SUM OF TWO FOURTH POWERS OF INTEGERS

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Introduction. Problems concerning the sum of two fourth powers of integers seem to be so difficult that little has been known since long years [3]. For instance, it is an important problem to determine whether there are infinitely many prime numbers which are represented in the form $p=a^4+b^4$. But nothing is known except that the density of such prime numbers is easily proved to be 0; accordingly it is difficult to obtain a necessary and sufficient condition under which p is represented in such a form.

In this paper we propose to derive several theorems on the above subject by investigating the sum of two fourth powers of integers in the biquadratic number field $R(\sqrt{i})$ or in its subfields. We shall use as main tools the decomposition law of prime numbers in $R(\sqrt{i})$ and the concrete expression of a fundamental unit in $R(\sqrt{i})$.

Hereinafter we say that an integer in a certain number field is of B type if it is represented as a sum of two fourth powers of integers belonging to the field and we denote by B. P. "the representation as a sum of two fourth powers of integers".

The contents of § 1 relate to the existence problem of B type prime numbers in the subfield of $R(\sqrt{i})$ different from the rational number field, § 2 to the uniqueness problem of B. P. and § 3 to B. P. of the product of several B type prime numbers.

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Notations and Preliminaries

(1) $i=\sqrt{-1}$, $R=$ rational number field.

(2) Minimal basis of $R(i)$, $R(\sqrt{2})$, $R(\sqrt{-2})$ or $R(\sqrt{i})$ are respectively $(1, i)$, $(1, \sqrt{2})$, $(1, \sqrt{-2})$ or $(1, \sqrt{i}, i, i\sqrt{i})$, and their class numbers are all 1.

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Hence every ideal in each field is principal.

(3) The fundamental unit in $R(\sqrt{i})$, as well as $R(\sqrt{2})$, is $\varepsilon = 1+\sqrt{2}$. For the sake of convenience, we put $\sqrt{i} = (1+i)/\sqrt{2}$, and so $\varepsilon = 1+\sqrt{2} = 1+(1-i)\sqrt{i}$. Hence, for $n = 0, \pm 1, \pm 2, \ldots$, $\varepsilon^n = r + s(1-i)\sqrt{i}$, where r and s are rational integers satisfying $r^2 - 2s^2 = (-1)^n$.

(4) Necessary and sufficient condition under which a rational prime number p is completely decomposed in $R(\sqrt{i})$ is $p \equiv 1 \pmod 8$.

(5) The notation $a|b$ or $a \nmid b$ respectively means that a is a divisor of b or not, where a and b are integers.

§1. Concerning the existence of B type prime numbers, we have the following results.

THEOREM 1. *There is no B type prime number* ($\notin R$) *in* $R(i)$.

Proof. Let a prime number $\pi(\notin R)$ in $R(i)$ be of B type and put

$$\pi = \alpha^4 + \beta^4 = (\alpha+\beta\sqrt{i})(\alpha-\beta\sqrt{i})(\alpha+\beta i\sqrt{i})(\alpha-\beta i\sqrt{i}); \quad \alpha, \beta \in R(i). \tag{1.1}$$

Since then π is prime in $R(i)$, at least two factors in the right hand side of (1.1) must be of the form $\pm(\sqrt{i})^k \varepsilon^n$, where $k = 0, 1, 2$ or 3 and $n = 0, \pm 1, \pm 2, \ldots$. Suppose, for instance, $\alpha + \beta\sqrt{i} = \pm(\sqrt{i})^k \varepsilon^n$ and put $\varepsilon^n = r + s(1-i)\sqrt{i}$, then

$$\alpha + \beta\sqrt{i} = \pm(\sqrt{i})^k \{r + s(1-i)\sqrt{i}\}. \tag{1.2}$$

Therefore

(1) If $k=0$, then $\alpha = \pm r$, $\beta = \pm s(1-i)$
(2) If $k=1$, then $\alpha = \pm s(1+i)$, $\beta = \pm r$
(3) If $k=2$, then $\alpha = \pm ri$, $\beta = \pm s(1+i)$
(4) If $k=3$, then $\alpha = \mp s(1-i)$, $\beta = \pm ri$.

