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Stability of Real C^*-Algebras

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Abstract. We will give a characterization of stable real C^*-algebras analogous to the one given for complex C^*-algebras by Hjelmborg and Rørdam. Using this result, we will prove that any real C^*-algebra satisfying the corona factorization property is stable if and only if its complexification is stable. Real C^*-algebras satisfying the corona factorization property include AF-algebras and purely infinite C^*-algebras. We will also provide an example of a simple unstable C^*-algebra, the complexification of which is stable.

1 Introduction

Let $\mathcal{K}$ be the real C^*-algebra consisting of compact operators on a real, infinite dimensional, separable Hilbert space. We say that a real C^*-algebra A is *stable* if it is isomorphic to $A \otimes \mathcal{K}$ (throughout this paper, tensor products will be over the real numbers unless otherwise indicated). As in the complex case, there is an isomorphism $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, implying that a real C^*-algebra A is stable if and only if it is isomorphic to $B \otimes \mathcal{K}$ for some real C^*-algebra B. In this paper, we investigate the relationship between the stability of a real C^*-algebra and the stability of its complexification $A_{\mathbb{C}} = \mathbb{C} \otimes A = A + iA$. The calculation

$$\mathbb{C} \otimes (A \otimes \mathcal{K}) = (\mathbb{C} \otimes A) \otimes_{\mathbb{C}} (\mathbb{C} \otimes \mathcal{K}) = A_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{K}_{\mathbb{C}}$$

shows that if A is stable as a real C^*-algebra, then $A_{\mathbb{C}}$ is stable as a complex C^*-algebra. The converse is not true in general, as we will see in Section 4.

However, we will present results which provide a converse for a large class of real C^*-algebras. More specifically, our main theorem states that any real C^*-algebra satisfying the corona factorization property is stable if the complexification is. We will prove that this class includes both real AF-algebras and real purely infinite C^*-algebras. These results appear in Section 3.

To facilitate the proofs of the results mentioned above, we will develop in Section 2 a set of equivalent characterizations of stability for real C^*-algebras as well as some permanence properties for the stable C^*-algebras that follow, which are analogous to those results in [8].

2 Characterization of Stability for Real C^*-Algebras

For a real C^*-algebra A, let $F(A)$ be the set of all positive elements in A that have a positive multiplicative identity in A; that is,

$$F(A) = \{a \in A_+ \mid ba = a \text{ for some } b \in A_+\}.$$

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For a and b in A, we write $a \sim b$ if a and b are Murray-von Neumann equivalent; that is, there exists x in A such that $x^*x = a$ and $xx^* = b$. Also, we write $a \perp b$ if $ab = ba = 0$.

Theorem 2.1 *Let A be a real σ-unital C^*-algebra. Then the following are equivalent:*

(S0) *A is stable.*

(S1) *For all $a \in F(A)$, there exists $b \in A_+$ such that $a \sim b$ and $a \perp b$.*

(S2) *For all $a \in F(A)$ and for every $\epsilon > 0$, there exist $b, c \in A_+$ such that $\|a - b\| < \epsilon$, $b \sim c$ and $\|bc\| < \epsilon$.*

(S3) *For all $a \in F(A)$ and for every $\epsilon > 0$, there exists a unitary $u \in \widetilde{A}$ such that $\|auau^*\| < \epsilon$.*

(S4) *There is a sequence $\{e_n\}_{n=1}^{\infty}$ of mutually orthogonal and equivalent projections in $\mathcal{M}(A)$ such that the infinite sum $\sum_{n=1}^{\infty} e_n$ converges to $1_{\mathcal{M}(A)}$ in the strict topology on $\mathcal{M}(A)$.*

(S5) *For all $a \in A_+$ and all $\epsilon > 0$, there exists $b \in A_+$ such that $\|ab\| < \epsilon$ and $a \sim b$.*

And if A has a countable approximate identity consisting of projections then each condition above is equivalent to the following:

(S6) *For all projections $p \in A$ there is a projection $q \in A$ such that $p \sim q$ and $p \perp q$.*

These characterizations are identical to those given in [8], except for (S3), which is a weaker version of their Proposition 2.2(d). Note that the proof of (c) $\Rightarrow$ (d) in [8, p. 157] uses the complex structure in an essential way. We will prove Theorem 2.1 in stages in the rest of this section, referring to [8] when the proofs are the same.

First we will present some corollaries, including the following collection of permanence properties.

Corollary 2.2 *Suppose that A is a real C^*-algebra.*

(P1) *If A is σ-unital and is an inductive limit of stable σ-unital real C^*-algebras, then A is stable.*

(P2) *If A is stable, then so is every ideal in A and every quotient of A.*

(P3) *If B is a stable real sub-C^*-algebra of A containing an approximate identity for A, then A is stable.*

(P4) *If A is a σ-unital and stable and if G is a countable discrete group acting on A, then $A \rtimes_\alpha G$ is stable.*

(P5) *If A is a σ-unital and stable and $a \in A$ is positive with norm at most 1, then $\overline{(1-a)A(1-a)}$ is stable.*

Proof The proofs of (P1) and (P2) are the same as in the complex case; see [8, Corollary 4.1] and [14, Corollary 2.3(ii)]. We will provide a proof of (P3) below, and the proof of (P4) follows from (P3) as in [8, Corollary 4.5].

For (P3), we show that A satisfies condition (S3) if B does. Let a be in $F(A)$, and let ϵ be a positive number. Since B has an approximate identity that is an approximate identity of A and since $F(B)$ is dense in B_+, there exists e in $F(B)$ with $\|e\| = 1$ such that $2\,\|a\|\,\|a - ae\| < \frac{\epsilon}{2}$. Then there exists a unitary w in $\widetilde{B}$ such that

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$(\|a\|^2 + 1)\,\|ewew^*\| < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned}\|awaw^*\| &\leq \|awaw^* - aewaw^*\| + \|aewaw^* - aeweaw^*\| + \|aeweaw^*\|\\ &\leq \|a - ae\|\,\|a\| + \|a\|\,\|a - ae\| + \|a\|\,\|ewe\|\,\|a\|\\ &= 2\,\|a - ae\|\,\|a\| + \|a\|^2\,\|ewew^*\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

For (P5), let $B = \overline{(1-a)A(1-a)}$ and let $b \in F(B) \subseteq F(A)$. Approximating $a + b$ with elements in $F(A)$ and using condition (S3), we obtain a sequence of unitaries $u_n \in \widetilde{A}$ such that $\lim_{n\to\infty} \|u_n(a+b)u_n^*(a+b)\| = 0$. The rest of the proof proceeds as in [8, Corollary 4.3]. ■

Recall that any complex C^*-algebra A can be considered a real C^*-algebra by forgetting the structure of complex scalar multiplication. Since the characterization of (S5) is the same as that given in [8] and makes no reference to the field of scalars used, we immediately have the following corollary.

