Forum of Mathematics, Sigma (2023), **Vol. 11**:e53 1–21
doi:10.1017/fms.2023.51

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RESEARCH ARTICLE

On Bloch's map for torsion cycles over non-closed fields

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Received: 9 December 2022; **Revised:** 30 March 2023; **Accepted:** 18 May 2023

2020 Mathematics Subject Classification: *Primary* – 14C15, 14C25; *Secondary* – 14D06

Abstract

We generalize Bloch's map on torsion cycles from algebraically closed fields to arbitrary fields. While Bloch's map over algebraically closed fields is injective for zero-cycles and for cycles of codimension at most two, we show that the generalization to arbitrary fields is only injective for cycles of codimension at most two but, in general, not for zero-cycles. Our result implies that Jannsen's cycle class map in integral ℓ-adic continuous étale cohomology is, in general, not injective on torsion zero-cycles over finitely generated fields. This answers a question of Scavia and Suzuki.

1. Introduction

Let X be a smooth variety over a field k and let ℓ be a prime invertible in k. In this paper, we study the ℓ-power torsion subgroup $\mathrm{CH}^i(X)[\ell^\infty]$ of the Chow group $\mathrm{CH}^i(X)$ of codimension-i cycles on X. Following some constructions in [Blo79, Sch21b], we define a cycle map

$$\lambda^i_X : \mathrm{CH}^i(X)[\ell^\infty] \longrightarrow \frac{H^{2i-1}(X_{\text{ét}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{M^{2i-1}(X)}, \tag{1.1}$$

where $M^{2i-1}(X) \subset H^{2i-1}(X_{\text{ét}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is defined as follows: we pick a finitely generated subfield $k_0 \subset k$ such that there is a k_0-variety X_0 with $X = X_0 \times_{k_0} k$ and let

$$M^{2i-1}(X) = \mathrm{im}\left(\varinjlim_{k'/k_0} N^{i-1} H^{2i-1}_{cont}((X_0 \times_{k_0} k')_{\text{ét}}, \mathbb{Q}_\ell(i)) \longrightarrow H^{2i-1}(X_{\text{ét}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \right),$$

where k' runs through all finitely generated subfields of k that contain k_0, and N^* denotes the coniveau filtration (cf. (2.2) below).

If k is algebraically closed and X is projective, then $M^{2i-1}(X) = 0$ by weight reasons (see Lemma 4.1 below) and the above map coincides with Bloch's map [Blo79]; see Lemma 4.5 below. If, moreover, $k = \mathbb{C}$, the map agrees on the subgroup of homologically trivial ℓ-power torsion cycles with Griffiths' Abel–Jacobi map [Gri69]; see [Blo79, Proposition 3.7].

Bloch's map over algebraically closed fields is injective on torsion cycles of codimension ≤ 2; the non-trivial case $i = 2$ is a theorem of Bloch and Merkurjev–Suslin; see [MS83, §18]. Moreover, for

any ground field k, Colliot-Thélène–Sansuc–Soulé showed in [CTSS83, Corollaire 3] that $\mathrm{CH}^2(X)[\ell^\infty]$ is isomorphic to a subquotient of $H^3(X_{\text{ét}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$. The following result generalizes the injectivity theorem of Bloch and Merkurjev–Suslin over algebraically closed fields by making the aforementioned result in [CTSS83] more precise; the argument relies on some results from [Sch21b].

Theorem 1.1. *Let X be a smooth variety over a field k and let ℓ be a prime invertible in k. Then λ^i_X from (1.1) is injective for $i = 1, 2$ and induces the following isomorphisms:*

$$\mathrm{CH}^1(X)[\ell^\infty] \simeq H^1(X_{\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))/M^1(X) \text{ and } \mathrm{CH}^2(X)[\ell^\infty] \simeq N^1H^3(X_{\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/M^3(X).$$

In the body of this paper, we prove a version of the above theorem that works for arbitrary algebraic schemes; see Theorem 4.2 and Corollary 4.4. To this end, one has to replace in the above discussion ordinary cohomology by Borel–Moore cohomology; cf. [Sch21b, §4].

Using some Lefschetz hyperplane argument, Bloch shows that the injectivity of λ^2_X implies the injectivity of $\lambda^{\dim X}_X$ over algebraically closed fields (see [Blo79]), hence Roitman's theorem [Roi80] away from the characteristic. Similarly, $\lambda^{\dim X}_X$ is known to be injective over finite fields; see, for example, [KS83, CTSS83]. However, even over algebraically closed fields, the map λ^i_X is for $2 < i < \dim X$, in general, not injective (e.g., [Schoe00, Tot97, SV05, RS10, Tot16, Sch21c, SS23]).

Roitman's theorem on the injectivity of $\lambda^{\dim X}_X$ over algebraically closed fields turned out to be a very robust statement, with generalizations to non-proper and even singular spaces (e.g., [Lev85, KS02, Gei10]). As mentioned above, there is also a generalization to finite fields [KS83, CTSS83]. In light of the injectivity of λ^2_X over arbitrary fields, it is natural to wonder if Roitman's theorem admits also a generalization to arbitrary fields (i.e., is $\lambda^{\dim X}_X$ always injective?) The main result of this paper answers this question negatively.

Theorem 1.2. *Let ℓ be a prime and let k be a field of characteristic different from ℓ. Then there is a finitely generated field extension K/k and a smooth projective threefold X over K such that λ^3_X from (1.1) is not injective.*

The proof of the above result will rely on an adaptation of Schoen's argument in [Schoe00], together with some results on the integral Hodge and Tate conjecture due to Kollár, Hassett–Tschinkel and Totaro. Taking products with projective spaces, we get the following.

Corollary 1.3. *Let $i \leq n$ be positive integers, let k be a field and let ℓ be a prime invertible in k. Then the map λ^i_X from (1.1) is injective for all smooth projective varieties X of dimension n over all finitely generated field extension of k if and only if $i \leq 2$.*

We will see in Lemma 4.7 below that over any field, the restriction of Jannsen's cycle class map in continuous étale cohomology [Jan88] to ℓ-power torsion cycles,

$$\mathrm{cl}^i_X : \mathrm{CH}^i(X)[\ell^\infty] \longrightarrow H^{2i}_{cont}(X_{\text{ét}}, \mathbb{Z}_\ell(i)), \tag{1.2}$$

factors through λ^i_X. In particular, (1.1) refines Jannsen's cycle class map on torsion cycles. As a consequence of Theorem 1.2, we immediately get the following, which answers [SS23, Question 1.7(a)].

Corollary 1.4. *For any $n \geq 3$ and any prime ℓ, there is a smooth projective n-fold X over a finitely generated field of characteristic different from ℓ such that Jannsen's cycle class map*

$$\mathrm{cl}^n_X : \mathrm{CH}^n(X)[\ell^\infty] \longrightarrow H^{2n}_{cont}(X_{\acute{e}t}, \mathbb{Z}_\ell(n))$$

is not injective on the subgroup of ℓ-power torsion classes.

Jannsen showed that in the case of divisors (i.e., codimension one cycles) his cycle class map for smooth projective varieties over finitely generated fields is injective integrally; see [Jan88, Remark 6.15(a)] and [Sch21b, (P7.2) in Proposition 6.6]. He further conjectured that after tensoring with $\mathbb{Q}$, his cycle class map is injective for cycles of arbitrary codimension over finitely generated fields. Scavia

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and Suzuki exhibited in [SS23] several interesting examples that show that tensoring with $\mathbb{Q}$ is really necessary (i.e., the integral Jannsen conjecture fails in general). The main case left open by [SS23] is that of zero-cycles, that we deal with in this paper. The case of zero-cycles is of particular importance because injectivity of $\lambda_X^{\dim X}$ on ℓ-power torsion zero-cycles for all smooth projective varieties X over finitely generated fields would imply the Rost nilpotence conjecture in characteristic zero by [Dia22] and the arguments in [Sch21b, §9.4] and [Sch22, §B.2] (see also [RS18]).

For the aforementioned application to the Rost nilpotence conjecture, it would be enough to prove injectivity of $\lambda_X^{\dim X}$ for varieties with a k-rational point (essentially because the argument in [Dia22] proceeds via base change to the function field, where a rational point is automatic). The examples in Theorem 1.2 have no rational point. However, a variant of our construction gives (at least for the prime $\ell = 2$) examples with rational points as well, which puts an end to the hope that the Rost nilpotence conjecture may be a consequence of the injectivity of $\lambda^{\dim X}$ on smooth projective varieties with a rational point.

Theorem 1.5. *There is a finitely generated field k of characteristic zero and a smooth projective threefold X over k such that X has a k-rational point and λ_X^3 from (1.1) is not injective for the prime $\ell = 2$.*

The examples in Theorem 1.2 have Kodaira dimension 2, while those in Theorem 1.5 have negative Kodaira dimension, as they are conic bundles over surfaces of positive Kodaira dimension. The question whether $\lambda_X^{\dim X}$ is injective for smooth projective varieties X that are geometrically rationally connected remains open (this question is interesting for the base change $X_{k(X)}$ of any rationally connected variety X over an algebraically closed field k which does not admit an integral decomposition of the diagonal; for examples of the latter, see the survey [Sch21a] and the references therein).

2. Preliminaries

We recall some of the notation and conventions from [Sch21b] that we will use. An algebraic scheme is a separated scheme of finite type over a field. A variety is an integral algebraic scheme. A field is finitely generated if it is finitely generated over its prime field. The n-torsion subgroup of an abelian group G is denoted by $G[n]$; the subgroup of elements annihilated by some power of n is denoted by $G[n^\infty]$. If $\varphi : H \to G$ is a morphism of abelian groups, we denote by slight abuse of notation $G/H := \operatorname{coker}(\varphi)$.

We fix a field k and a prime ℓ invertible in k. For an algebraic scheme X of dimension d_X over k, we denote by $\mathrm{CH}^i(X) := \mathrm{CH}_{d_X - i}(X)$ the Chow group of cycles of dimension $d_X - i$ on X. For $A \in \{\mathbb{Z}/\ell^r, \mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{Q}_\ell/\mathbb{Z}_\ell\}$, we let

$$H^i(X, A(n)) := H^i_{BM}(X, A(n))$$

be twisted Borel–Moore pro-étale cohomology; see [Sch21b, (6.13)-(6.15)] and [Sch21b, Proposition 6.6]. Some of the most important properties of this functor are collected in [Sch21b, Section 4]. If X is smooth and equi-dimensional, then we have canonical identifications

$$H^i(X, \mathbb{Z}/\ell^r(n)) \simeq H^i(X_{\text{ét}}, \mu_{\ell^r}^{\otimes n}), \quad H^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \varinjlim_r H^i(X_{\text{ét}}, \mu_{\ell^r}^{\otimes n}), \tag{2.1}$$

$$H^i(X, \mathbb{Z}_\ell(n)) \simeq H^i_{cont}(X_{\text{ét}}, \mathbb{Z}_\ell(n)) \text{ and } H^i(X, \mathbb{Q}_\ell(n)) \simeq H^i_{cont}(X_{\text{ét}}, \mathbb{Q}_\ell(n)) = H^i_{cont}(X_{\text{ét}}, \mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell;$$

see [Sch21b, Lemma 6.5].