In these four cases, we have $\pi = \pm(r^4 - 4s^4)$ after all. But this is contrary to the assumption $\pi \notin R$. Next we suppose that the left hand side of (1.2) is equal to $\alpha - \beta\sqrt{i}$, $\alpha + \beta i\sqrt{i}$ or $\alpha - \beta i\sqrt{i}$. In these cases proof will be similarly carried out by interchanging β with $-\beta$ or with $\pm\alpha$.

THEOREM 1′. *The only B type prime numbers* ($\notin R$) *in* $R(\sqrt{2})$ *are* $5\varepsilon^{4m}$ ($m = \pm 1, \pm 2, \ldots$) *and there is no B type prime number* ($\notin R$) *in* $R(\sqrt{-2})$.

Proof. In the case of $R(\sqrt{2})$, let a prime number $\pi(\notin R)$ in $R(\sqrt{2})$ be of B type and put

$$\pi = \alpha^4 + \beta^4 = (\alpha + \beta\sqrt{i})(\alpha - \beta\sqrt{i})(\alpha + \beta i\sqrt{i})(\alpha - \beta i\sqrt{i}), \tag{1.3}$$

where $\alpha, \beta \in R(\sqrt{2})$ and $(\alpha, \beta) = 1$. Since then π is prime in $R(\sqrt{2})$, at least two factors in the right hand side of (1.3) must be of the form $\pm(\sqrt{i})^k \varepsilon^m$, where $k = 0, 1, 2$ or 3 and $m = 0, \pm 1, \pm 2, \ldots$. First we treat the case:

$$\alpha + \beta\sqrt{i} = \pm(\sqrt{i})^k \varepsilon^m. \tag{1.4}$$

It is easily seen that (1.4) is impossible for $k = 0, 1$, so we closely examine the remaining cases of $k = 2, 3$.

(1) If $k = 2$, then taking square of the both sides of (1.4) we have $2\alpha\beta\sqrt{i} = -(\varepsilon^{2m} + \alpha^2 + \beta^2 i)$, whence

$$4\alpha^2\beta^2 i = (\varepsilon^{2m} + \alpha^2)^2 - \beta^4 + 2\beta^2(\varepsilon^{2m} + \alpha^2)i.$$

Since α, β and $\varepsilon \in R(\sqrt{2})$ and $\beta \neq 0$, we obtain $\alpha^2 = \varepsilon^{2m}$ and so $\beta^2 = 2\varepsilon^{2m}$. Hence $\pi = \alpha^4 + \beta^4 = 5\varepsilon^{4m}$ $(m \neq 0)$.

(2) If $k = 3$, then (1.4) gives the relation $\alpha = \pm(\sqrt{i})^3\varepsilon^m - \beta\sqrt{i}$. Hence $\alpha^2 = (\beta^2 - \varepsilon^{2m})i \pm 2\varepsilon^m\beta$. This means $\alpha^2 = \pm 2\varepsilon^m\beta$ and $\beta^2 - \varepsilon^{2m} = 0$. Consequently $\beta^4 = \varepsilon^{4m}$, $\alpha^4 = 4\varepsilon^{4m}$ and $\pi = \alpha^4 + \beta^4 = 5\varepsilon^{4m}$ $(m \neq 0)$.

Assuming that the left hand side of (1.4) is respectively equal to $\alpha - \beta\sqrt{i}$, $\alpha + \beta i\sqrt{i}$ or $\alpha - \beta i\sqrt{i}$, the proof will be similarly carried out through interchanging β with $-\beta$ or $\pm\alpha$.