Corollary 2.3 *Let A be a σ-unital complex C^*-algebra. Then $A \otimes_{\mathbb{C}} \mathcal{K}_{\mathbb{C}}$ is isomorphic to A as complex C^*-algebras if and only if $A \otimes \mathcal{K}$ is isomorphic to A as real C^*-algebras.*

Proof of equivalence of (S1), (S2), and (S3) The statement (S1) $\Rightarrow$ (S2) is immediate. The proofs of [8, Lemmas 2.3 and 2.5] work the same way in the real case to prove (S2) $\Rightarrow$ (S3). Also, the proof of (S3) $\Rightarrow$ (S1) is the same as the proof of the (b) $\Rightarrow$ (c) implication of [8, Proposition 2.2], including Lemma 2.4. ■

We define an element a in a real C^*-algebra A to be *strictly positive* if it is strictly positive in the complexification $A_{\mathbb{C}}$, that is, if $\phi(a) > 0$ for every nonzero positive linear functional ϕ on $A_{\mathbb{C}}$. As in [8] we define

$$f_\epsilon(x) = \begin{cases} 0 & 0 \leq t \leq \epsilon \\ \epsilon^{-1}t - 1 & \epsilon \leq t \leq 2\epsilon \\ 1 & 2\epsilon \leq t, \end{cases}$$

and for every strictly positive element a we define

$$F_a(A) = \{b \in A_+ \mid \exists \epsilon > 0 : f_\epsilon(a)b = b\}.$$

If $a \in A$ is strictly positive, then the sequence $\{f_{1/n}(a)\}$ is an approximate identity for A. Indeed, as in [11, Proposition 3.10.5] it is an approximate identity for $A_{\mathbb{C}}$. Also, note that if $\{e_n\}$ is an approximate identity for A, then $a = \sum_{n=1}^{\infty} e_n/2^n$ is a strictly positive element of A (again see [11, Proposition 3.10.5]).

Lemma 2.4 *Let A be a σ-unital real C^*-algebra that satisfies* (S1). *Then the following hold for every strictly positive element a:*

(i) *For all $b \in F_a(A)$, there exists $c \in F_a(A)$ with $b \sim c$ and $b \perp c$.*
(ii) *For all $\epsilon > 0$, there exists a projection $p \in \mathcal{M}(A)$ satisfying $1 - p \perp f_\epsilon(a)$, $p \sim 1$, and $1 - p \gtrsim 1$.*

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Proof If a is a positive element in a real C^*-algebra, then $\overline{aAa}$ is hereditary. (Although the hereditary C^*-algebra generated by a may be strictly smaller. In the language of [16], $\overline{aAa}$ is the regular hereditary C^*-algebra generated by a.) With this in mind, the proof is the same as the proof of [8, Lemma 2.6]. ∎

Proof of equivalence of (S0), (S1), and (S4) Using Lemma 2.4, we can use the same argument as in the proof of [8, Theorem 2.1]. ∎

Proof of equivalence of (S0) and (S5) Condition (S5) directly implies (S2). For the other direction, let a be a positive element of a stable C^*-algebra A, and let $\epsilon > 0$. We may assume that $\|a\| \leq 1$ and $a \neq 0$. Also, we may assume that $\epsilon < 1$. Since $F(A)$ is dense in A_+, there exists a_0 in $F(A)$ such that $\|a - a_0\| < \frac{\epsilon}{8}$.

By (S3), there exists a unitary u in $\widetilde{A}$ such that $\|a_0 u a_0 u^*\| < \frac{\epsilon}{2}$. Set $x = ua^{\frac{1}{2}}$ and set $b = xx^* = uau^*$. Then $a \sim b$ and

$$\begin{aligned} \|ab\| &\leq \|a_0ua_0u^*\| + \|ab - aua_0u^*\| + \|aua_0u^* - a_0ua_0u^*\| \\ &\leq \tfrac{\epsilon}{2} + \|a\|\,\|a - a_0\| + \|a - a_0\|\,\|a_0\| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{8} + \tfrac{\epsilon}{8}\big(1 + \tfrac{\epsilon}{8}\big) < \epsilon. \end{aligned}$$ ∎

Proof of equivalence of (S0) and (S6) This is proved in the same was as in Section 3 of [8]. ∎

3 The Corona Factorization Property

As in [10], we define the corona factorization property for a real C^*-algebra as follows. A projection is *norm-full* in A if the only ideal in A containing p is A itself.

Definition 3.1 A real C^*-algebra A has the *corona factorization property* if every norm-full projection p in the multiplier algebra $\mathcal{M}(A \otimes \mathcal{K})$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(A\otimes\mathcal{K})}$.

Note that A has the corona factorization property if and only if $A \otimes \mathcal{K}$ has the corona factorization property. The goal of this section is to prove the following theorems. Recall from [16] that a real C^*-algebra A is *purely infinite* if the subalgebra $\overline{aAa}$ contains an infinite projection for every $a \in A^+$.

Theorem 3.2 *Let A be a real C^*-algebra with the corona factorization property. Then A is stable if and only if $A_{\mathbb{C}}$ is stable.*

Theorem 3.3 *The following classes of real C^*-algebras satisfiy the corona factorization property:*

(i) *AF-algebras*
(ii) *separable, simple, purely infinite C^*-algebras A such that $A_{\mathbb{C}}$ is also purely infinite.*

Corollary 3.4 *A real C^*-algebra A in either of the classes mentioned above is stable if and only if $A_{\mathbb{C}}$ is stable.*

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Peter Stacey has pointed out that in the case of AF-algebras, the same result can be obtained via a K-theoretic argument using scale.

We note that Zhang's Dichotomy for real purely infinite C^*-algebras follows using Corollary 3.4 from the same result in the complex case ([19, Theorem 1.2]) with the additional assumption that $A_{\mathbb{C}}$ is purely infinite. Unfortunately, the question of whether or not the complexification of a purely infinite real C^*-algebra is purely infinite is open (see [16] for partial results).

However, we note that simply by repeating the argument of [1, Section 27.5] we can prove Zhang's Dichotomy in full generality.

