)) \to \ldots.$$

3.7 CLAIM: $H^{2r+1}(M(j,k-1)) = 0$ for $j \geqslant k$, $k \geqslant 2$. Assuming this claim, we then have from (3.6) the following short exact sequence

$$0 \to H^{2i}(M(j,k-1)) \to H^{2i}(M(j,k)) \to H^{2i-2^j-2^k}(M(j,k-1)) \to 0.$$

Therefore $H^{2i}(M(j,k)) = H^{2i}(M(j,k-1)) \oplus H^{2i-2^j-2^k}(M(j,k-1))$. Thus we have

$$(3.7)_{(j,k)}\qquad P_2(M(j,k),\,d) = \left(1 + t^{2^j+2^k}\right)P_2(M(j,k-1),\,d)$$

for $k > 1$. Now for $k \geqslant 2$,

$$(3.8)_{(k,1)}\qquad (M(k-1,k-1),\,d) \cong (M(k,1),\,d) \otimes (\mathbb{Z}_2[R(k,0),\,R(k,1)],\,d).$$

Since $d(R(k,0)) = R(k,1)$, for $k \geqslant 2$

$$P_2(M(k-1,\,k-1),\,d) = \left(1 - t^{2(2^k+2)}\right)^{-1} P_2(M(k,1),\,d).$$

Thus for $k \geqslant 2$

$$(3.9)_k\qquad P_2(M(k,1),\,d) = \left(1 - t^{2^{k+1}+2^2}\right)P_2(M(k-1,\,k-1),\,d).$$

3.10 PROOF OF CLAIM 3.7:

$$\begin{aligned} P_2(M(1,1),\,d) &= P_2\left(H^*(BSO;\mathbb{Z}_2),\,\mathrm{Sq}^1\right)\left(1-t^4\right)\left(1-t^8\right) \\ &= Q(t)\cdot\left(1-t^4\right)\left(1-t^8\right). \end{aligned}$$

Thus $H^{2i+1}(M(1,1),\,d) = 0$. $H^{2i+1}(M(j,k-1)) = 0$ for $j > k$, $k \geqslant 2$ is proved by induction on $(j,k-1)$. It $k = 2$, then it follows from $(3.9)_j$ and induction hypothesis. If $k > 2$ then if follows from the exact sequence (3.6) and the induction hypothesis.

Now from $(3.7)_{(j,k)}$ we have

$$P_2(M(j,k),\,d) = \left(1 + t^{2^j+2^k}\right)\left(1 + t^{2^j+2^{k-1}}\right)\ldots\left(1 + t^{2^j+2^2}\right)P_2(M(j,1),\,d).$$

This together with $(3.9)_j$ gives us:

$$\begin{aligned} P_2(M(j,k),\,d) = & \prod_{k\geqslant r\geqslant 2}\left(1+t^{2^j+2^r}\right)\prod_{j-1\geqslant s\geqslant r\geqslant 2}\left(1+t^{2^s+2^r}\right) \\ & \prod_{j\geqslant r\geqslant 2}\left(1-t^{2^{r+1}+2^2}\right)P_2(M(1,1),d) \\ & \prod_{\substack{j-1\geqslant s\geqslant r\geqslant 2\\ s=j,k\geqslant r\geqslant 2}}\left(1+t^{2^s+2^r}\right)\prod_{j\geqslant r\geqslant 2}\left(1-t^{2^{r+1}+2^2}\right)Q(t)\cdot(1-t^4)(1-t^8) \\ = & \ (1-t^4)Q(t)\prod_{\substack{j-1\geqslant s>r\geqslant 2\\ s=j,k\geqslant r\geqslant 2}}\left(1+t^{2^s+2^r}\right)\prod_{j\geqslant r\geqslant 2}\left(1-t^{2^{r+1}+2^2}\right) \\ & \cdot\left(1-t^{2^{j+1}}\right); \end{aligned}$$

$$P_2(M(j,j),\,d) = U(t)\prod_{\substack{j-1\geqslant s\geqslant r\geqslant 2\\ s=j,j>r\geqslant 2}}\left(1+t^{2^s+2^r}\right)\prod_{j\geqslant r\geqslant 2}\left(1-t^{2^{r+1}+2^2}\right)\cdot\left(1-t^{2^{j+2}}\right).$$

Since

$$\prod_{s>r\geqslant 2}\left(1+t^{2^s+2^r}\right)\prod_{r\geqslant 2}\left(1-t^{2^{r+1}+2^2}\right)=1$$

we see that

$$P_2\left(\varinjlim M(k,j),\,d\right) = (1-t^4)Q(t) = \prod_{j\geqslant 2}\left(1-t^{4j}\right)^{-1}.$$

Therefore we have

THEOREM 3.11.

$$P_2\big(H^*(BO\langle 8\rangle;\mathbb{Z}_2),Sq^1\big) = P_0(H^*(BO\langle 8\rangle;\mathbb{Q})).$$

From Theorem 3.11 and [**1**] we have

THEOREM 3.12. *The* 2*-primary component of* $H^*(BO\langle 8\rangle;\mathbb{Z})$ *has order* 2 *only.*

4. THE MOD p COHOMOLOGY OF $BO\langle 8\rangle$ FOR ODD PRIME p

Consider the mod p Leray-Serre cohomology spectral sequence for $K(\mathbb{Z},3)\longrightarrow BO\langle 8\rangle\longrightarrow$ BSpin

By Cartan [**3**] $H^*(K(\mathbb{Z},3),\,\mathbb{Z}_p)$ is an anti-commutative algebra generated by

$$\{\mathcal{P}^{p^k}\mathcal{P}^{p^{k-1}}\dots\mathcal{P}^p\mathcal{P}^1\iota_3,\,\beta\mathcal{P}^{p^k}\mathcal{P}^{p^{k-1}}\dots\mathcal{P}^1\iota_3 \mid k\geqslant 0\}$$

where $\mathcal{P}^{p^i}$ are the Steenrod reduced mod p operations and β is the Bockstein operation associated with the exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ and ι_3 is the fundamental class of $K(\mathbb{Z}, 3)$.