For $j \geq 0$, we denote by $F_j X$ the pro-scheme given by the inverse limit of all open subsets $U \subset X$ with $\dim(X \setminus U) < \dim X - j$. If $U \hookrightarrow X$ is an open immersion with $\dim U = \dim X$, then there are restriction maps $H^i(X, A(n)) \to H^i(U, A(n))$. As in [Sch21b, Section 5], we define

$$H^i(F_j X, A(n)) := \varinjlim_{U \subset X} H^i(U, A(n)),$$

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where U runs through all open subsets of X that make up the pro-scheme F_jX above. The coniveau filtration N^* on $H^i(X, A(n))$ is then given by

$$N^jH^i(X, A(n)) := \ker(H^i(X, A(n)) \to H^i(F_{j-1}X, A(n))); \tag{2.2}$$

see [Sch21b, (5.1)].

For $m \geq j$, there are natural restriction maps $H^i(F_mX, A(n)) \to H^i(F_jX, A(n))$, and we denote the image of this map by $F^mH^i(F_jX, A(n))$; see [Sch21b, Definition 5.3]. For a scheme point $x \in X$, we let $H^i(x, A(n)) := H^i(F_0\overline{\{x\}}, A(n))$, where $\overline{\{x\}} \subset X$ denotes the closure of x; note that $H^0(x, A(0)) = A \cdot [x]$, where $[x] \in H^0(x, A(0))$ denotes the fundamental class of x; cf. [Sch21b, (P3) in Definition 4.2 and Proposition 6.6].

The Gysin sequence induces the following important long exact sequence (see [Sch21b, Lemma 5.8]):

$$\longrightarrow H^i(F_jX, A(n)) \longrightarrow H^i(F_{j-1}X, A(n)) \xrightarrow{\partial} \bigoplus_{x \in X^{(j)}} H^{i+1-2j}(x, A(n-j)) \xrightarrow{\iota_*} H^{i+1}(F_jX, A(n)). \tag{2.3}$$

Since $H^i(x, A(n)) = 0$ for $i < 0$, we deduce $H^i(X, A(n)) = H^i(F_jX, A(n))$ for $j \geq \lceil i/2 \rceil$; see [Sch21b, Corollary 5.10].

Let $\mathrm{CH}^i(X)_{\mathbb{Z}_\ell} := \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. Then there is a cycle class map

$$\mathrm{cl}^i_X : \mathrm{CH}^i(X)_{\mathbb{Z}_\ell} \longrightarrow H^{2i}(X, \mathbb{Z}_\ell(i));$$

see [Sch21b, (7.1)]. This map is induced by the following pushforward map that appears in the above long exact sequence (2.3):

$$\bigoplus_{x \in X^{(i)}} H^0(x, \mathbb{Z}_\ell(0)) = \bigoplus_{x \in X^{(i)}} \mathbb{Z}_\ell[x] \xrightarrow{\iota_*} H^{2i}(F_iX, \mathbb{Z}_\ell(i)) = H^{2i}(X, \mathbb{Z}_\ell(i)).$$

If X is smooth and equi-dimensional, then the above cycle class map agrees with Jannsen's cycle class map in continuous étale cohomology from [Jan88]; see [Sch21b, Lemma 9.1].

There is a natural coniveau filtration N^* on $\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$, given by the condition that a cycle $[z]$ lies in N^j if and only if there is a closed subset $Z \subset X$ of codimension j such that z is rationally equivalent to a homologously trivial cycle on Z; that is,

$$[z] \in \mathrm{im}(\ker(\mathrm{cl}^{i-j}_Z) \longrightarrow \mathrm{CH}^i(X)_{\mathbb{Z}_\ell});$$

see [Sch21b, Definition 7.3]. In view of [Sch21b, Definition 7.2 and Lemma 7.4], we define

$$A^i(X)_{\mathbb{Z}_\ell} := \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}/N^{i-1}\,\mathrm{CH}^i(X)_{\mathbb{Z}_\ell} \quad \text{and} \quad A^i(X)[\ell^\infty] := A^i(X)_{\mathbb{Z}_\ell}[\ell^\infty].$$

If the field k is finitely generated, then $A^i(X)_{\mathbb{Z}_\ell} = \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ by [Sch21b, Lemma 7.5].

Remark 2.1. Let X be an algebraic k-scheme of dimension d_X. Assume that there is a closed embedding $\iota : X \hookrightarrow Y$ into a smooth equi-dimensional algebraic k-scheme Y of dimension d_Y. Then the Borel–Moore cohomology groups $H^i(X, A(n)) = H^i_{BM}(X, A(n))$ above may be identified with ordinary cohomology groups with support as follows:

$$H^i_{BM}(X, A(n)) = H^{i+2c}_X(Y_{\text{proét}}, \widehat{A}(n+c)),$$

where $c = d_Y - d_X$; see also [Sch22, Appendix A]. We explain the above identification in the case where $A = \mathbb{Z}_\ell$; the general case is similar. To this end, let $\pi_X : X \to \mathrm{Spec}\, k$ be the structure map and

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note that by definition in [Sch21b, (6.13)-(6.15)], we have

$$H^i_{BM}(X, \mathbb{Z}_\ell(n)) = \mathrm{R}^{i-2d_X}\,\Gamma(X_{\text{proét}}, \pi_X^! \widehat{\mathbb{Z}}_\ell(n - d_X)).$$

If $\pi_Y : Y \to \operatorname{Spec} k$ denotes the structure morphism, then $\pi_X = \pi_Y \circ \iota$ and so

$$\pi_X^! = (\pi_Y \circ \iota)^! = \iota^! \circ \pi_Y^! = \iota^! \circ \pi_Y^*(d_Y)[2d_Y],$$

where we used that $\pi_Y^! = \pi_Y^*(d_Y)[2d_Y]$ by Poincaré duality; see, for example, [Sch21b, Lemma 6.1(4)]. Hence,

$$H^i_{BM}(X, \mathbb{Z}_\ell(n)) = \mathrm{R}^{i-2d_X+2d_Y}\,\Gamma(Y_{\text{proét}}, \iota^! \widehat{\mathbb{Z}}_\ell(n - d_X + d_Y)) = H^{i+2c}_X(Y_{\text{proét}}, \mathbb{Z}_\ell(n + c)),$$

where $c := d_Y - d_X$, and where the above right-hand side denotes ordinary pro-étale cohomology with support; see [Sch22, (A.6)] and [BS15].

3. An auxiliary cycle map

3.1. Construction of $\tilde{\lambda}^i_{tr}$

In this subsection, we construct a map $\tilde{\lambda}^i_{tr}$ which is closely related to the transcendental Abel–Jacobi map in [Sch21b] and which will be used in Section 4.1 below to construct λ^i_X from (1.1).

To begin, we will need the following simple lemma.

Lemma 3.1. *Let X be an algebraic scheme over a field k and let $A \in \{\mathbb{Z}/\ell^r, \mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{Q}_\ell/\mathbb{Z}_\ell\}$, where ℓ is a prime invertible in k. Then the natural restriction map $H^{2i-1}(X, A(i)) \to H^{2i-1}(F_{i-1}X, A(i))$ is injective. Using this to identify each $N^j H^{2i-1}(X, A(i))$ with a subgroup of $H^{2i-1}(F_{i-1}X, A(i))$, we get a natural isomorphism*

$$\iota_*\left(\ker\left(\partial \circ \iota_* : \bigoplus_{x \in X^{(i-1)}} H^1(x, A(1)) \to \bigoplus_{x \in X^{(i)}} [x]A\right)\right) \simeq N^{i-1}H^{2i-1}(X, A(i)). \tag{3.1}$$

Proof. By (2.3), $H^{2i-1}(F_jX, A(i)) \to H^{2i-1}(F_{j-1}X, A(i))$ is injective for all $j \geq i$, as $H^l(x, A(n)) = 0$ for $l < 0$ and $x \in X$. Since $F_jX = X$ for $j > \dim X$, we deduce that $H^{2i-1}(X, A(i)) \to H^{2i-1}(F_{i-1}X, A(i))$ is injective, which proves the injectivity claim in the lemma.

Note next that the following sequence is exact by (2.3):

$$\bigoplus_{x \in X^{(i-1)}} H^1(x, A(1)) \xrightarrow{\iota_*} H^{2i-1}(F_{i-1}X, A(i)) \longrightarrow H^{2i-1}(F_{i-2}X, A(i)).$$

By the compatibility of the Gysin long exact sequence with proper pushforwards (see [Sch21b, (P2)]), we find that

$$\iota_*\left(\ker\left(\partial \circ \iota_* : \bigoplus_{x \in X^{(i-1)}} H^1(x, A(1)) \to \bigoplus_{x \in X^{(i)}} [x]A\right)\right) \subset H^{2i-1}(F_{i-1}X, A(i))$$

agrees with the image of the map $N^{i-1}H^{2i-1}(X, A(i)) \to H^{2i-1}(F_{i-1}X, A(i))$. This concludes the proof of the lemma. □

Let X be an algebraic k-scheme and let ℓ be a prime invertible in k. By [Sch21b, Lemma 8.1], there is a canonical isomorphism

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$$\psi_r : A^i(X)[\ell^r] \xrightarrow{\simeq} \frac{\ker\left(\partial \circ \iota_* : \bigoplus_{x\in X^{(i-1)}} H^1(x, \mu_{\ell^r}^{\otimes 1}) \longrightarrow \bigoplus_{x\in X^{(i)}} [x]\mathbb{Z}/\ell^r\right)}{\ker\left(\partial \circ \iota_* : \bigoplus_{x\in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1)) \longrightarrow \bigoplus_{x\in X^{(i)}} [x]\mathbb{Z}_\ell\right)}, \tag{3.2}$$

which we describe in what follows explicitly. To this end, note that an ℓ^r-torsion class in $A^i(X)_{\mathbb{Z}_\ell}$ corresponds to a cycle $[z] \in \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ such that $\ell^r[z] \in N^{i-1}\,\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$. By [Sch21b, Definition 7.2 and Lemma 7.4], this means that there is a class

$$\xi \in \bigoplus_{x\in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1))$$

with $\partial(\iota_*\xi) = \ell^r \cdot z$. It follows that the reduction $\bar{\xi}$ of ξ modulo ℓ^r has trivial residues and we have $\psi_r([z]) = [\bar{\xi}]$.

Since ξ has trivial residues modulo ℓ^r, there is a class $\gamma \in H^{2i-1}(X, \mu_{\ell^r}^{\otimes i})$ with

$$\gamma = \iota_*\bar{\xi} \in F^i H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i}).$$

Note that γ is uniquely determined by $\bar{\xi}$ because the map

$$H^{2i-1}(X, \mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i})$$

is injective by Lemma 3.1. Using this, we will, in what follows, implicitly identify $\iota_*\bar{\xi}$ and γ with each other.