Theorem 3.5 (Zhang's Dichotomy) *Every real σ-unital, simple, and purely infinite C^*-algebra A is either unital or stable.*

Following Cuntz in [7], we consider the following conditions that can be placed on a set $\mathcal{P}$ of projections of a real C^*-algebra A.

(Π_1) If p and q are in $\mathcal{P}$ and p is orthogonal to q, then $p + q$ are in $\mathcal{P}$.
(Π_2) If p is an element of $\mathcal{P}$ and q is a projection of A such that $p \sim q$, then q is an element of $\mathcal{P}$.
(Π_3) For all p and q in $\mathcal{P}$, there exists e in $\mathcal{P}$ such that $p \sim e$, $e < q$, and $q - e$ is an element of $\mathcal{P}$.
(Π_4) If p and q are projections in A with $p \leq q$ and $p \in \mathcal{P}$, then $q \in \mathcal{P}$.

If p is a projection, then $[p]$ will denote the Murray–von Neumann equivalence class represented by p. Using similar techniques as in [7, Theorem 1.4] one can show that the following holds.

Theorem 3.6 *Let A be a real C^*-algebra with a non-empty subset $\mathcal{P}$ of projections in A satisfying* (Π_1), (Π_2), *and* (Π_3) *above. Then $G = \{[p] : p \in \mathcal{P}\}$ is a group with the natural addition $[p] + [q] = [p' + q']$, where p' and q' are elements of $\mathcal{P}$ chosen such that $p \sim p'$, $q \sim q'$, and p' is orthogonal to q'. Moreover, if A is a unital real C^*-algebra and in addition $\mathcal{P}$ satisfies* (Π_4)*, then the obvious map from G to $K_0(A)$ is a group isomorphism.*

As in the complex case, we say that a projection p in a real C^*-algebra A is *properly infinite* if there are orthogonal projections p_1 and p_2 with $p_1 \sim p_2 \sim p$.

Lemma 3.7 *If the set $\mathcal{P}$ of properly infinite norm-full projections in a real C^*-algebra A is nonempty, then it satisfies* (Π_1), (Π_2), (Π_3), *and* (Π_4).

Proof (Π_1) Suppose p and q are elements of $\mathcal{P}$ and p is orthogonal to q. Then there exist projections $p_1, p_2 \leq p$ and $q_1, q_2 \leq q$ such that p_1 is orthogonal to p_2, q_1 is orthogonal to q_2, $p_1 \sim p_2 \sim p$, and $q_1 \sim q_2 \sim q$.

Set $r_1 = p_1 + q_1$, $r_2 = p_2 + q_2$, and $r = p + q$. A computation shows that r_1 is orthogonal to r_2, $r_1 \sim r_2 \sim r$, and $r_1, r_2 \leq r$.

(Π_2) Let p be an element of $\mathcal{P}$ and p' be a projection such that $p \sim q$. Since p is an element of $\mathcal{P}$, there exist orthogonal projections p_1 and p_2 such that $p_1, p_2 \leq p$ and $p_1 \sim p_2 \sim p$. Since $p \sim q$, there exists v in A such that $vv^* = p$ and $v^*v = q$.

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Let $q_1 = v^* p_1 v$ and $q_2 = v^* p_2 v$. A computation shows that q_1 is orthogonal to q_2, that $q_1 \sim q_2 \sim q$, and that $q_1, q_2 \leq q$.

(Π_4) Let p be an element of $\mathcal{P}$ and let q be a projection such that $p \leq q$. Since p is norm-full in A, there is a positive integer n such that we have

$$q \sim q' \leq \operatorname{diag}(p, \ldots, p)$$

in $M_n(A)$. Since p is an element of $\mathcal{P}$, there exist n mutually orthogonal projections $p_1, \ldots, p_n$ in pAp all of which are Murray–von Neumann equivalent to p. Thus $\operatorname{diag}(p, \ldots, p) \sim p_1 + \cdots + p_n \leq p$. Therefore, $q \precsim p$, *i.e.*, q is Murray–von Neumann equivalent to a subprojection of p. Since p is an element of $\mathcal{P}$, there exist orthogonal projections p_1 and p_2 in pAp such that $p_1 \sim p_2 \sim p$. Therefore, $q \precsim p_1 \leq p$ and $q \precsim p_2 \leq p$. Hence, there exist orthogonal projections q_1 and q_2 in $pAp \subset qAq$ such that $q_1 \sim q_2 \sim q$. Since $p = pq$ and p is norm-full in A, q is norm-full in A. Hence, q is an element of $\mathcal{P}$.

(Π_3) Let p and q be elements of $\mathcal{P}$. As in the proof of (Π_4), we have that $p \precsim q$. Therefore, $p \sim e \leq q$ for some projection e, which is in $\mathcal{P}$ by (Π_2). Therefore, there exist orthogonal projections e_1 and e_2 such that $e_1 \sim e_2 \sim e$ and $e_1, e_2 < e$. Note that $e_1 + e_2 \leq e \leq q$. So, $e_2 \leq q - e_1 < q$. Since $e \sim e_2$, by (Π_2) we have that e_2 is an element of $\mathcal{P}$. By (Π_4), we have that $q - e_1$ is an element of $\mathcal{P}$. ■

Proposition 3.8 *Let A be a real C^*-algebra and let p and q be norm-full, properly infinite, projections of A. Then $[p]_0 = [q]_0$ if and only if p is Murray–von Neumann equivalent to q.*

Proof By (Π_3), we may assume that p and q are orthogonal. Let $r = p + q$. The projection r is norm-full, so by [2, Proposition 9], we have $[p]_0 = [q]_0$ in $K_0(A)$ if and only if $[p]_0 = [q]_0$ in $K_0(rAr)$. By Theorem 3.6 and Lemma 3.7, this holds if an only if p and q are Murray–von Neumann equivalent. ■

Lemma 3.9 *Let A be a real C^*-algebra. Then $\mathcal{M}(A \otimes \mathcal{K})$ contains a unital copy of $\mathcal{O}_n^{\mathbb{R}}$ for all n.*

Proof There is a faithful representation of the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$ on a separable real Hilbert space (see [15, p. 4]). Therefore, $B(H) = \mathcal{M}(\mathcal{K})$ contains a unital copy of $\mathcal{O}_n^{\mathbb{R}}$. Then the unital embedding $\mathcal{M}(\mathcal{K}) \hookrightarrow \mathcal{M}(A \otimes \mathcal{K})$ completes the proof. ■

Proposition 3.10 ([5, Theorem 4.23]) *Let A be a stable real C^*-algebra. Then a projection p in $\mathcal{M}(A)$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(A)}$ if and only if pAp is a norm-full, stable sub-C^*-algebra of A.*

Proof It is clear that if p is Murray–von Neumann equivalent to $1_{\mathcal{M}(A)}$, then pAp is norm-full in A and stable. Conversely, suppose pAp is norm-full and stable in A. Then $(pAp)_{\mathbb{C}} \cong pA_{\mathbb{C}}p$ is norm-full in $A_{\mathbb{C}}$ and stable. Then by [5, Theorem 4.23], the projection p is Murray–von Neumann equivalent to $1_{\mathcal{M}(A_{\mathbb{C}})}$ and hence is norm-full in $\mathcal{M}(A_{\mathbb{C}}) \cong \mathcal{M}(A)_{\mathbb{C}}$. It follows that p is norm-full in $\mathcal{M}(A)$.