Now by Thomas [5], $H^*(\mathrm{BSpin}; \mathbb{Z}_p) \cong \mathbb{Z}_p[P_1, P_2, \ldots]$ where $P_i \in H^{4i}(\mathrm{BSpin}; \mathbb{Z}_p)$ is the mod p Pontrjagin class.

Let $\{E_r, d_r\}$ be the mod p Leray-Serre spectral squence for $BO\langle 8\rangle \to \mathrm{BSpin}$. Then $E_2 \cong H^*(\mathrm{BSpin}; \mathbb{Z}_p) \otimes H^*(K(3, \mathbb{Z}), \mathbb{Z}_p)$. Then:

$$d_4\iota_3 = \frac{1}{2}P_1;$$
$$d_{2(p^{k+1}+1)}\left(\mathcal{P}^{p^k} \ldots \mathcal{P}^{p^{k-1}} \mathcal{P}^p \mathcal{P}^l \iota_3\right) = \frac{1}{2}\mathcal{P}^{p^k} \ldots \mathcal{P}^1 P_1;$$
$$d_{2(p^{k+1}+1)+1}\left(\beta\mathcal{P}^{p^k} \ldots \mathcal{P}^1 \iota_3\right) = 0.$$

By Wu [6] $\mathcal{P}^{p^k} \ldots \mathcal{P}^p \mathcal{P}^1 P_1 = P_{(p^{k+1}+1)/2}$ modulo decomposables. We shall now define a system of generators for $H^*(\mathrm{BSpin}; \mathbb{Z}_p)$. Let

$$v_0 = P_1,$$
$$v_k = \mathcal{P}^{p^{k-1}} \ldots \mathcal{P}^P \mathcal{P}^1 P_1 \text{ for } k \geqslant 1.$$

Then $\dim v_k = 2(p^k + 1)$. Thus $\{v_0, v_1, v_2, \ldots\}$ together with P_i for $i \neq \frac{p^k+1}{2} k \geqslant 0$ generates $H^*(\mathrm{BSpin}; \mathbb{Z}_p)$.

By the usual spectral sequence argument we have:

THEOREM 4.1. *$H^*(BO\langle 8\rangle; \mathbb{Z}_p) \cong H^*(\mathrm{BSpin}; \mathbb{Z}_p)/L \otimes \Lambda$ where Λ is the subalgebra generated by $\{\beta\mathcal{P}^{p^k} \ldots \mathcal{P}^1\iota_3,\ k \geqslant 0\}$ and L is the ideal generated by $\{v_0, v_1, \ldots, v_k, \ldots\}$.*

Let $R_k = H^*(\mathrm{BSpin}; \mathbb{Z}_p)/(v_0, v_1, \ldots v_k)$. For $k \geqslant 0$ we have the following exact sequence

$$0 \longrightarrow \sum^{2(p^{k+1}+1)} R_k \xrightarrow{\cdot v_{k+1}} R_k \longrightarrow R_{k+1} \longrightarrow 0.$$

Thus $P_p(R_{k+1}, t) + t^{2(p^{k+1}+1)} P_p(R_k, t) = P_p(R_k, t)$. That is $P_p(R_{k+1}, t) = \left(1 - t^{2(p^{k+1}+1)}\right) P_p(R_k, t)$. Hence we have

THEOREM 4.2.

$$\begin{aligned} P_p(R_k, t) &= \prod_{j\geqslant 1}^{k} \left(1 - t^{2(p^j+1)}\right) P_0(R_0, t) \\ &= \prod_{j\geqslant 1}^{k} \left(1 - t^{2(p^j+1)}\right) \prod_{j\geqslant 2} \left(1 - t^{4j}\right)^{-1}. \end{aligned}$$

It follows from Theorem 4.1 and Theorem 4.2 that

$$P_p(H^*(\mathrm{BSpin};\mathbb{Z}_p)/L,t) = \prod_{j\geqslant 1}\left(1-t^{2(p^j+1)}\right)\prod_{j\geqslant 2}\left(1-t^{4j}\right)^{-1}.$$

Now $P_p(\Lambda,t) = \prod_{k\geqslant 1}\left(1-t^{2(p^k+1)}\right)^{-1}$. Therefore

$$\begin{aligned}P_p(H^*(BO\langle 8\rangle;\mathbb{Z}_p),\, t) &= P_p(H^*(\mathrm{BSpin};\mathbb{Z}_p)/L,t)P_p(\Lambda,t)\\ &= \prod_{j\geqslant 2}\left(1-t^{4j}\right)^{-1} = P_0(H^*(BO\langle 8\rangle;\mathbb{Q}),t).\end{aligned}$$

Thus we prove:

Theorem 4.3. *$H^*(BO\langle 8\rangle;\mathbb{Z})$ has no p-torsion for odd prime p.*

Theorem 1.1 now follows from Theorem 3.12 and Theorem 4.3.

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