The image of $\gamma = \iota_*\bar{\xi}$ in $H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is well-defined (i.e., depends only on the class $[z]$) up to elements of the form $\iota_*\bar{\zeta}$ for

$$\zeta \in \ker\left(\partial \circ \iota_* : \bigoplus_{x\in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1)) \longrightarrow \bigoplus_{x\in X^{(i)}} [x]\mathbb{Z}_\ell\right). \tag{3.3}$$

By Lemma 3.1, the subgroup of $H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ generated by classes of the form $\iota_*\bar{\zeta}$ with ζ as in (3.3) agrees with the image of $N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i))$, and so we find that the class

$$\tilde{\lambda}^i_{tr}([z]) := -[\iota_*\bar{\xi}] \in H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i))$$

is well-defined, giving rise to a map

$$\tilde{\lambda}^i_{tr} : A^i(X)[\ell^\infty] \longrightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i)). \tag{3.4}$$

(The minus sign is necessary to make our map compatible with Bloch's map; cf. [Blo79, p. 112] and [Sch21b, Proposition 8.3].)

3.2. *Basic properties of* $\tilde{\lambda}^i_{tr}$

Lemma 3.2. *The restriction of* $\tilde{\lambda}^i_{tr}$ *to the subgroup of classes* $[z] \in A^i(X)[\ell^\infty]$ *with* $\mathrm{cl}^i_X(z) = 0$ *coincides with the transcendental Abel–Jacobi map* λ^i_{tr} *from [Sch21b, §7.5].*

Proof. Assume that $\mathrm{cl}^i_X(z) = 0$. Following the construction in [Sch21b, §7.5], there is a class $\alpha \in H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))$ with $\partial\alpha = z$ and a class $\beta \in H^{2i-1}(X, \mathbb{Z}_\ell(i))$ with

$$\beta = \ell^r\alpha - \iota_*\xi \in F^i H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i)) \tag{3.5}$$

for some $\xi \in \bigoplus_{x\in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1))$. We then think about β/ℓ^r as a class in $H^{2i-1}(X, \mathbb{Q}_\ell(i))$ and project this further to a class $[\beta/\ell^r] \in H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$. By definition in *loc. cit.*, this class represents $\lambda^i_{tr}([z])$:

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$$\lambda^i_{tr}([z]) = [\beta/\ell^r] \in \frac{H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i))}.$$

We aim to see that this coincides with $\tilde{\lambda}^i_{tr}([z])$ defined above. To this end, note that (3.5) implies $\partial\iota_*\xi = \ell^r\partial\alpha = \ell^r \cdot z$ and so

$$\tilde{\lambda}^i_{tr}([z]) = -[\iota_*\bar{\xi}],$$

where $\bar{\xi}$ is the reduction modulo ℓ^r of $\xi \in \bigoplus_{x\in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1))$. By (3.5), we have $\bar{\beta} = -\iota_*\bar{\xi}$ and so the claim in the lemma reduces to the simple observation that the following diagram is commutative:

$$\begin{array}{ccccc} H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i)) & \xrightarrow{\cdot(1/\ell^r)} & H^{2i-1}(F_{i-1}X, \mathbb{Q}_\ell(i)) & \longrightarrow & H^{2i-1}(F_{i-1}X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \\ \downarrow{\scriptstyle \text{mod } \ell^r} & & & & \downarrow{\scriptstyle =} \\ H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i}) & & \longrightarrow & & \operatorname{colim}_s H^{2i-1}(F_{i-1}X, \mu_{\ell^s}^{\otimes i}) \end{array}.$$

This concludes the proof of the lemma. □

It is shown in [Sch21b] that λ^i_{tr} is injective for $i \le 2$; cf. [Sch21b, Theorem 9.4 and Corollary 9.5] or [Sch21b, Theorem 1.8(2)]. The following result extends this to $\tilde{\lambda}^i_{tr}$ as follows.

Proposition 3.3. *Let X be an algebraic scheme over a finitely generated field k and let ℓ be a prime invertible in k. Then*

$$\tilde{\lambda}^i_{tr} : A^i(X)[\ell^\infty] \longrightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i))$$

is injective for $i \le 2$.

Proof. Let $[z_0] \in \ker(\tilde{\lambda}^i_{tr})$. We first show that $[z_0]$ lies in the kernel of the cycle class map. By (3.2), we find that there is a class $\xi \in \bigoplus_{x\in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1))$ such that $\ell^r z_0 = \partial(\iota_*\xi)$ and such that $\tilde{\lambda}^i_{tr}([z_0]) = -[\iota_*\bar{\xi}] = 0$. Especially, we may assume $\iota_*\bar{\xi} \in \operatorname{im}(N^{i-1}H^{2i-1}(X, \mathbb{Z}_\ell(i)) \longrightarrow H^{2i-1}(X, \mu_{\ell^r}^{\otimes i}))$ and so by (3.1), we obtain $\iota_*\bar{\xi} = \iota_*\bar{\zeta}$ for some

$$\zeta \in \ker\left(\partial \circ \iota_* \colon \bigoplus_{x\in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1)) \longrightarrow \bigoplus_{x\in X^{(i)}} [x]\mathbb{Z}_\ell\right).$$

Exactness of the Bockstein sequence then gives $\iota_*\xi = \iota_*\zeta + \ell^r\alpha$ for some $\alpha \in H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))$. Since $\partial\iota_*\zeta = 0$, we get that $z_0 = \partial\alpha$ and so $\mathrm{cl}^i_X([z_0]) = \iota_*[z_0] = 0$ by exactness of (2.3). Hence, z_0 has trivial cycle class on X, as claimed. By Lemma 3.2, it then follows that z_0 lies in the kernel of the transcendental Abel–Jacobi map from [Sch21b]: $\lambda^i_{tr}([z_0]) = 0$. In other words, we have proved that $\ker(\tilde{\lambda}^i_{tr}) = \ker(\lambda^i_{tr})$ (but note that the two maps are defined on different domains: λ^i_{tr} is defined on homologically trivial ℓ^∞-torsion cycles, while $\tilde{\lambda}^i_{tr}$ is defined on arbitrary ℓ^∞-torsion cycles). Since $\ker(\tilde{\lambda}^i_{tr}) = \ker(\lambda^i_{tr})$, [Sch21b, Theorem 1.8(2)] yields

$$\ker(\tilde{\lambda}^i_{tr}) = H^{2i-2}_{i-3,nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/G^i H^{2i-2}_{i-3,nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)),$$

where $H^i_{j,nr}(X, A(n)) = \operatorname{im}(H^i(F_{j+1}X, A(n)) \to H^i(F_jX, A(n)))$ denotes the j-th refined unramified cohomology; cf. [Sch21b]. Since $F_jX = \emptyset$ for $j < 0$, we get that $H^i_{j,nr}(X, A(n)) = 0$ for $j < 0$ and so $\ker(\tilde{\lambda}^i_{tr}) = 0$ for $i < 3$. This proves the proposition. □

The following result is motivated by the description of the image of λ^i_{tr} in [Sch21b, Proposition 7.16].

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Proposition 3.4. *Let X be an algebraic k-scheme and let ℓ be a prime invertible in k. Then the map $\tilde{\lambda}^i_{tr}$ from (3.4) satisfies*

$$\operatorname{im}(\tilde{\lambda}^i_{tr}) = \frac{N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i))}.$$

Proof. Recall from Section 3.1 that $\tilde{\lambda}^i_{tr}$ was constructed as the direct limit of maps

$$\tilde{\lambda}^i_r \colon A^i(X)[\ell^r] \longrightarrow \frac{H^{2i-1}(X, \mu_{\ell^r}^{\otimes i})}{N^{i-1}H^{2i-1}(X, \mathbb{Z}_\ell(i))}$$

over all positive integers $r \geq 1$. As taking direct limits is an exact functor, it suffices to show that

$$\operatorname{im}(\tilde{\lambda}^i_r) = \frac{N^{i-1}H^{2i-1}(X, \mu_{\ell^r}^{\otimes i})}{N^{i-1}H^{2i-1}(X, \mathbb{Z}_\ell(i))} \quad \text{for all } r \geq 1. \tag{3.6}$$

Let $[z_0] \in A^i(X)[\ell^r]$. Then by (3.2) and the construction in Section 3.1, there is a class

$$\xi \in \bigoplus_{x \in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1))$$

such that $\partial(\iota_*\xi) = \ell^r z_0$ and such that $\tilde{\lambda}^i_r([z_0]) = -[\iota_*\overline{\xi}]$. Since the following sequence

$$\bigoplus_{x \in X^{(i-1)}} H^1(x, \mu_{\ell^r}^{\otimes 1}) \xrightarrow{\iota_*} H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i-1}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})$$

is exact by (2.3), we find $\iota_*\overline{\xi} \in N^{i-1}H^{2i-1}(X, \mu_{\ell^r}^{\otimes i})$. This proves the inclusion "$\subseteq$" for (3.6).

However, if $\beta \in N^{i-1}H^{2i-1}(X, \mu_{\ell^r}^{\otimes i})$ then, by (3.1), we can pick

$$\overline{\xi} \in \ker\left(\partial \circ \iota_* \colon \bigoplus_{x \in X^{(i-1)}} H^1(x, \mu_{\ell^r}^{\otimes 1}) \longrightarrow \bigoplus_{x \in X^{(i)}} [x]\mathbb{Z}/\ell^r\right)$$

such that $\iota_*\overline{\xi} = \beta$. Hilbert 90 implies that $H^1(x, \mathbb{Z}_\ell(1)) \to H^1(x, \mu_{\ell^r}^{\otimes i})$ is surjective for all $x \in X$; see [Sch21b, (P6) in Definition 4.4 and Proposition 6.6]. Hence, there is a class $\xi \in \bigoplus_{x \in X^{(i-1)}} H^1(x, \mathbb{Z}_\ell(1))$ whose reduction modulo ℓ^r is $\overline{\xi} \in \bigoplus_{x \in X^{(i-1)}} H^1(x, \mu_{\ell^r}^{\otimes i})$. In particular, $0 = \partial\beta = \partial(\iota_*\overline{\xi})$ yields $\partial(\iota_*\xi) = \ell^r z_0$ for some $z_0 \in \bigoplus_{x \in X^{(i)}} [x]\mathbb{Z}_\ell$ and thus, $\tilde{\lambda}^i_r(-[z_0]) = [\iota_*\overline{\xi}] = [\beta]$. This finishes the proof of the proposition. □

By Lemma 3.2, $\tilde{\lambda}^i_{tr}$ and λ^i_{tr} agree on ℓ-power torsion classes with trivial cycle class. Moreover, in the proof of Proposition 3.3, we showed that $\tilde{\lambda}^i_{tr}$ and λ^i_{tr} have the same kernel. The following lemma shows more generally that $\tilde{\lambda}^i_{tr}$ factors through the cycle class map.