Now Lemma 3.9 implies that p is properly infinite, since it is the unit of $\mathcal{M}(pAp) \cong p\mathcal{M}(A)p$. As $K_0(\mathcal{M}(A)) = 0$ by [2, Theorem 4], Proposition 3.8 implies that p is Murray–von Neumann equivalent to $1_{\mathcal{M}(A)}$. ■

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Proof of Theorem 3.2. The "only if" direction is clear. We prove the "if" direction. Suppose $A_{\mathbb{C}}$ is stable as a complex C^*-algebra. Then by Corollary 2.3, $A_{\mathbb{C}}$ is stable as a real C^*-algebra. Note that $A_{\mathbb{C}}$ can be embedded into $\mathrm{M}_2(A)$ such that there exists an approximate identity of $A_{\mathbb{C}}$ that is an approximate identity of $\mathrm{M}_2(A)$. Therefore, by Corollary 2.2(P3), $\mathrm{M}_2(A)$ is stable.

We claim that $\mathrm{M}_2(A)$ stable implies A stable. Let $\{e_{ij}\}_{i,j=1}^{\infty}$ be the standard system of matrix units of $\mathcal{K}$, and let $p = 1_{\mathcal{M}(A)} \otimes e_{11} \in \mathcal{M}(A) \otimes \mathcal{K}$. Then $A \cong p(A \otimes \mathcal{K})p$ is a norm-full hereditary real sub-C^*-algebra of $A \otimes \mathcal{K}$. By Lemma 3.9, there are isometries s_1 and s_2 in $\mathcal{M}(A \otimes \mathcal{K})$ such that $s_1 s_1^* + s_2 s_2^* = 1_{\mathcal{M}(A)}$. Letting $q = s_1 p s_1^* + s_2 p s_2^*$, we have $q(A \otimes \mathcal{K})q \cong \mathrm{M}_2(A)$. Since $p(A \otimes \mathcal{K})p$ is a norm-full real sub-C^*-algebra of $A \otimes \mathcal{K}$, so is $q(A \otimes \mathcal{K})q$. Therefore, by Proposition 3.10, q is Murray–von Neumann equivalent to $1_{\mathcal{M}(A\otimes\mathcal{K})}$. Hence, q is a norm-full projection of $\mathcal{M}(A \otimes \mathcal{K})$. Hence, p is a norm-full projection of $\mathcal{M}(A \otimes \mathcal{K})$. Since $A \otimes \mathcal{K}$ satisfies the corona factorization property, p is Murray–von Neumann equivalent to $1_{\mathcal{M}(A\otimes\mathcal{K})}$. Hence $A = p(A \otimes \mathcal{K})p \cong A \otimes \mathcal{K}$. ■

We now turn to the proof that real AF-algebras satisfy the corona factorization property. We begin by considering the case of a complex AF-algebra. The following result is closely related to, but does not exactly fall under the scope of, [10, Proposition 2.1].

Lemma 3.11 *Let A be a stable complex AF-algebra. Then for every norm-full projection $p \in \mathcal{M}(A)$, p is Murray–von Neumann equivalent to $1_{\mathcal{M}(A)}$.*

Proof Let p be a norm-full projection in $\mathcal{M}(A)$. We will first show that pAp is stable. Since it is AF, it is enough to show that pAp has no bounded trace by [14, Proposition 3.4].

Suppose on the contrary that pAp has a bounded trace. Since p is a norm-full projection in $\mathcal{M}(A)$, pAp is full in A. By [4, Corollary 2.6], $pAp \otimes \mathcal{K}$ is isomorphic to A. Hence, we can extend τ to a lower-semicontinuous trace on A, which in turn extends to a trace $\widetilde{\tau}$ on $\mathcal{M}(A)_+$. Note that $\widetilde{\tau}(1_{\mathcal{M}(A)}) = \infty$, since otherwise $\widetilde{\tau}$ restricts to a bounded trace on A, contradicting the fact that A is stable.

Since p is a norm-full projection in $\mathcal{M}(A)$, there exist $x_1, \dots, x_n \in \mathcal{M}(A)$ such that

$$1_{\mathcal{M}(A)} = \sum_{k=1}^{n} x_k p x_k^* .$$

Hence,

$$\widetilde{\tau}(1_{\mathcal{M}(A)}) \leq \widetilde{\tau}(p) \sum_{k=1}^{n} \|x_k x_k^*\| ,$$

which implies that $\widetilde{\tau}(p) = \infty$.

Since A is a complex AF-algebra, there exists an approximate identity $\{e_n\}_{n\in\mathbb{N}}$ of A consisting of projections.

Since τ is a bounded trace on pAp,

$$|\tau(pe_n p)| \leq \|\tau\| \, \|pe_n p\| \leq \|\tau\| \, \|p\| .$$

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Hence,

$$\widetilde{\tau}(p) = \limsup_{n\to\infty} \tau(e_n p e_n) = \limsup_{n\to\infty} \tau(p e_n p) \leq \|\tau\| \, \|p\| < \infty,$$

which is a contradiction to the fact that $\widetilde{\tau}(p) = \infty$.

We have just shown that pAp is a stable complex AF-algebra. Therefore, $\mathcal{M}(pAp) \cong p\mathcal{M}(A)p$ is properly infinite. Hence, p is a norm-full, properly infinite projection in $\mathcal{M}(A)$. Thus, p is Murray–von Neumann equivalent to $1_{\mathcal{M}(A)}$. ■

Lemma 3.12 *Let A be a real AF-algebra and p, q be projections in A. Then $[p] < [q]$ holds in $K_0(A)$ if and only if $[p] < [q]$ in $K_0(A_{\mathbb{C}})$.*

Consequently, since A and $A_{\mathbb{C}}$ both have cancellation, p is Murray–von Neumann equivalent to a proper sub-projection of q in A if and only if p is Murray–von Neumann equivalent to a proper sub-projection of of q in $A_{\mathbb{C}}$.

