CORRESPONDENCE.

The Hamiltonian Revival.

To the Editor of the *Mathematical Gazette*.

Sir,—In his fascinating article " The Hamiltonian Revival ", in your issue of July last, Professor E. T. Whittaker turned aside to deal an uncalled-for stroke against two great men and part of their work in the following passage :

> " Then those who were in the outer circles of Hamilton's influence—*e.g.* Willard Gibbs in America and Heaviside in England—wasted their energies in devising bastard derivatives of the quaternion calculus—dyadics and vector-analysis—which reproduced its most regrettable feature, namely the limitations imposed by its close association with the geometry of ordinary space, and which represented no advance, but rather a retrogression, from the point of view of general theory."

Doubtless many others besides myself read these words with surprise and regret, because no one can be unaware of the growing appreciation and use of vectors (and to a less extent of dyadics), both as a means of expression and as a tool in mathematical technique.

The word *bastard* may be used in either a technical sense or merely as a term of abuse, but its user may reasonably be asked to justify its application, and also the charge of *wasted* energies. Professor Whittaker indeed indicates what he regards as the " most regrettable feature " of vector analysis and dyadics, and I hope to comment on that criticism ; but before doing so it would be well to know whether this is the sole justification of his condemnatory words, or if not, what are the other counts in his indictment.

E. A. Milne.

To the Editor of the *Mathematical Gazette*.

Sir,—May I take the occasion presented by Professor Milne's letter to explain more fully what was in my mind when I wrote the sentence he objects to.

First, let me recall a definition : Any set of objects of thought may be called *generalised numbers*, provided we can define what is meant by the " equality " of any two of them, and can also define the operations of " addition " and " multiplication " with respect to them, and provided also that the set of objects form a " group " with respect to these operations, *i.e.* the effect of combining two of the objects by means of one of the operations is to produce an object which also belongs to the set. The operations should satisfy certain conditions which need not be given here.

Thus, matrices of any order n may be regarded as generalised numbers ; for we can define the equality, addition, and multiplication of matrices ; and the result of adding or multiplying two matrices is to produce another matrix.

Another example of generalised numbers is quaternions ; for we can define the equality, addition, and multiplication of quaternions ; and the result of adding or multiplying two quaternions is to produce another quaternion.

Whenever we have a set of generalised numbers, we can work out a calculus based on them.

Now vectors are *not* generalised numbers ; for the product of two vectors is not a vector. It is true that in vector-analysis we meet with a quantity called a vector-product of two vectors (namely, a vector at right angles to their plane and proportional to the area of the triangle formed by them), but this has not the properties of a true product, since if $\overrightarrow{\alpha\beta}$ denotes the vector-product of $\vec{\alpha}$ and $\vec{\beta}$, then the equation $\overrightarrow{\alpha\beta}=\vec{\gamma}$ does not fix $\vec{\alpha}$ when $\vec{\beta}$ and $\vec{\gamma}$ are known.

The fact that vectors are not generalised numbers makes it impossible to create a new calculus in which vectors shall be the only objects considered. As a matter of fact, the so-called vector-analysis is not a new calculus, but merely a syncopated form of ordinary Cartesian analysis.

Hamilton saw all this, and realised that a true calculus of vectors could exist only within a framework which would have to be provided by a set of generalised numbers ; and what he did in discovering quaternions was, precisely, to introduce this framework.

His idea was a very simple one, namely, that the quotient of two vectors was to be regarded as an object of a new kind, to be called a *quaternion*. He then showed that quaternions so defined were generalised numbers, and found that quaternion addition and multiplication satisfied the associative and distributive laws, and the commutative law of addition, but not the commutative law of multiplication—an epoch-making discovery. In terms of quaternions, all the properties of vectors, of rotations in space, etc., can be very simply represented : thus, vectors may be regarded as " quadrantal " quaternions, *i.e.* quaternions whose " angle " is a right angle ; and as to rotations in space, we have the simple formula, that if $\vec{a}$ is a vector, then the vector which is obtained from $\vec{a}$ by rotating it about any axis in space, through any angle, is represented by $q\vec{a}q^{-1}$, where q is a quaternion which specifies the rotation.

Both Gibbs and Heaviside spent a good deal of time and energy in attacking quaternions, notably in the controversy of 1892–93 ; and I think I was justified in describing this energy as " wasted ". Their other works show that they were indeed great men, but they did not rise to the idea of treating vectors by means of a calculus of generalised numbers of a new type : all they wanted was some way of writing ordinary Cartesian analysis without putting the axes of coordinates in evidence. Up to a point, this aim is achieved by " vector-analysis " : but the scope of " vector-analysis " is, for the reasons given above, very limited. When I called it a "bastard" derivative of quaternions, I meant that it was the progeny of quaternions, but not in the legitimate line of succession and evolution. I am, etc., E. T. WHITTAKER.

THE RESEARCH METHOD IN TEACHING MATHEMATICS.

To the Editor of the *Mathematical Gazette.*

SIR,—Miss Knowles reminds us of one of the distinguishing advantages of mathematics as a school subject, namely that it is something for a boy to do, rather than something he must learn. This advantage is shared by carpentry, music and drawing, by English if it is taught the right way, and to a more limited extent by foreign languages.

The Spens Committee have missed this vital point, for they seek to reduce the time allowed for the subject, at the same time increasing the field to be covered. They recommend a more descriptive treatment. In other words they would have it become the kind of subject that can be taught by dictating notes, a "crammable" subject, a thing to learn, not a thing to do.

It is significant that "general science" is considered a good choice for a not very able boy who wishes to pass his School Certificate examination. Yours faithfully, E. H. LOCKWOOD.

Felsted School, Ross-on-Wye.

1357. He (the Archbishop's grandfather) goes to Cambridge for study in 1780, but writes in his diary: "The libraries of Cambridge not well supplied with books: no studies in any credit there but mathematical ones." (p. 10.)

1358. She (the Archbishop's mother) taught arithmetic, with very little knowledge of arithmetic herself, by steady repetition. She had a key to the sums in the arithmetic book, giving the answers. If a sum was brought to her and the answer was wrong, she drew her pencil through it and made no further remark. It had to be done again till it was done right. The sum of today was repeated tomorrow, and so on, until perfect accuracy was obtained. —(From a Memorandum by the Archbishop's sister.) (p. 17.)

1359. Euclid was the same. She (his mother) did not understand a word. He began to do so as he advanced in the subject, and could substitute one expression for another, or change the order of letters. She interposed and corrected him. He would reply impatiently "It was all the same." "Say it", she ordered, "precisely as it is here," touching the book.—(From Memorandum by the Archbishop's sister.) (p. 18.)

1360. . . . The Archbishop told him that on his ninth birthday, to the best of his recollection, after he had gone to bed, his mother happened incidentally to mention to his father that she had carried out his orders to teach the boy Euclid, and that he knew his Euclid. "What! all of it? Can he say any proposition?" "Yes, he knows it all." The father, naturally disbelieving this, had the child woke up, when he repeated, sitting in bed, a long proposition." (p. 18.)

Gleanings 1357–1360 from *Memoirs of Archbishop Frederick Temple,* I. [Per Mr. A. F. Mackenzie.]