Lemma 3.5. *Let X be an algebraic k-scheme which admits a closed embedding into a smooth equidimensional algebraic k-scheme (e.g., X is quasi-projective). Then the composition of $\tilde{\lambda}^i_{tr}$ from (3.4) with the negative of the Bockstein map*

$$\delta : H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \longrightarrow H^{2i}(X, \mathbb{Z}_\ell(i))$$

agrees with the cycle class map $\mathrm{cl}^i_X : A^i(X)[\ell^\infty] \longrightarrow H^{2i}(X, \mathbb{Z}_\ell(i))$.

Proof. We follow the proof of [CTSS83, Proposition 1]. By topological invariance of the pro-étale site, we may up to replacing k by its perfect closure assume that k is perfect (cf. [BS15, Lemma 5.4.2]).

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Let $[z_0] \in A^i(X)[\ell^\infty]$. Then there is a closed subset $W \subset X$ of pure codimension $i-1$ and a class $\xi \in H^1(F_0W, \mathbb{Z}_\ell(1))$ such that $\partial\xi = \ell^r z_0$ for some integer $r \geq 0$. In particular, $\tilde{\lambda}^i_{tr}([z_0]) = -[\iota_*\bar{\xi}]$, where $\bar{\xi} \in H^1(W, \mu_{\ell^r})$ is the reduction of ξ modulo ℓ^r and $\iota\colon W \hookrightarrow X$ the obvious closed embedding. Since the Bockstein map is compatible with proper pushforwards (cf. [Sch21b, (P5) in Definition 4.4 and Proposition 6.6]), the lemma follows if we can show that

$$-\delta(\bar{\xi}) = \mathrm{cl}^1_W(z_0) \in H^2(W, \mathbb{Z}_\ell(1)). \tag{3.7}$$

The above statement does not depend on the ambient space X. Since X can be embedded into a smooth equi-dimensional k-scheme by assumption, we may thus from now on assume without loss of generality that X is smooth and equi-dimensional.

In what follows, for a closed subset $Z \subset X$ and $A \in \{\mathbb{Z}/\ell^r, \mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{Q}_\ell/\mathbb{Z}_\ell\}$, the group $H^i_Z(X, A(n))$ stands for ordinary pro-étale cohomology with support; if $c = \dim X - \dim Z$, then

$$H^{i-2c}_{BM}(Z, A(n)) = H^i_Z(X, A(n))$$

by Remark 2.1. The equation (3.7) then translates into the claim that

$$\alpha_1 := -\delta(\bar{\xi}) \in H^{2i}_W(X, \mathbb{Z}_\ell(i)) \quad \text{and} \quad \alpha_2 := \mathrm{cl}^1_W(z_0) \in H^{2i}_W(X, \mathbb{Z}_\ell(i)) \quad \text{coincide.} \tag{3.8}$$

We aim to describe the class $\alpha_2 = \mathrm{cl}^1_W(z_0)$. Since $\xi \in H^1(F_0W, \mathbb{Z}_\ell(1))$, we may pick a closed subset $W' \subset W$ of pure codimension one such that ξ admits a lift to $H^1(W \backslash W', \mathbb{Z}_\ell(1)) = H^{2i-1}_{W\backslash W'}(X \backslash W', \mathbb{Z}_\ell(i))$. Then $\partial\xi = \ell^r z_0 \in H^{2i}_{W'}(X, \mathbb{Z}_\ell(i)) = \bigoplus_{x \in W'^{(0)}} [x]\mathbb{Z}_\ell$, and α_2 is the image of z_0 in $H^{2i}_W(X, \mathbb{Z}_\ell(i))$.

In order to show $\alpha_1 = \alpha_2$, we take a Cartan-Eilenberg injective resolution

$$0 \longrightarrow I^\bullet \longrightarrow J^\bullet \xrightarrow{p} K^\bullet \longrightarrow 0$$

for the following short exact sequence of sheaves on $X_{\text{proét}}$

$$0 \longrightarrow \hat{\mathbb{Z}}_\ell(i) \xrightarrow{\times \ell^r} \hat{\mathbb{Z}}_\ell(i) \longrightarrow \mu^{\otimes i}_{\ell^r} \longrightarrow 0$$

and we consider the following commutative diagram of complexes of abelian groups.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H^0_{W'}(X, I^\bullet) & \longrightarrow & H^0_W(X, I^\bullet) & \longrightarrow & H^0_{W\backslash W'}(X \setminus W', I^\bullet) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H^0_{W'}(X, J^\bullet) & \longrightarrow & H^0_W(X, J^\bullet) & \longrightarrow & H^0_{W\backslash W'}(X \setminus W', J^\bullet) & \longrightarrow & 1 \\
 & & \downarrow p & & \downarrow p & & \downarrow p & & \\
1 & \longrightarrow & H^0_{W'}(X, K^\bullet) & \longrightarrow & H^0_W(X, K^\bullet) & \longrightarrow & H^0_{W\backslash W'}(X \setminus W', K^\bullet) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

As higher cohomology (with support) of injective objects vanishes, we find that both the rows and columns of the above diagram are exact. We shall denote the differential of these complexes by the letter d.

Let $\eta \in H^0_W(X, K^{2i-1})$ be a lift of $\bar{\xi} \in H^{2i-1}_W(X, \mu^{\otimes i}_{\ell^r})$. We claim that there exists a lift $\omega \in H^0_W(X, J^{2i-1})$ of η such that the image ω' of ω in $H^0_{W\backslash W'}(X \setminus W', J^{2i-1})$ satisfies $d\omega' = 0$. Indeed,

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let η' denote the image of η in $H^0_{W\setminus W'}(X\setminus W', K^{2i-1})$. Since $\overline{\xi}$ is the reduction modulo ℓ^r of a class $\xi \in H^{2i-1}_{W\setminus W'}(X\setminus W', \mathbb{Z}_\ell(i))$ and the reduction map

$$H^{2i-1}_{W\setminus W'}(X\setminus W', \mathbb{Z}_\ell(i)) \longrightarrow H^{2i-1}_{W\setminus W'}(X\setminus W', \mu_{\ell^r}^{\otimes i})$$

is induced by

$$p\colon H^0_{W\setminus W'}(X\setminus W', J^\bullet) \longrightarrow H^0_{W\setminus W'}(X\setminus W', K^\bullet),$$

we can find a class $\tau_1' \in H^0_{W\setminus W'}(X\setminus W', J^{2i-1})$ with $d\tau_1' = 0$ such that $p(\tau_1') = \eta' + d(\kappa')$ for some $\kappa \in H^0_W(X, K^{2i-2})$ with image $\kappa' \in H^0_{W\setminus W'}(X\setminus W', K^{2i-2})$. Replacing η with $\eta + d\kappa$, we may assume $p(\tau_1') = \eta'$. Let $\tau_1 \in H^0_W(X, J^{2i-1})$ be a lift of τ_1'. Exactness of the above diagram yields $\eta - p(\tau_1) = p(\iota_*\tau_2)$ for some $\tau_2 \in H^0_{W'}(X, J^{2i-1})$ and we take $\omega := \tau_1 + \iota_*\tau_2$, where $\iota_*\tau_2$ denotes the image of τ_2 in $H^0_W(X, J^{2i-1})$. Then $p(\omega) = \eta$ and the image ω' of ω in $H^0_{W\setminus W'}(X\setminus W', J^{2i-1})$ satisfies $d\omega' = d\tau_1' = 0$, as claimed above.

Using that the Bockstein map $\delta\colon H^{2i-1}_W(X, \mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i}_W(X, \mathbb{Z}_\ell(i))$ is the coboundary map of the middle vertical short exact sequence in the diagram above, we deduce $\alpha_1 = [d\omega]$, where we regard $d\omega$ as an element in $H^0_W(X, I^{2i})$.

Finally, we recall the construction of the class α_2, so as to verify our claim (i.e., $\alpha_1 = \alpha_2$). Indeed, the class ω' corresponds to a lift ξ of $\overline{\xi} \in H^{2i-1}_{W\setminus W'}(X\setminus W', \mu_{\ell^r}^{\otimes i})$ in $H^{2i-1}_{W\setminus W'}(X\setminus W', \mathbb{Z}_\ell(i))$, and an easy diagram chase gives that $\partial\xi \in H^{2i}_{W'}(X, \mathbb{Z}_\ell(i))$ is the cohomology class of $d\omega \in H^0_{W'}(X, J^{2i})$. Since $p(d\omega) = d\eta = 0$, we deduce $d\omega \in H^0_{W'}(X, I^{2i})$ and the corresponding class in $H^{2i}_{W'}(X, \mathbb{Z}_\ell(i))$ agrees with z_0. Hence, the image of z_0 in $H^{2i}_W(X, \mathbb{Z}_\ell(i))$ is the cohomology class of $d\omega \in H^0_W(X, I^{2i})$, showing $\alpha_1 = \alpha_2$, as we want. This concludes the proof of the lemma. □

Lemma 3.6. *Let X_j for $j = 1, 2$ be smooth projective k-varieties and let ℓ be a prime invertible in k. Let $z_j \in \mathrm{CH}^{i_j}(X_j)$ be classes such that z_2 is ℓ^r-torsion. Then*

$$\tilde{\lambda}^{i_1+i_2}_{tr}(z_1 \times z_2) = \mathrm{pr}_1^*\Big(\overline{\mathrm{cl}^{i_1}_{X_1}(z_1)}\Big) \cup \mathrm{pr}_2^*\Big(\tilde{\lambda}^{i_2}_{tr}([z_2])\Big),$$

where $\overline{\mathrm{cl}^{i_1}_{X_1}(z_1)} \in H^{2i_1}(X_1, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i_1))$ denotes the reduction modulo ℓ^r of $\mathrm{cl}^{i_1}_{X_1}(z_1)$, that is, the image of $\mathrm{cl}^{i_1}_{X_1}(z_1)$ via the composition

$$H^{2i_1}(X_1, \mathbb{Z}_\ell(i_1)) \longrightarrow H^{2i_1}(X_1, \mu_{\ell^r}^{\otimes i_1}) \longrightarrow \mathrm{colim}_s\, H^{2i_1}(X_1, \mu_{\ell^s}^{\otimes i_1}) = H^{2i_1}(X_1, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i_1)).$$

Proof. By topological invariance of the étale, resp. pro-étale site, we may up to replacing k by its perfect closure assume that k is perfect (cf. [BS15, Lemma 5.4.2]).

We will frequently cite properties of cohomology with support from [Sch22, Appendix A], which applies to our setting by Remark 2.1 above.