Proof We have $K_{-1}(A) = 0$ for any finite dimensional real C^*-algebra, so the same is true for any real AF-algebra. Thus the exact sequence ([15, Theorem 1.4.7])

$$K_{n-1}(A) \to K_n(A) \to K_n(A_{\mathbb{C}}) \to K_{n-2}(A) \to K_{n-1}(A)$$

implies that the inclusion $A \hookrightarrow A_{\mathbb{C}}$ induces a monomorphism on the ordered groups $(K(A), K^+(A)) \to (K(A_{\mathbb{C}}), K^+(A_{\mathbb{C}}))$. ■

Proof of Theorem 3.3(i) Let A be a real AF-algebra, which we may assume is stable. Let p be a norm-full projection in $\mathcal{M}(A) \subset \mathcal{M}(A_{\mathbb{C}})$. Since p is a norm-full projection in $\mathcal{M}(A)$, we have that p is a norm-full projection in $\mathcal{M}(A_{\mathbb{C}}) = \mathcal{M}(A)_{\mathbb{C}}$. Hence, by Lemma 3.11, p is Murray–von Neumann equivalent in $\mathcal{M}(A_{\mathbb{C}})$ to $1_{\mathcal{M}(A_{\mathbb{C}})} = 1_{\mathcal{M}(A)}$.

Let $\{e_n\}_{n\in\mathbb{N}}$ and $\{p_n\}_{n\in\mathbb{N}}$ be sequences of finite rank orthogonal projections in A such that

$$\sum_{n=1}^{\infty} e_n = 1_{\mathcal{M}(A)} \quad \text{and} \quad \sum_{n=1}^{\infty} p_n = p$$

where the sums converge in the strict topology of $\mathcal{M}(A)$, hence also in the strict topology of $\mathcal{M}(A_{\mathbb{C}})$.

Set $q_n = \sum_{k=1}^{n} p_k$. We will inductively construct a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that e_k is Murray–von Neumann equivalent to a proper projection of $q_{n_k} - q_{n_{k-1}}$. Since $e_1 \leq 1_{\mathcal{M}(A)}$ and $1_{\mathcal{M}}(A) \sim p$ in $\mathcal{M}(A_{\mathbb{C}})$, there exists n_1 and a unitary u in the unitization of $A_{\mathbb{C}}$ such that $ue_1u^* < q_{n_1}$. By Lemma 3.12, e_1 is Murray–von Neumann equivalent to a proper sub-projection of q_{n_1} in A.

Consider the projections $1_{\mathcal{M}(A)} - e_1$ and $p - q_{n_1}$. A computation shows that these projections are norm-full projections. Hence, by Lemma 3.11, $1_{\mathcal{M}(A)} - e_1 \sim 1_{\mathcal{M}(A)} \sim p - q_{n_1}$ in $\mathcal{M}(A_{\mathbb{C}})$. Since e_2 is a sub-projection of $1_{\mathcal{M}(A)} - e_1$, there exists $n_2 > n_1$ and a unitary v such that $ve_2v^* < q_{n_2} - q_{n_1}$. Thus, by Lemma 3.12, e_2 is Murray–von Neumann equivalent to a proper sub-projection of $q_{n_2} - q_{n_1}$ in A.

We now continue this process by considering the norm-full projections $1_{\mathcal{M}(A)} - e_1 - e_2$ and $p - q_{n_2}$. To get $n_3 > n_2$ such that e_3 is Murray–von Neumann equivalent

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to a proper sub-projection of $q_{n_3} - q_{n_2}$. Continuing this process we get the desired sequence $\{n_k\}_{k\in\mathbb{N}}$.

Let $v_k \in A$ such that $v_k^* v_k = e_k$ and $v_k v_k^* \leq q_{n_k} - q_{n_{k-1}}$. Set $v = \sum_{k=1}^{\infty} v_k$, where the sum converges in the strict topology of $\mathcal{M}(A)$. Since $\{e_k\}_{k\in\mathbb{N}}$ and $\{q_{n_k} - q_{n_{k-1}}\}_{k\in\mathbb{N}}$ are collections of mutually orthogonal projections, we have that

$$v^* v = \sum_{k=1}^{\infty} e_k = 1_{\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes A)} \quad \text{and} \quad vv^* \leq \sum_{k=1}^{\infty} (q_{n_k} - q_{n_{k-1}}) = \sum_{k=1}^{\infty} p_k = p$$

Therefore, p is a norm-full properly infinite projection in $\mathcal{M}(A)$. Hence, p is Murray–von Neumman equivalent to $1_{\mathcal{M}(A)}$. ■

Finally, we work toward the proof that a real purely infinite C^*-algebra whose complexification is also purely infinite satisfies the corona factorization property. As observed by Stacey in [16], the proof in the complex case of the equivalence of the properties of real rank zero, FS, and HP (in [6]) carries over to give us the same theorem in the real case. Also, the proof of the following is the same as in the complex case in [6].

Proposition 3.13 *Let A be a real C^*-algebra. For p in $\mathcal{M}(A)$, $RR(A) = 0$ if and only if $RR(pAp) = 0$ and $RR((1_{\mathcal{M}(A)} - p)A(1_{\mathcal{M}(A)} - p)) = 0$.*

Proposition 3.14 *Let A be a σ-unital real C^*-algebra with real rank zero. Then there exists a sequence of pairwise orthogonal projections $\{p_k\}_{k=1}^{\infty}$ in A such that $\{\sum_{k=1}^{n} p_k\}_{n=1}^{\infty}$ is an approximate identity of A consisting of projections.*

Consequently, $\sum_{k=1}^{\infty} p_k = 1_{\mathcal{M}(A)}$ where the sum converges in the strict topology.