Accordingly it has been decided that $\pi = 5\varepsilon^{4m}$ is a necessary condition for a prime number π in $R(\sqrt{2})$ to be of B type. This is clearly sufficient, for $5\varepsilon^{4m} = (2\varepsilon^m)^4 + (\varepsilon^m)^4$.

In the case of $R(\sqrt{-2})$, we have (1.4) with $\alpha, \beta \in R(\sqrt{-2})$ and can conclude quite similarly that $\pi = 5\varepsilon^{4m}$ $(m \neq 0)$ are the only B type prime numbers in $R(\sqrt{-2})$. But $5\varepsilon^{4m} \notin R(\sqrt{-2})$.

THEOREM 1″. *There is no B type prime number* $(\notin R, R(\sqrt{2}))$ *in* $R(\sqrt{i})$.

The proof of this theorem shall well be omitted, because it is essentially the same as in the case of theorem 1 in spite of a comparatively complicated computation.

§ **2.** It seems very difficult to determine in a general form the number of

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B. P. of a given integer in a field. Until now the following results with regard to a product of two prime numbers has only been obtained.

THEOREM 2. *Let r, s be rational prime numbers which are either* 2 *or of the form* $8h+1$ *and further let the product rs be of B type, then the B. P. of rs is unique.*

Proof. Assume $rs = x_1^4 + y_1^4 = x_2^4 + y_2^4$ under the condition $(x_1, y_1) = (x_2, y_2) = 1$ and consider the following decompositions in $R(\sqrt{i})$,

(2.1) $$x_1^4 + y_1^4 = (x_1 + y_1\sqrt{i})(x_1 - y_1\sqrt{i})(x_1 + y_1 i\sqrt{i})(x_1 - y_1 i\sqrt{i}).$$

(2.2) $$x_2^4 + y_2^4 = (x_2 + y_2\sqrt{i})(x_2 - y_2\sqrt{i})(x_2 + y_2 i\sqrt{i})(x_2 - y_2 i\sqrt{i}).$$

On the other hand, decompose respectively r, s into $r = \pi\bar{\pi}$, $s = \sigma\bar{\sigma}$ in $R(i)$, and respectively π, $\bar{\pi}$, σ, and $\bar{\sigma}$ into $\pi = \pi_1\pi_2$, $\bar{\pi} = \bar{\pi}_1\bar{\pi}_2$, $\sigma = \sigma_1\sigma_2$ and $\bar{\sigma} = \bar{\sigma}_1\bar{\sigma}_2$ in $R(\sqrt{i})$: i.e.

(2.3) $$r = \pi\bar{\pi} = \pi_1\pi_2\bar{\pi}_1\bar{\pi}_2$$

(2.4) $$s = \sigma\bar{\sigma} = \sigma_1\sigma_2\bar{\sigma}_1\bar{\sigma}_2.$$

Now each factor in the right hand sides of (2.3) and (2.4) is distributed into each factor of the right hand sides of (2.1) and (2.2). (Consider the norm from $R(\sqrt{i})$ to R). Notations being suitably selected, we may assume that

(2.5) $$\pi_1\sigma_1 \mid (x_1 + y_1\sqrt{i}).$$

Here let us examine other factors;

(1) Suppose first that $\pi_1\sigma_1$ is also contained in a factor of (2.2), for instance, in $x_2 + y_2\sqrt{i}$. Then, using (2.5), we obtain $\pi_1\sigma_1 \mid (x_1y_2 - x_2y_1)$ which causes the following relation

(2.6) $$rs \mid (x_1y_2 - x_2y_1).$$

As $|x_1y_2 - x_2y_1| < rs$ unless $|x_1| = |y_1| = |x_2| = |y_2| = 1$, it follows from (2.6) that $x_1y_2 - x_2y_1 = 0$, which yields $x_2 = \pm x_1$, $y_2 = \pm y_1$, for $(x_1, y_1) = (x_2, y_2) = 1$. Accordingly B. P. is unique. If $\pi_1\sigma_1$ is contained in one of any other three factors of (2.2) than $x_2 + y_2\sqrt{i}$, a similar proof is available.