There is a closed subset $W_2 \subset X$ of pure codimension $i_2 - 1$ and a class $\xi \in H^1(F_0W_2, \mathbb{Z}_\ell(1))$ such that $\partial\xi = \ell^r z_2$. We then have $\tilde{\lambda}^{i_2}_{tr}([z_2]) = -[(\iota_2)_*\overline{\xi}]$, where $\overline{\xi}$ denotes the reduction of ξ modulo ℓ^r and $(\iota_2)_*$ denotes the pushforward induced by the closed embedding $\iota_2 : W_2 \hookrightarrow X_2$. If $\iota_1 : W_1 \hookrightarrow X_1$ denotes the inclusion of the support of z_1, then $c := \mathrm{cl}^0_{W_1}(z_1) \in H^0(W_1, \mathbb{Z}_\ell(0)) = H^0(F_0W_1, \mathbb{Z}_\ell(0))$ satisfies $\mathrm{cl}^{i_1}_{X_1}([z_1]) = (\iota_1)_*c$. We may then consider the class

$$p_1^*c \cup p_2^*\xi \in H^1(F_0(W_1 \times W_2), \mathbb{Z}_\ell(1)),$$

where p_i^* denotes the pullback induced by the projection map $p_i : W_1 \times W_2 \to W_i$. Note that we use here the reduction step that k is perfect as it implies that for each i, W_i is generically smooth and equidimensional, in which case Borel–Moore cohomology agrees with ordinary cohomology. In particular, the cup product used above exists.

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The residue of the above class is given by

$$\partial\big(p_1^* c \cup p_2^* \xi\big) = p_1^* c \cup \partial\big(p_2^* \xi\big) = \ell^r \cdot (z_1 \times z_2);$$

cf. [Sch21a, Lemma 2.4]. Hence,

$$\tilde{\lambda}_{tr}^{i_1+i_2}(z_1 \times z_2) = -(\iota_1 \times \iota_2)_*(\overline{p_1^* c \cup p_2^* \xi}).$$

In particular, the lemma follows once we have proven the following claim:

$$-(\iota_1 \times \iota_2)_*(\overline{p_1^* c \cup p_2^* \xi}) = \mathrm{pr}_1^*((\iota_1)_* \overline{c}) \cup \mathrm{pr}_2^*\Big(-(\iota_2)_* \overline{\xi}\Big) = \mathrm{pr}_1^*\Big(\overline{\mathrm{cl}_{X_1}^{i_1}(z_1)}\Big) \cup \mathrm{pr}_2^*\Big(\tilde{\lambda}_{tr}^{i_2}([z_2])\Big), \tag{3.9}$$

where pr_i^* denotes the pullback induced by the projection $\mathrm{pr}_i : X_1 \times X_2 \to X_i$. The second equality in (3.9) is clear and so it suffices to prove the first. By linearity, it suffices to prove this in the case where W_1 is irreducible and $c = 1 \cdot [W_1] \in H^0(W_1, \mathbb{Z}_\ell(0))$ is the fundamental class. It then suffices to prove

$$(\iota_1 \times \iota_2)_*(\overline{p_2^* \xi}) = \mathrm{pr}_1^*\Big(\overline{\mathrm{cl}_{X_1}^{i_1}(W_1)}\Big) \cup \mathrm{pr}_2^*\Big((\iota_2)_* \overline{\xi}\Big). \tag{3.10}$$

We consider the closed embeddings

$$f : W_1 \times W_2 \to X_1 \times W_2 \ \text{ and } \ g : X_1 \times W_2 \to X_1 \times X_2.$$

Note that $\iota_1 \times \iota_2 = g \circ f$. Let further $q_1 : X_1 \times W_2 \to X_1$ and $q_2 : X_1 \times W_2 \to W_2$ denote the projections. Then $p_2 = q_2 \circ f$ and so

$$(\iota_1 \times \iota_2)_*(\overline{p_2^* \xi}) = g_* f_*(f^* q_2^* \overline{\xi}).$$

By the projection formula (see, for example, [Sch22, Lemma A.19]), applied to f, we thus get

$$(\iota_1 \times \iota_2)_*(\overline{p_2^* \xi}) = g_*(f_* 1 \cup q_2^* \overline{\xi}).$$

Note that in the above equation,

$$f_* 1 = g^* \, \mathrm{cl}_{X_1 \times X_2}^{i_1}(W_1 \times X_2) \in H^{2i_1}(X_1 \times W_2, \mu_{\ell^r}^{\otimes i_1}).$$

Hence,

$$(\iota_1 \times \iota_2)_*(\overline{p_2^* \xi}) = g_*(g^* \, \mathrm{cl}_{X_1 \times X_2}^{i_1}(W_1 \times X_2) \cup q_2^* \overline{\xi}).$$

Applying the projection formula with respect to g thus yields

$$(\iota_1 \times \iota_2)_*(\overline{p_2^* \xi}) = \mathrm{cl}_{X_1 \times X_2}^{i_1}(W_1 \times X_2) \cup g_* q_2^* \overline{\xi}.$$

To prove (3.10), it thus suffices to show that

$$\mathrm{cl}_{X_1 \times X_2}^{i_1}(W_1 \times X_2) = \mathrm{pr}_1^*\Big(\mathrm{cl}_{X_1}^{i_1}(W_1)\Big) \ \text{ and } \ g_* q_2^* \overline{\xi} = \mathrm{pr}_2^*\Big((\iota_2)_* \overline{\xi}\Big). \tag{3.11}$$

The first identity follows directly by the compatibility of pullbacks in cohomology and Chow groups via the cycle class map (see, for example, [Sch22, Lemma A.21]). The second identity follows from the compatibility of pullbacks and pushforwards as outlined, for example, in [Sch22, Lemma A.12(2)], applied to the commutative diagram

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$$\begin{array}{ccc} X_1 \times (W_2 \setminus Z_2) & \longrightarrow & X_1 \times (X_2 \setminus Z_2) \\ \downarrow & & \downarrow \\ W_2 \setminus Z_2 & \longrightarrow & X_2 \setminus Z_2, \end{array}$$

where $Z_2 \subset W_2$ is a closed subset that is nowhere dense and which contains the singular locus of W_2. This concludes the proof of the lemma. □

Corollary 3.7. *Let X_j for $j = 1, 2$ be smooth projective k-varieties and let ℓ be a prime invertible in k. Let $z_j \in \mathrm{CH}^{i_j}(X_j)$ be classes such that z_2 is ℓ^r-torsion for some positive integer r. Assume that $\mathrm{cl}^{i_1}_{X_1}(z_1)$ is zero modulo ℓ^r. Then the ℓ^r-torsion cycle*

$$z := z_1 \times z_2 \in \mathrm{CH}^{i_1+i_2}(X_1 \times X_2)$$

lies in the kernel of $\tilde{\lambda}^{i_1+i_2}_{tr}$.

Proof. This is an immediate consequence of Lemma 3.6. □

4. Passing to the limit over finitely generated subfields

4.1. Construction of λ^i_X

Let X be an algebraic scheme over a field k and let $k_0 \subset k$ be a finitely generated subfield such that there is a variety X_0 over k_0 with $X = X_0 \times_{k_0} k$. For any finitely generated subfield $k' \subset k$ with $k_0 \subset k'$, we consider $X_{k'} := X_0 \times_{k_0} k'$. We then have

$$\mathrm{CH}^i(X) = \varinjlim_{k'/k_0} \mathrm{CH}^i(X_{k'}) \text{ and } \mathrm{CH}^i(X)[\ell^\infty] = \varinjlim_{k'/k_0} A^i(X_{k'})[\ell^\infty],$$

where the limit runs through all finitely generated subfields $k' \subset k$ with $k_0 \subset k'$ and where we used $\mathrm{CH}^i(X_{k'})_{\mathbb{Z}_\ell} = A^i(X_{k'})_{\mathbb{Z}_\ell}$ for k' finitely generated; cf. [Sch21b, Lemma 7.5]. Using this, we define λ^i_X as the direct limit of $\tilde{\lambda}^i_{tr}$ (see Section 3.1), applied to $X_{k'}$ for all finitely generated fields k' as above:

$$\lambda^i_X := \varinjlim_{k'/k_0} \tilde{\lambda}^i_{tr} : \mathrm{CH}^i(X)[\ell^\infty] = \varinjlim_{k'/k_0} A^i(X_{k'})[\ell^\infty] \longrightarrow \varinjlim_{k'/k_0} \frac{H^{2i-1}(X_{k'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X_{k'}, \mathbb{Q}_\ell(i))} = \frac{H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{M^{2i-1}(X)},$$

where we use that

$$H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \simeq \varinjlim_{k'/k_0} H^{2i-1}(X_{k'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)),$$

and where

$$M^{2i-1}(X) := \mathrm{im}\left(\varinjlim_{k'/k_0} N^{i-1}H^{2i-1}(X_{k'}, \mathbb{Q}_\ell(i)) \longrightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \right).$$

If X is smooth and equi-dimensional (e.g., a smooth variety), then Borel–Moore cohomology agrees with ordinary cohomology (see (2.1)) and the above map yields a cycle map as in (1.1).

Lemma 4.1. *Assume that X is a smooth projective k-variety. In the following special cases, the group $M^{2i-1}(X)$ can be computed explicitly as follows:*

(1) *If k is algebraically closed, then $M^{2i-1}(X) = 0$ for all i;*
(2) *If k is a finite field, then $M^{2i-1}(X) = 0$ for all i;*

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(3) *If $i = 1$, k is arbitrary and X is geometrically integral, then*

$$M^1(X) = \mathrm{im}(H^1(\mathrm{Spec}\, k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \to H^1(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))).$$

Proof. Assume first that k is algebraically closed. By the Weil conjectures proven by Deligne, the group $H^{2i-1}(X, \mathbb{Q}_\ell(i))$ does not contain any nontrivial element that is fixed by the absolute Galois group of a finitely generated subfield $k' \subset k$. Hence, the natural map $H^{2i-1}(X_{k'}, \mathbb{Q}_\ell(i)) \to H^{2i-1}(X, \mathbb{Q}_\ell(i))$ is zero for any finitely generated field $k' \subset k$. This implies $M^{2i-1}(X) = 0$ as we want in (1).

Assume now that k is a finite field. The Weil conjectures imply that $H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is a finite group (see [CTSS83, Théorème 2]) and so the map $H^{2i-1}(X, \mathbb{Q}_\ell(i)) \to H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is zero. This implies $M^{2i-1}(X) = 0$, as claimed.