Proof Let $\{e_n\}_{n=1}^{\infty}$ be an approximate identity of A. Since $RR(A) = 0$, every hereditary sub-C^*-algebra has an approximate identity consisting of projections (property (HP)). So there exists a projection p_1 in A such that $\|(1_{\mathcal{M}(A)} - p_1)e_1\| < 1$. Repeating the argument in the corner algebra $(1_{\mathcal{M}(A)} - p_1)A(1_{\mathcal{M}(A)} - p_1)$, there exists a projection p_2 orthogonal to p_1 such that

$$\big\|(1_{\mathcal{M}(A)} - p_2)(1_{\mathcal{M}(A)} - p_1)e_2^2(1_{\mathcal{M}(A)} - p_1)\big\| < \left(\tfrac{1}{2}\right)^2.$$

So, $\|(1_{\mathcal{M}(A)} - (p_1 + p_2))e_2\| < \frac{1}{2}$. Repeating this same process, we inductively define pairwise orthogonal projections $\{p_n\}_{n=1}^{\infty}$ such that

$$\left\| \left(1_{\mathcal{M}(A)} - \sum_{k=1}^{n} p_k \right) e_n \right\| < \tfrac{1}{n}.$$

A computation shows that $\{\sum_{k=1}^{n} p_k\}_{n=1}^{\infty}$ is an approximate identity of A. ■

Lemma 3.15 *Let A be a σ-unital, non-unital, real C^*-algebra with real rank zero. If p is a projection in $\mathcal{M}(A)$, then there exists a sequence of pairwise orthogonal projections $\{e_n\}_{n=1}^{\infty}$ in A such that $p = \sum_{n=1}^{\infty} e_n$, where the sum converges in the strict topology.*

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Proof By Proposition 3.13, pAp is a real C^*-algebra with real rank zero. By Proposition 3.14, there exists a sequence of pairwise orthogonal projections $\{e_n\}_{n=1}^{\infty}$ in pAp such that $\sum_{k=1}^{\infty} e_k = 1_{\mathcal{M}(pAp)}$ in the strict topology of $\mathcal{M}(pAp)$.

Set $q_n = \sum_{k=1}^{n} e_k$ and note that $pq_n = q_n p = q_n$ for all n. For any element $a \in A$ and any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|pa^*ap(q_n - p)\| < \epsilon^2.$$

Then we get that

$$\begin{aligned}\|aq_n - ap\|^2 &= \|(q_n - p)a^*a(q_n - p)\| = \|(q_n - p)pa^*ap(q_n - p)\| \\ &\leq \|pa^*ap(q_n - p)\| < \epsilon^2.\end{aligned}$$

Therefore, $aq_n \to ap$ for all $a \in A$, where the convergence is in the norm of A. By a similar argument or by applying adjoints, we get $q_n a \to pa$ for all $a \in A$. Hence $\sum_{k=1}^{\infty} e_k = p$ in the strict topology of $\mathcal{M}(A)$. ∎

Proof of Theorem 3.3(ii) Assume that A is simple and stable and that $A_{\mathbb{C}}$ is purely infinite. We first consider the case in which $A_{\mathbb{C}}$ is also simple.

Let p be a norm-full projection in $\mathcal{M}(A \otimes \mathcal{K})$. Let π denote the canonical projection from $\mathcal{M}(A \otimes \mathcal{K})$ onto $\mathcal{Q}(A \otimes \mathcal{K})$. We will first show that $\pi(p)$ is an infinite projection in $\mathcal{Q}(A \otimes \mathcal{K})$.

Since $A \otimes \mathcal{K}$ is a real C^*-algebra with real rank zero, by Lemma 3.15 there exists a sequence of pairwise orthogonal projections $\{e_k\}_{k=1}^{\infty}$ in $A \otimes \mathcal{K}$ such that $p = \sum_{k=1}^{\infty} e_k$, where the convergence of the sum is in the strict topology. Since $A \otimes \mathcal{K}$ is a real, simple, purely infinite C^*-algebra, there exists a sequence of pairwise orthogonal projections $\{f_k\}_{k=1}^{\infty}$ of $A \otimes \mathcal{K}$ such that

(i) f_k is Murray–von Neumann equivalent to e_{k+1} and
(ii) f_k is a proper subprojection of e_k

for all k in $\mathbb{N}$.

Choose v_k in $A \otimes \mathcal{K}$ such that $v_k v_k^* = e_{k+1}$ and $v_k^* v_k = f_k$. Set $v = \sum_{k=1}^{\infty} v_k$. A computation shows that the sum converges in the strict topology and that

$$vv^* = \sum_{k=1}^{\infty} v_k v_k^* = \sum_{k=1}^{\infty} e_{k+1} = p - e_1 \text{ and}$$

$$v^*v = \sum_{k=1}^{\infty} v_k^* v_k = \sum_{k=1}^{\infty} f_k = p - \sum_{k=1}^{\infty}(e_k - f_k).$$

Since $e_k - f_k$ are nonzero projections for all k, the sum $\sum_{k=1}^{\infty}(e_k - f_k)$ is not an element of $A \otimes \mathcal{K}$. Therefore, $\pi(vv^*) = \pi(p)$ and

$$\pi(p - v^*v) = \pi\Big(\sum_{k=1}^{\infty}(e_k - f_k)\Big) \neq 0.$$

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We have just shown that $\pi(p)$ is an infinite projection.

By [16, Theorem 4.4], $\mathcal{Q}(A \otimes \mathcal{K})$ is a simple, real C^*-algebra (here we make use of the assumption that $A_{\mathbb{C}}$ is simple). We show that it is also purely infinite. Indeed $\mathcal{Q}(A_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{K}_{\mathbb{C}}) \cong \mathcal{Q}(A \otimes \mathcal{K})_{\mathbb{C}}$ is purely infinite by [18, Theorem 1.3] (here we require the assumption that $A_{\mathbb{C}}$ is both simple and purely infinite). So, by [16, Theorem 3.3] (together with [3]), $\mathcal{Q}(A \otimes \mathcal{K})$ is purely infinite. By [16, Proposition 4.1], there exists x in $\mathcal{M}(A \otimes \mathcal{K})$ such that $\pi(xpx^*) = 1_{\mathcal{Q}(A\otimes\mathcal{K})}$.

It is easy to see that for every y in $A \otimes \mathcal{K}$, there exists an infinite rank projection $r \in 1 \otimes \mathcal{M}(\mathcal{K}) \subset \mathcal{M}(A \otimes \mathcal{K})$ such that $\|yr\| < \epsilon$. Letting $s \in 1 \otimes \mathcal{M}(\mathcal{K})$ be an isometry such that $ss^* = r$, we have $\|s^*ys\| < \epsilon$.