It suffices by a limit argument to prove the last claim in the case where k is an arbitrary finitely generated field. Let $G = \mathrm{Gal}_k$ be the absolute Galois group of k and let $\bar{k}$ be an algebraic (or separable) closure of k. Then the Hochschild–Serre spectral sequence from [Jan88] yields an exact sequence

$$H^1(\mathrm{Spec}\, k, \mathbb{Q}_\ell(1)) \longrightarrow H^1(X, \mathbb{Q}_\ell(1)) \longrightarrow H^1(X_{\bar{k}}, \mathbb{Q}_\ell(1))^G,$$

where we use that $H^0(X_{\bar{k}}, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell(1)$ because X is geometrically integral. The Weil conjectures proven by Deligne [Del74] imply that $H^1(X_{\bar{k}}, \mathbb{Q}_\ell(1))^G = 0$ and so the first map in the above sequence is surjective. Moreover, Kummer theory implies that $H^1(\mathrm{Spec}\, k, \mathbb{Q}_\ell(1)) \to H^1(\mathrm{Spec}\, k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ is surjective. Since $N^0H^1(X, \mathbb{Q}_\ell(1)) = H^1(X, \mathbb{Q}_\ell(1))$, we finally conclude

$$M^1(X) = \mathrm{im}(H^1(X, \mathbb{Q}_\ell(1)) \to H^1(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))) = \mathrm{im}(H^1(\mathrm{Spec}\, k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \to H^1(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))),$$

as we want. This finishes the proof of the lemma. □

4.2. Basic properties of λ^i_X

Theorem 4.2. *Let X be an algebraic scheme over a field k and let ℓ be a prime invertible in k. Then the cycle map*

$$\lambda^i_X : \mathrm{CH}^i(X)[\ell^\infty] \longrightarrow \frac{H^{2i-1}_{BM}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{M^{2i-1}(X)}$$

is injective for $i \leq 2$.

Proof. This follows from Proposition 3.3, where we recall our convention that $H^*(X, A(n)) = H^*_{BM}(X, A(n))$ denotes Borel–Moore cohomology. □

The following result generalizes [MS83, (18.4)].

Lemma 4.3. *Let X be an algebraic scheme over a field k and let ℓ be a prime invertible in k. Then the image of λ^i_X is given by*

$$\mathrm{im}(\lambda^i_X) = \frac{N^{i-1}H^{2i-1}_{BM}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{M^{2i-1}(X)}.$$

Proof. This follows from Proposition 3.4, together with the construction of λ^i_X via direct limit in Section 4.1. □

Corollary 4.4. *Let X be an algebraic scheme over a field k and let ℓ be a prime invertible in k. Then λ^i_X induces for $i \in \{1, 2\}$ isomorphisms*

$$\mathrm{CH}^1(X)[\ell^\infty] \simeq \frac{H^1_{BM}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))}{M^1(X)} \quad \textit{and} \quad \mathrm{CH}^2(X)[\ell^\infty] \simeq \frac{N^1H^3_{BM}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))}{M^3(X)}.$$

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Proof. By Lemma 4.3, $\mathrm{im}(\lambda^i_X) = N^{i-1}H^{2i-1}_{BM}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/M^{2i-1}(X)$. Hence, the result is an immediate consequence of Theorem 4.2. □

Comparing the above construction with Bloch's map from [Blo79], we get the following.

Lemma 4.5. *If k is algebraically closed and X is a smooth projective variety over k, then $M^{2i-1}(X) = 0$ for all i and the map*

$$\lambda^i_X : \mathrm{CH}^i(X)[\ell^\infty] \longrightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

agrees with Bloch's map from [Blo79]. In particular, if $k = \mathbb{C}$, then λ^i_X restricted to the subgroup of homologically trivial cycles can be identified with Griffiths Abel–Jacobi map from [Gri69].

Proof. The vanishing of $M^{2i-1}(X)$ follows from Lemma 4.1. Using this, it is straightforward to check that our map λ^i_X coincides over algebraically closed fields with Bloch's map from [Blo79]. The comparison with Griffiths' map thus follows from [Blo79, Proposition 3.7]. This concludes the proof of the lemma. □

Lemma 4.6. *Let X be an algebraic scheme over a field k and let $k_0 \subset k$ be a finitely generated subfield such that there is a variety X_0 over k_0 with $X = X_0 \times_{k_0} k$. Let ℓ be a prime invertible in k. Assume that λ^i_X is not injective. Let $L \subset k$ be any subfield with $k_0 \subset L$. Then there is a subfield $k' \subset k$ with $L \subset k'$, such that k'/L is finitely generated and*

$$\lambda^i_{X'} : \mathrm{CH}^i(X')[\ell^\infty] \longrightarrow \frac{H^{2i-1}(X', \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{M^{2i-1}(X')}$$

is not injective, where $X' := X_0 \times_{k_0} k'$.

Proof. By assumption, there is a nontrivial element $[z] \in \mathrm{CH}^i(X)[\ell^\infty]$ with $\lambda^i_X([z]) = 0$. We can choose a finitely generated extension k'/L such that z is defined over k' and so we get a class

$$[z'] \in \mathrm{CH}^i(X') \text{ with } [z'_k] = [z] \in \mathrm{CH}^i(X)[\ell^\infty],$$

where $X' = X_0 \times_{k_0} k'$. Up to possibly replacing k' by a larger finitely generated extension of L, we can assume that $[z']$ is ℓ^∞-torsion and so there is a class

$$\lambda^i_{X'}([z']) \in \frac{H^{2i-1}(X', \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{M^{2i-1}(X')}.$$

Note that $[z'_k] = [z]$ lies in the kernel of λ^i_X. Since λ^i_X and $\lambda^i_{X'}$ are defined via direct limits over all finitely generated subfields of k and k', respectively, we see that up to possibly replacing k' by a larger finitely generated extension of L, we may assume that $\lambda^i_{X'}([z']) = 0$ while $[z'] \in \mathrm{CH}^i(X')$ is still nontrivial because its base change to k is nontrivial. This proves the lemma. □

Next, we have the following.

Lemma 4.7. *Let X be an algebraic k-scheme which admits a closed embedding into a smooth equidimensional algebraic k-scheme (e.g., X is quasi-projective) and let ℓ be a prime invertible in k. The composition of λ^i_X with the projection*

$$\frac{H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{M^{2i-1}(X)} \twoheadrightarrow \frac{H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{H^{2i-1}(X, \mathbb{Q}_\ell(i))},$$

followed by the injection

$$-\delta : \frac{H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{H^{2i-1}(X, \mathbb{Q}_\ell(i))} \hookrightarrow H^{2i}(X, \mathbb{Z}_\ell(i))$$

induced by the Bockstein map δ, agrees with the cycle class map $\mathrm{cl}^i_X : \mathrm{CH}^i(X)[\ell^\infty] \longrightarrow H^{2i}(X, \mathbb{Z}_\ell(i))$.

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Proof. This is a direct consequence of Lemma 3.5. □

Lemma 4.8. *Let X_j for $j = 1, 2$ be smooth projective k-varieties and let ℓ be a prime invertible in k. Let $z_j \in \mathrm{CH}^{i_j}(X_j)$ be classes such that z_2 is ℓ^r-torsion for some positive integer r. Assume that $\mathrm{cl}^{i_1}_{X_1}(z_1)$ is zero modulo ℓ^r. Then the ℓ^r-torsion cycle*

$$z := z_1 \times z_2 \in \mathrm{CH}^{i_1+i_2}(X_1 \times X_2)$$

lies in the kernel of the cycle map $\lambda^{i_1+i_2}_X$ from (1.1).

Proof. Let k_0 be a finitely generated subfield over which X_j and the cycle z_j are both defined for $j = 1, 2$. Since z_2 is ℓ^r-torsion, we may assume that it is already ℓ^r-torsion on X_0 (i.e., when viewed as a cycle over k_0). Note, moreover, that

$$H^{2i_1}(X_1, \mu_{\ell^r}^{\otimes i_1}) = \varinjlim_{k'/k_0} H^{2i_1}(X_{1k'}, \mu_{\ell^r}^{\otimes i_1}),$$

where k' runs through all finitely generated subfields of k that contain k_0, and $X_{1k'}$ denotes the base change to k' of a fixed form of X_1 over k_0. We can therefore also assume that the cycle class of z_1 over k_0 is zero modulo ℓ^r. The claim in the lemma follows then from Corollary 3.7. □

5. Schoen's argument over non-closed fields

The following proposition extends (some version of) the main result from [Schoe00] to non-closed fields.

Proposition 5.1. *Let k be an algebraically closed field and let C be a smooth irreducible curve over k. Let ℓ be a prime and let r be a positive integer. Let E be an elliptic curve over $k(C)$ whose j-invariant is transcendental over k. Up to replacing C by a (possibly ramified) finite cover, the following holds for any k-variety B, where we denote by $K = k(B \times C)$ the function field of $B \times C$: There is a class $\tau \in \mathrm{CH}_0(E_K)$ of order ℓ^r such that for any smooth projective variety Y over $k(B)$, the kernel of the exterior product map*

$$\mathrm{CH}^i(Y) \otimes \mathbb{Z}/\ell^r \longrightarrow \mathrm{CH}^{i+1}(Y_K \times_K E_K)[\ell^r], \quad z \longmapsto [z_K \times \tau]$$

is contained in the image of

$$\mathrm{CH}^i(Y)_{\mathrm{tors}} \otimes \mathbb{Z}/\ell^r \longrightarrow \mathrm{CH}^i(Y) \otimes \mathbb{Z}/\ell^r.$$

Proof. Replacing C by the normalization of a projective closure, we may assume that C is smooth and projective. Note that k is algebraically closed and that the j-invariant of E is transcendental over k. Using this, the same argument as in [Schoe00, Lemma 2.7] shows that up to replacing C by a finite cover, we may assume that the curve E_K admits a regular projective model $\mathcal{E}$ over $C_{k(B)} := C \times_k k(B)$, with the following properties:

(i) $\mathcal{E} \to C_{k(B)}$ is a minimal elliptic surface over $k(B)$;
(ii) there is a k-rational point on C with induced point $0 \in C_{k(B)}$, such that the fibre F of $\mathcal{E} \to C_{k(B)}$ above 0 is of type $I_{\ell^r N}$ for some $N \geq 1$;
(iii) if we denote by $F_i, i = 0, 1, \ldots, \ell^r N - 1$ the components of F, then $F_i^2 = -2$ for all i and $F_i \cdot F_{i-1} = 1$ for all i if $\ell^r N \neq 2$ and $F_0 F_1 = 2$ if $\ell^r N = 2$, where the index has to be read modulo $\ell^r N$;
(iv) the model $\mathcal{E} \to C_{k(B)}$ admits two sections s_0, s_1 such that $s_1 - s_0$ restricts to a zero-cycle $\tau \in \mathrm{CH}_0(E_K)$ of order ℓ^r and such that s_0 meets F_0, while the (unique) component of F that meets s_1 is of the form F_{mN} with m coprime to ℓ.

The same argument as in [Schoe00, Lemma 2.8] then shows that there is a divisor D on $\mathcal{E}$ which is supported on the special fibre F and such that the following holds.

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(v) if D' is another divisor on $\mathcal{E}$ that is supported on some fibres of $\mathcal{E} \to C_{k(B)}$, then $D' \cdot D \equiv 0 \mod \ell^r N$;
(vi) $(s_1 - s_0) \cdot D = mN \cdot \chi$ for some integer m that is coprime to ℓ^r and some zero-cycle $\chi \in \mathrm{CH}_0(F)$ of degree 1.

To conclude the argument, we consider the model

$$\mathcal{X} := Y \times_{k(B)} \mathcal{E} = (Y \times_{k(B)} C_{k(B)}) \times_{C_{k(B)}} \mathcal{E} \longrightarrow C_{k(B)}$$

with special fibre $X_0 = Y \times_{k(B)} F$ and generic fibre $X_\eta = Y_K \times_K E_K$. Since Y is smooth over $k(B)$ and $\mathcal{E}$ is regular, the model $\mathcal{X}$ is regular as well. Let $z \in \mathrm{CH}^i(Y)$. Since $s_1 - s_0$ is a divisor class on $\mathcal{E}$, we can consider the exterior product cycle

$$\xi := z \times (s_1 - s_0) \in \mathrm{CH}^{i+1}(\mathcal{X}).$$

The restriction $\xi_\eta \in \mathrm{CH}^{i+1}(Y_K \times_K E_K)$ of this cycle to the generic fibre of $\mathcal{X} \to C_{k(B)}$ agrees with

$$[z_K \times \tau] \in \mathrm{CH}^{i+1}(Y_K \times_K E_K),$$

where $z_K \in \mathrm{CH}^i(Y_K)$ denotes the base change of z and $\tau \in \mathrm{CH}_0(E_K)$ is the class of order ℓ^r from (iv).

To prove the proposition, we assume that $\xi_\eta = 0$, and we then aim to show that z is the sum of a torsion class and a class that is ℓ^r-divisible. To this end, note that the localization formula together with the assumption $\xi_\eta = 0$ implies that there are closed points $c_1, \dots, c_n \in C_{k(B)}$ with $c_j \neq 0$ for all j such that

$$\xi \in \mathrm{im}\left(\mathrm{CH}^i(X_0) \oplus \bigoplus_j \mathrm{CH}^i(X_{c_j}) \longrightarrow \mathrm{CH}^{i+1}(\mathcal{X})\right).$$

Hence,

$$\xi = \xi' + \xi'' \text{ where } \xi' \in \mathrm{im}\left(\mathrm{CH}^i(X_0) \longrightarrow \mathrm{CH}^{i+1}(\mathcal{X})\right) \text{ and } \xi'' \in \mathrm{im}\left(\bigoplus_j \mathrm{CH}^i(X_{c_j}) \longrightarrow \mathrm{CH}^{i+1}(\mathcal{X})\right).$$

Note that $X_0 = Y \times_{k(B)} F$. Since F is a cycle of smooth rational curves, the natural map

$$(\mathrm{CH}^i(Y) \otimes \mathrm{CH}^0(F)) \oplus (\mathrm{CH}^{i-1}(Y) \otimes \mathrm{CH}^1(F)) \longrightarrow \mathrm{CH}^i(X_0)$$

is surjective. It follows that we can write

$$\xi' = \sum_j \xi_{1j} \times D'_j + \xi_2 \times \zeta$$

for some $\xi_{1j} \in \mathrm{CH}^i(Y)$, $D'_j \in \mathrm{CH}^0(F)$, $\xi_2 \in \mathrm{CH}^{i-1}(Y)$, and $\zeta \in \mathrm{CH}^1(F)$. Let $Y \times D$ denote the pullback of the divisor D via the natural projection $\mathrm{pr}_2 : \mathcal{X} \to \mathcal{E}$. Since $\mathcal{X}$ is regular, we may consider the intersection product $\xi \cdot (Y \times D) \in \mathrm{CH}^{i+2}(\mathcal{X})$. Since D is supported on the special fibre F, the support of ξ'' is disjoint from the support of $Y \times D$ and so $\xi'' \cdot (Y \times D) = 0$. Hence,

$$\xi \cdot (Y \times D) = \left(\sum_j \xi_{1j} \times D'_j + \xi_2 \times \zeta\right) \cdot (Y \times D) \in \mathrm{CH}^{i+2}(\mathcal{X}).$$

By item (v), $(\xi_{1j} \times D'_j) \cdot (Y \times D) \equiv 0 \mod \ell^r N$ for all j and so

$$\xi \cdot (Y \times D) = \xi_2 \times (\zeta \cdot D) = 0 \in \mathrm{CH}^{i+2}(\mathcal{X})/\ell^r N$$

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because $\zeta \cdot D = 0$ for degree reasons. Hence,

$$\xi \cdot (Y \times D) \in \ell^r N \cdot \mathrm{CH}^{i+2}(\mathcal{X}).$$

However, $\xi = z \times (s_1 - s_0)$ and so item (vi) implies

$$\xi \cdot (Y \times D) = z \times mN\chi.$$

Consider the natural proper map

$$\pi : \mathcal{X} = Y \times_{k(B)} \mathcal{E} \longrightarrow Y$$

of $k(B)$-varieties. Then the above computations show that

$$\pi_*(\xi \cdot (Y \times D)) = mN \cdot z = \ell^r N \cdot z' \in \mathrm{CH}^i(Y)$$

for some $z' \in \mathrm{CH}^i(Y)$. Hence,

$$t := m \cdot z - \ell^r \cdot z' \in \mathrm{CH}^i(Y) \tag{5.1}$$

is N-torsion. Since m is coprime to ℓ, we find that the class of z in $\mathrm{CH}^i(Y)/\ell^r$ may be represented by a torsion class, as we want. This concludes the proof of the proposition. □

For later use, we record here the following variant of the above argument.

Proposition 5.2. *Let k be an arbitrary field. Let Y be a smooth projective k-variety. Then there is an elliptic curve E over $K = k(\mathbb{P}^1)$ (in fact, the Legendre elliptic curve $y^2 = x(x-1)(x-t)$) and a class $\tau \in \mathrm{CH}_0(E)$ of order 2 such that the exterior product map*

$$\mathrm{CH}^i(Y) \otimes \mathbb{Z}/2 \longrightarrow \mathrm{CH}^{i+1}(Y_K \times_K E)[2], \quad z \longmapsto [z_K \times \tau]$$

is injective.

Proof. We let E be the generic fibre of the Legendre family $y^2 = x(x-1)(x-t)$, where t denotes an affine coordinate on $\mathbb{P}^1$. Let $\mathcal{E} \to \mathbb{P}^1$ be a minimal elliptic surface whose generic fibre is E. The special fibre F above $t = 0$ is then of type I_2: $F = F_0 \cup F_1$ with $F_j^2 = -2$ and $F_0 \cdot F_1 = 2$. Note further that each 2-torsion point of E is K-rational (see [IG59]) and it extends to a section of $\mathcal{E} \to \mathbb{P}^1$. It follows, in particular, that there are sections s_0, s_1 of $\mathcal{E} \to \mathbb{P}^1$ such that s_j meets F_j and such that $s_1 - s_0$ restricts to a zero-cycle $\tau \in \mathrm{CH}_0(E)$ of order 2. Let $z \in \mathrm{CH}^i(Y)$ be nonzero modulo 2. Let $z_K \in \mathrm{CH}^i(Y_K)$ denote the base change of z. To prove the proposition, it is then enough to show that the 2-torsion class

$$z_K \times \tau \in \mathrm{CH}^{i+1}(Y_K \times_K E)$$

is nonzero. This follows by exactly the same argument as in the proof of Proposition 5.1 above. The main difference is that in the current situation, the special fibre F is of type I_2 and so the integer N in the proof of Proposition 5.1 equals 1, so that the class t in (5.1) is not only torsion, but in fact zero. □

Remark 5.3. Let C be a smooth irreducible projective curve over a field k and let ℓ be a prime. The proof of Proposition 5.2 shows more generally that if E is an elliptic curve over $K = k(C)$ whose ℓ-torsion points are all K-rational and, in addition, there is a K-rational point $0 \in C$ so that the fibre F of the minimal model $\mathcal{E} \to C$ above $t = 0$ is of type I_ℓ, then there is a class $\tau \in \mathrm{CH}_0(E)$ of order ℓ such that for any smooth projective k-variety Y, the exterior product map

$$\mathrm{CH}^i(Y) \otimes \mathbb{Z}/\ell \longrightarrow \mathrm{CH}^{i+1}(Y_K \times_K E)[\ell], \quad z \longmapsto [z_K \times \tau]$$

is injective.

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A concrete example for the prime $\ell = 3$ is given by the elliptic curve E over $\mathbb{Q}(\omega)(\mathbb{P}^1)$ ($\omega^3 = 1, \omega \neq 1$) defined by the degree 3 equation

$$x_0^3 + x_1^3 + x_2^3 = 3\lambda x_0 x_1 x_2$$

in $\mathbb{P}^2$; see [IG59].

6. A construction of Kollár, Hassett–Tschinkel and Totaro

In this section, we use some arguments of Kollár [Ko92], Hassett–Tschinkel (cf. [Tot13, Introduction]) and Totaro [Tot13] to prove the following.

Proposition 6.1. *Let k be an algebraically closed field and let ℓ be a prime invertible in k. If the characteristic of k is positive, assume that k has positive transcendence degree over its prime field. There is smooth projective hypersurface $S \subset \mathbb{P}^3_{k(\mathbb{P}^1)}$ such that the cycle class map*

$$\mathrm{CH}_0(S)/\ell \longrightarrow H^4(S, \mu_\ell^{\otimes 2})$$

is not injective up to torsion. That is, there is a zero-cycle $z \in \mathrm{CH}_0(S)$ with trivial cycle class in $H^4(S, \mu_\ell^{\otimes 2})$ such that $z \notin \mathrm{CH}_0(S)_{\mathrm{tors}} + \ell \cdot \mathrm{CH}_0(S)$.

Proof. Our examples arise due to the failure of the integral Hodge and Tate conjectures for certain hypersurfaces in $\mathbb{P}^1_k \times_k \mathbb{P}^3_k$ of bidegree $(3, \ell^2)$. To this end, let $\lambda \in k$ be transcendental over the prime field if k has positive characteristic and let $\lambda := \ell$ if k has characteristic zero. We then consider the smooth hypersurface $\mathcal{X} \subset \mathbb{P}^1_{u,t} \times \mathbb{P}^3_{x_0,x_1,x_2,x_3}$ over k, defined by the equation

$$u^3 x_0^{\ell^2} + tu^2 x_1^{\ell^2} + t^2 u x_2^{\ell^2} + t^3 x_3^{\ell^2} + \lambda(u^3 x_3^{\ell^2} - t^3 x_0^{\ell^2} + u^3 x_2^{\ell^2} - t^3 x_1^{\ell^2}) = 0.$$

Let $S := \mathcal{X}_\eta$ be the generic fibre of the first projection $\mathrm{pr}_1 \colon \mathcal{X} \to \mathbb{P}^1_k$. We aim to show that the cycle class map $\mathrm{cl}^2_S \colon \mathrm{CH}_0(S)/\ell \longrightarrow H^4(S, \mu_\ell^{\otimes 2})$ is not injective up to torsion.