Applying this observation to the element $1_{\mathcal{M}(A\otimes\mathcal{K})} - xpx^*$ of $A \otimes \mathcal{K}$, we obtain an isometry s in $\mathcal{M}(A \otimes \mathcal{K})$ such that

$$\|1_{\mathcal{M}(A\otimes\mathcal{K})} - s^*xpx^*s\| = \|s^*(1_{\mathcal{M}(A\otimes\mathcal{K})} - xpx^*)s\| < 1.$$

There exists a positive element y of $\mathcal{M}(A \otimes \mathcal{K})$ such that

$$y^{-\frac{1}{2}}s^*xpx^*sy^{-\frac{1}{2}} = 1_{\mathcal{M}(A\otimes\mathcal{K})}.$$

Hence, $1_{\mathcal{M}(A\otimes\mathcal{K})}$ is Murray–von Neumann equivalent to a subprojection of p. Since $1_{\mathcal{M}(A\otimes\mathcal{K})}$ is properly infinite, p is properly infinite. Hence, p is a norm-full, properly infinite projection of $\mathcal{M}(A \otimes \mathcal{K})$. Therefore, p is Murray–von Neumann equivalent to $1_{\mathcal{M}(A\otimes\mathcal{K})}$.

In the case that $A_{\mathbb{C}}$ is not simple, let ψ be the conjugate linear automorphism on $A_{\mathbb{C}}$ defined by $\psi(a + ib) = a - ib$. Take A_1 to be a minimal non-trivial ideal and take $A_2 = \psi(A_1)$. Then it is easily proven that $A_{\mathbb{C}} \cong A_1 \oplus A_2$. Furthermore, the homomorphism from A to A_i, given by the inclusion in $A_{\mathbb{C}}$, composed with the projection on A_i, is an isomorphism of real C^*-algebras. It follows that each A_i is a simple and purely infinite complex C^*-algebra. By [10, Proposition 2.1], each A_i has the coronal factorization property. It follows that A has the corona factorization property. ∎

4 Examples

In this section we prove the following theorem.

Theorem 4.1 (i) *There exists a nonstable real C^*-algebra B such that $\mathbb{C}\otimes B$ is stable.*

(ii) *There exists a nonstable real C^*-algebra B such that $\mathbb{C} \otimes B$ is not stable, but $M_2(B)$ is stable.*

Furthermore, in either case, B can be taken so that B and $\mathbb{C} \otimes B$ are simple and have stable rank equal to one.

The following lemma is the real analog of [13, Proposition 3.6] and has the same proof, using Theorem 2.1.

Lemma 4.2 *Suppose A is a real C^*-algebra with projections $e, p_1, p_2, \dots$ and let*

$$B = \overline{\bigcup_{j=1}^{\infty} q_j(A \otimes \mathcal{K})q_j}, \tag{4.1}$$

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where $q_j = p_1 \oplus p_2 \oplus \cdots \oplus p_j \in M_j(A) \subset A \otimes \mathcal{K}$.

(i) *If* $e \otimes 1_n \sim p_j \otimes 1_n$ *for all* j, *then* $M_n(B)$ *is stable.*

(ii) *If, in addition to* (i), e *is not equivalent to a subprojection of* $q_j \otimes 1_{n-1}$ *for any* j *then* $M_{n-1}(B)$ *is not stable.*

Lemma 4.3 *There exists a real* C^**-algebra* A *with projections* $e, p_1, p_2, \ldots$ *such that*

(i) $e \sim p_j$ *in* $\mathbb{C} \otimes A$ *for all* j;

(ii) e *is not equivalent in* A *to a subprojection of* $p_1 \oplus p_2 \oplus \cdots \oplus p_j$ *for any* j.

Proof Let $\mathbb{T}$ be the unit circle in the complex plane. Let $C = M_2(\mathbb{R}) \otimes C(\mathbb{T}, \mathbb{R})$, and let $e, p \in C$ be projections corresponding respectively to the one-dimensional trivial bundle θ_1 and to the Möbius bundle μ over $\mathbb{T}$. Then $e \nsim p$ in C, since these bundles are not isomorphic, but as the complexification of these two bundles are isomorphic, we have $e \sim p$ in $\mathbb{C} \otimes C$.

In K-theory we have $[p] \neq [e]$ but $c([p]) = c([e])$. Indeed, under the appropriate homomorphisms we can identify $[p] = (1,1)$ and $[e] = (1,0)$ in $K_0(C) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, and $c([p]) = c([e]) = 1$ in $K_0(\mathbb{C} \otimes C) \cong \mathbb{Z}$. Furthermore the Stieffel–Whitney classes are $\mathrm{sw}(\mu) = 1 + x$ and $\mathrm{sw}(\theta_1) = 1$, where $H^*(\mathbb{T}; \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^2)$.

Let $A = M_2(\mathbb{R}) \otimes \big(\bigotimes_{i=1}^{\infty} C(\mathbb{T}, \mathbb{R})\big)$ and let $\iota_j \colon C \to A$ be the unital homomorphism induced by inclusion of $C(\mathbb{T}, \mathbb{R})$ into the j-th factor of $\bigotimes_{i=1}^{\infty} C(\mathbb{T}, \mathbb{R})$. Let $p_j = \iota_j(p)$, and let e also denote the image of e in A under ι_1. In $\mathbb{C} \otimes A$ we have $e \sim p_j$ for all j, establishing (i).

For (ii) assume that $e \sim f$ in A, where f is a subprojection of $p_1 \oplus p_2 \oplus \cdots \oplus p_n$. Then there is a projection f' such that $f \oplus f' = p_1 \oplus \cdots \oplus p_n$. Let ν, ν', and μ_i denote the vector bundles over T^∞ corresponding to the projections f, f', and p_i respectively. Since $e \sim f$, we have $\mathrm{sw}(\nu) = 1$.

Then in $H^*(\mathbb{T}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ we have

$$\begin{aligned} \mathrm{sw}(\nu') = \mathrm{sw}(\nu)\,\mathrm{sw}(\nu') &= \mathrm{sw}(\mu_1)\,\mathrm{sw}(\mu_2) \cdots \mathrm{sw}(\mu_n) \\ &= (1 + x_1)(1 + x_2) \cdots (1 + x_n) . \end{aligned}$$

This implies $\mathrm{sw}_n(\nu') = \prod_{i=1}^{n} x_i$. But this is impossible, since the vector bundle ν' has dimension less than n. ■

Lemma 4.4 *There exists a real* C^**-algebra* A *with projections* $e, p_1, p_2, \ldots$ *such that*

(i) $e \otimes 1_2 \sim p_j \otimes 1_2$ *in* $M_2(A)$ *for all* j;

(ii) e *is not equivalent in* $\mathbb{C} \otimes A$ *to a subprojection of* $p_1 \oplus p_2 \oplus \cdots \oplus p_j$ *for any* j.