Step 1. Let $n := \gcd\{\deg(z) | z \in \mathrm{CH}_0(S)\}$ be the index of S. If $\mathcal{Z}^4(\mathcal{X})$ denotes the cokernel of the cycle class map $\mathrm{cl}^2_{\mathcal{X}} \colon \mathrm{CH}^2(\mathcal{X})_{\mathbb{Z}_\ell} \to H^4(\mathcal{X}, \mathbb{Z}_\ell(2))$, then $\mathcal{Z}^4(\mathcal{X}) \simeq \mathbb{Z}_\ell/n\mathbb{Z}_\ell$.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^3(\mathbb{P}^1 \times_k \mathbb{P}^3)_{\mathbb{Z}_\ell} & \xrightarrow{\mathrm{cl}^3} & H^6(\mathbb{P}^1 \times_k \mathbb{P}^3, \mathbb{Z}_\ell(3)) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i_*} \\ \mathrm{CH}^2(\mathcal{X})_{\mathbb{Z}_\ell} & \xrightarrow{\mathrm{cl}^2_{\mathcal{X}}} & H^4(\mathcal{X}, \mathbb{Z}_\ell(2)), \end{array}$$

where $i_* \colon H^4(\mathcal{X}, \mathbb{Z}_\ell(2)) \to H^6(\mathbb{P}^1 \times_k \mathbb{P}^3, \mathbb{Z}_\ell(3))$ is an isomorphism by the Weak Lefschetz theorem [Mil80, Theorem VI.7.1]. Recall that $H^6(\mathbb{P}^1 \times_k \mathbb{P}^3, \mathbb{Z}_\ell(3))$ is the free $\mathbb{Z}_\ell$-module of rank 2 generated by $\ell_1 := \mathrm{cl}^3(\mathbb{P}^1_k \times pt)$ and $\ell_2 := \mathrm{cl}^3(pt \times L)$, where $L \subset \mathbb{P}^3_k$ is any one-dimensional linear subspace. Let $L \subset \mathbb{P}^3_k$ be the subspace defined by the equations $x_0 = 0$ and $x_3 = \zeta x_2$, where $\zeta \in k$ with $\zeta^{\ell^2} = -1$. An easy check shows $(1 : 0) \times L \subset \mathcal{X}$. The 1-cycle $\tilde{\ell}_2 := [(1 : 0) \times L] \in \mathrm{CH}^2(\mathcal{X})$ clearly then satisfies $i_* \mathrm{cl}^2_{\mathcal{X}}(\tilde{\ell}_2) = \ell_2$. Pick $z \in \mathrm{CH}^2(\mathcal{X})$ such that $\deg(z|_S) = n$. By replacing z with $z' := z - m\tilde{\ell}_2$, where the integer m is determined by $\mathrm{pr}_{2*}(z) = m[L]$ with $\mathrm{pr}_2 : \mathcal{X} \to \mathbb{P}^3$, we may assume that $\mathrm{pr}_{2*}(z) = 0$. It follows that $i_* \mathrm{cl}^2_{\mathcal{X}}(z) = n\ell_1$. To conclude, we notice that the image of the composite

$$\mathrm{CH}^2(\mathcal{X})_{\mathbb{Z}_\ell} \longrightarrow H^4(\mathcal{X}, \mathbb{Z}_\ell(2)) \xrightarrow{\mathrm{pr}_{1*}} H^0(\mathbb{P}^1_k, \mathbb{Z}_\ell(0)) = \mathbb{Z}_\ell$$

is $n\mathbb{Z}_\ell$, where n is the index of S. □

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Step 2. We have $n \in \ell\mathbb{Z}_\ell$.

Proof. The proof is essentially the same as in [Tot13, Theorem 2.1], where the case $\ell = 2$ and $k = \bar{\mathbb{Q}}$ is treated. We briefly recall the argument for the reader's convenience.

Note that $\mathcal{X}$ degenerates via $\lambda \to 0$ to the hypersurface

$$u^3 x_0^{\ell^2} + tu^2 x_1^{\ell^2} + t^2 u x_2^{\ell^2} + t^3 x_3^{\ell^2} = 0$$

in $\mathbb{P}^1 \times \mathbb{P}^3$ over some algebraically closed field extension $F/\bar{\mathbb{F}}_q$, where $q = \ell$ if $\mathrm{char}(k) = 0$ and $q = \mathrm{char}(k)$, otherwise. A straightforward specialization argument enables us to reduce our task to proving that the hypersurface

$$Y := \{x_0^{\ell^2} + tx_1^{\ell^2} + t^2 x_2^{\ell^2} + t^3 x_3^{\ell^2} = 0\}$$

in $\mathbb{P}^3$ over $F((t))$ has no rational point over any extension $F((s))$ of $F((t))$ whose degree is not divisible by ℓ. For a contradiction, we assume that there is an extension $F((s))/F((t))$ of degree d with $\gcd(d, \ell) = 1$ and Laurent series $t(s), x_i(s) \in F((s))$ satisfying the equation of Y. Then the valuations of the 4 terms in the equation, given by $\mathrm{ord}_s(t^i x_i^{\ell^2}) \equiv id \bmod \ell^2$, are all different, forcing all the $x_i(s)$ to be zero. This certainly cannot be a point of the projective space, contradicting our hypothesis. □

Finally, we are in the position to conclude the proof of Proposition 6.1. By Step 1, we can pick a 1-cycle $\alpha \in \mathrm{CH}^2(\mathcal{X})$ such that $\mathrm{cl}^2_{\mathcal{X}}(\alpha) = n\ell_1$. We set $z := \alpha|_S \in \mathrm{CH}^2(S)$ and note that $\mathrm{cl}^2_S(z) = 0 \in H^4(S, \mu_\ell^{\otimes 2})$ because $n \in \ell\mathbb{Z}_\ell$ by Step 2. We claim that $z \notin \mathrm{CH}^2(S)_{\mathrm{tors}} + \ell\,\mathrm{CH}^2(S)$. Indeed, if we write $z = z_1 + \ell z_2$ for some $z_1 \in \mathrm{CH}^2(S)_{\mathrm{tors}}$ and $z_2 \in \mathrm{CH}^2(S)$, then $n = \deg(z) = \ell \deg(z_2)$. This yields a contradiction, as the integer n is the index of S. □

7. Proof of the main results

Proof of Theorem 1.1. The injectivity of λ^i_X for $i = 1, 2$ follows from Theorem 4.2. The isomorphisms in question then follow from the description of the image of λ^i_X in Corollary 4.4. In the case $i = 1$, we note in addition that $N^0 H^i(X, A(n)) = H^i(X, A(n))$. □

Proof of Theorem 1.2. Let ℓ be a prime and let k be a field of characteristic different from ℓ. We aim to construct a finitely generated field extension K/k and a smooth projective threefold X over K such that

$$\lambda^3_X : \mathrm{CH}^3(X)[\ell^\infty] \longrightarrow H^5(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/M^5(X)$$

is not injective. To prove this, we are free to replace k by a finitely generated field extension and so we can assume that k has positive transcendence degree over its prime field. Lemma 4.6 allows us to reduce further to the case where k is algebraically closed and of positive transcendence degree over its prime field. It then follows from Proposition 6.1 that there is a smooth projective hypersurface $S \subset \mathbb{P}^3_{k(\mathbb{P}^1)}$ which carries a zero-cycle $z_1 \in \mathrm{CH}_0(S)$ with trivial cycle class in $H^4(S, \mu_\ell^{\otimes 2})$ such that

$$z_1 \notin \mathrm{CH}_0(S)_{\mathrm{tors}} + \ell \cdot \mathrm{CH}_0(S). \tag{7.1}$$

Let C be any smooth curve over k and let E be an elliptic curve over $k(C)$ whose j-invariant is transcendental over k (e.g., we could take $C = \mathbb{P}^1$ and let E be the Legendre elliptic curve). We then apply Proposition 5.1 to $C, E, B := \mathbb{P}^1$, and to the $k(B)$-variety $Y := S$. We let $K := k(B \times C)$ and find that up to replacing C by a finite cover, there is a zero-cycle $\tau \in \mathrm{CH}^1(E_K)$ of order ℓ such that the kernel of the exterior product map

$$\mathrm{CH}^2(S) \otimes \mathbb{Z}/\ell \longrightarrow \mathrm{CH}^3(S_K \times_K E_K)[\ell], \quad z \longmapsto [z_K \times \tau] \tag{7.2}$$

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is contained in the image of

$$\mathrm{CH}^2(S)_{\mathrm{tors}} \otimes \mathbb{Z}/\ell \longrightarrow \mathrm{CH}^2(S) \otimes \mathbb{Z}/\ell.$$

We finally set $X := S_K \times_K E_K$ and consider the zero-cycle

$$z := [z_{1K} \times \tau] \in \mathrm{CH}^3(X).$$

Since τ has order ℓ, the zero-cycle z is ℓ-torsion. By (7.1) and the above description of the kernel of the exterior product map in (7.2), we find that z is nontrivial. However, since the cycle class of z_1 in $H^4(S, \mu_\ell^{\otimes 2})$ vanishes while τ is ℓ-torsion, Lemma 4.8 implies that

$$\lambda_X^3(z) = 0 \in H^5(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/M^5(X).$$

This concludes the proof of the theorem. □

Proof of Corollary 1.3. One direction is Theorem 1.1. The other direction follows from Theorem 1.2 by taking products with projective spaces. □

Proof of Corollary 1.4. This follows from Lemma 4.7 and Theorem 1.2. □

Proof of Theorem 1.5. By [PS96, Theorem 8.5], there is a smooth projective surface Y (given as a conic fibration over a hyperelliptic curve) over a field k such that $Y(k) \neq \emptyset$ and such that the cycle class map $\mathrm{CH}^2(Y)/2 \to H^4(Y, \mu_2^{\otimes 2})$ is not injective. In *loc. cit.*, the example is defined over $\mathbb{Q}_3$; a straightforward limit argument then allows us to assume that $k \subset \mathbb{Q}_3$ is a finitely generated subfield. We pick a zero-cycle $z \in \mathrm{CH}^2(Y)$ that is nonzero modulo 2 but its cycle class $\mathrm{cl}_Y^2(z)$ is zero modulo 2. Let now $K = k(\mathbb{P}^1)$. Then Proposition 5.2 implies that there is an elliptic curve E over K (in fact, the Legendre elliptic curve) and a class $\tau \in \mathrm{CH}_0(E)$ of order 2 such that the cycle $z_K \times \tau \in \mathrm{CH}^3(X)[2]$ is nonzero, where X is the smooth projective threefold $Y_K \times_K E$. It follows from Lemma 4.8 that $\lambda_X^3(z_K \times \tau) = 0$, proving the noninjectivity of λ_X^3 for the prime $\ell = 2$. Since E is an elliptic curve, it has a rational point, and since Y has a rational point as well, we find that $X(K) \neq \emptyset$, as we want. □

Acknowledgements. We thank the referee for his or her useful comments. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 948066 (ERC-StG RationAlgic).

Competing interest. The authors have no competing interest to declare.

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