Proof Let $C = M_4(C(S^2, \mathbb{R}))$, and let $e, p \in \mathbb{C} \otimes C$ be projections corresponding respectively to the (complex) 2-dimensional trivial bundle θ_2 and the direct sum $2\beta_U$ of two copies of the complex 1-dimensional Bott bundle over S^2. Then $e \nsim p$ in $\mathbb{C} \otimes C$, but we will show that $e \otimes 1_2 \sim p \otimes 1_2$ in $M_2(C)$. Indeed, a real vector bundle of dimension n over S^2 is determined by the homotopy type of its clutching map $S^1 \to GL(\mathbb{R}^n)$. But since $\pi_1(GL(\mathbb{R}^n)) = \mathbb{Z}_2$ for all $n \geq 3$, we have $p \oplus p \sim e \oplus e$.

In K-theory we have $K_0(\mathbb{C} \otimes C) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K_0(C) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. With this identification, we have $[e] = (2,0)$, $[p] = (2,2)$ and $[r(e)] = [r(p)] = (2,0)$.

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Similar to the proof to Lemma 4.3, let $A = M_4(\mathbb{R}) \otimes \left(\bigotimes_{i=1}^{\infty} C(S^2, \mathbb{R})\right)$, and we have corresponding projections $e, p_j \in A$. The projections p_j correspond to vector bundles μ_n over $(S^2)^\infty$. In $M_2(A)$ we have $r(e) \sim r(p_j)$ for all j, and in $\mathbb{C} \otimes A$ we have that e is not a subprojection of $p_1 \oplus p_2 \oplus \cdots \oplus p_j$, since the chern classes are given by $\operatorname{ch}(\theta_2) = 1$ and $\operatorname{ch}(\mu_1 \oplus \mu_2 \oplus \cdots \oplus \mu_j) = \prod_{i=1}^{j}(1+x_j)$, where $H^*((S^2)^\infty) = \mathbb{Z}[x_1, x_2, \dots,]/(x_1^2, x_2^2, \dots)$. ■

Proof of Theorem 4.1: non-simple case Let A be the real C^*-algebra from Lemma 4.3 and define B as in equation (4.1). Since $e \sim p_j$ holds in $\mathbb{C} \otimes A$, we have $e \otimes 1_2 \sim p_j \otimes 1_2$ in $M_2(A)$. Then by Lemma 4.2(ii) (with $n = 2$), we have that B is not stable. Now note that

$$B_{\mathbb{C}} \cong \overline{\bigcup_{j=1}^{\infty} q_j(A_{\mathbb{C}} \otimes \mathcal{K})q_j}.$$

Therefore Lemma 4.2(i) applied to $\mathbb{C} \otimes A$ shows that $\mathbb{C} \otimes B$ is stable. This proves part (i).

For part (ii), let A be the real C^*-algebra from Lemma 4.4 and define B as in equation (4.1). Since $e \otimes 1_2 \sim p_j \otimes 1_2$ in $M_2(\mathbb{C} \otimes A)$, Lemma 4.2(ii) immediately gives that $\mathbb{C} \otimes B$ is not stable. Finally, since $e \otimes 1_2 \sim p_j \otimes 1_2$ holds in $M_2(A)$, part (i) of the same lemma tells us that $M_2(B)$ is stable. ■

Proof of Theorem 4.1: simple case We now describe how to construct examples that are simple. We will use the same construction as Rørdam in [13, Section 5], following Villadsen in [17]. The only differences are our choice of the initial topological space X and that we will be using functions with values in matrices over $\mathbb{R}$ instead of $\mathbb{C}$. To begin, choose sequences $\{k_i\}, \{m_i\}$, and $\{d_i\}$ as in [13]. Let $X = \mathbb{T}$ and define $X_i = X^{d_i}$ and $A_i = M_{2m_i}(C(X_i, \mathbb{R}))$. Then we get connecting maps $\phi_i \colon A_i \to A_{i+1}$ defined by

$$\phi_i(f)(x) = \operatorname{diag}((f \circ \pi_1^i)(x), (f \circ \pi_2^i)(x), \dots, (f \circ \pi_i^i)(x), f(x_i))$$

for any $f \in A_i$ and $x \in X_{i+1}$. In this expression, $\pi_1, \dots, \pi_i$ are certain projections from X_{i+1} onto distinct factors of X_i and $x_i \in X_i$ are points selected such that for each i, the union of the images of $x_j \in X_j$ under all possible compositions of projections $X_j \to X_i$ for all $j \geq i$ is dense in X_i.

Then as in [13] the limit $A = \lim_{i\to\infty} A_i$ is simple, as is the complexification $\mathbb{C}\otimes A$. Furthermore, using the projections $e, p \in A_1$ from the proof of Lemma 4.3 we obtain, as in the proof of [13, Proposition 5.2], a sequence of projections $e, p_1, p_2, \dots \in A$ satisfying the statement of Lemma 4.3.

The C^*-algebra B obtained as in equation (4.1) is a real regular hereditary sub-C^*-algebra (in the sense of [16]) of the simple C^*-algebra $A \otimes \mathcal{K}$. The proof of [9, Theorem 3.2.8] carries over to the case of real C^*-algebras to show that a real hereditary sub-C^*-algebra of a simple C^*algebra is simple. Therefore, B is simple, as is $\mathbb{C} \otimes B$.

Finally, we note that [17, Proposition 10] implies that the C^*-algebra $A_{\mathbb{C}}$ above has stable rank one, and [12, Theorem 3.3] shows that $M_n(A_{\mathbb{C}})$ has real rank one for all n. This property is preserved by direct limits, so $A_{\mathbb{C}} \otimes \mathcal{K}$ has real rank one. The method

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of proof of Lemma 3.4 can be used to show that the corner algebras $q_j(A_{\mathbb{C}} \otimes \mathcal{K})q_j$ then have real rank one. Hence $B_{\mathbb{C}}$ has real rank one. All of these results cited apply in the case of real C^*-algebras (with the same proofs) to show that B also has real rank one.

Finally, to create a simple, stable rank one, real C^*-algebra satisfying Theorem 4.1(ii) we repeat the above construction using the space $X = S^2$ and the projections e and p described in the proof of Lemma 4.4. ■

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