(2) Suppose secondly that $\pi_1\sigma_1$ is not contained in any factor of (2.2). Then, notations being suitably chosen, we can assume

(2.7) $$\pi_1\sigma_2 \mid (x_2 + y_2\sqrt{i}).$$

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Therefore, using (2.5) and (2.7), we have

(2.8) $$r \mid (x_1 y_2 - x_2 y_1)$$

similarly to the above case (1). Here again we closely examine the factor of (2.2) which contains σ_1.

(i) If $\sigma_1 \mid (x_2 - y_2\sqrt{i})$, then from (2.5) we obtain $s \mid (x_1 y_2 + x_2 y_1)$ and the relation $rs \mid \{(x_1 y_2)^2 - (x_2 y_1)^2\}$ through (2.8). Since $|(x_1 y_2)^2 - (x_2 y_1)^2| < (x_1 y_2)^2 + (x_2 y_1)^2 < \frac{1}{2}(x_1^4 + y_2^4 + x_2^4 + y_1^4) = rs$, we have $(x_1 y_2)^2 = (x_2 y_1)^2$. Therefore $x_2 = \pm x_1$, $y_2 = \pm y_1$.

(ii) If $\sigma_1 \mid (x_2 + y_2 i\sqrt{i})$, then $s \mid (x_1 x_2 + y_1 y_2)$ after all. Hence, $rs \mid (x_1 x_2 + y_1 y_2)(x_1 y_2 - x_2 y_1)$ holds by (2.8). Since $|(x_1 x_2 + y_1 y_2)(x_1 y_2 - x_2 y_1)| \leqq \frac{1}{2}\left\{(x_1 x_2 + y_1 y_2)^2 + (x_1 y_2 - x_2 y_1)^2\right\} < rs$, we have $(x_1 x_2 + y_1 y_2)(x_1 y_2 - x_2 y_1) = 0$, which leads to the same conclusion.

(iii) If $\sigma_1 \mid (x_2 - y_2 i\sqrt{i})$, then a similar proof is available.

(3) In other remaining cases where $\pi_1 \sigma_2$ is contained in any one of $(x_2 - y_2\sqrt{i})$, $(x_2 + y_2 i\sqrt{i})$ or $(x_2 - y_2 i\sqrt{i})$, we can also obtain similar proofs.

THEOREM 2′. *If a product $\pi\sigma (\notin R)$ of two prime numbers π, σ in $R(i)$ is of B type, then its B. P. is unique.*

Proof. Under conditions $\alpha, \beta \in R(i)$ and $(\alpha, \beta) = 1$, we put

(2.9) $$\pi\sigma = \alpha^4 + \beta^4 = (\alpha^2 + \beta^2 i)(\alpha^2 - \beta^2 i).$$

If $(1+i, \pi\sigma) = 1$, then $(\alpha^2 + \beta^2 i, \alpha^2 - \beta^2 i) = 1$ and if $(1+i, \pi\sigma) \neq 1$, then $\alpha^2 + \beta^2 i$ and $\alpha^2 - \beta^2 i$ have the common factor $1+i$ at least, and so $\pi\sigma = \pm(1+i)^2 = \pm 2i$. If further any one of the factors in (2.9) is ± 1 or $\pm i$, then, quite similarly to (1.2), we get $\pi\sigma = \pm(r^4 - 4s^4)$. This is, however, contrary to the assumption $\pi\sigma \notin R$. Hence π and σ must be contained separately in two factors of (2.9). If $\pi = \pm(\alpha^2 + \beta^2 i)$ and $\sigma = \pm(\alpha^2 - \beta^2 i)$, then

(2.10) $$\alpha^2 = \pm(\pi + \sigma)/2, \qquad \beta^2 = \mp i(\pi - \sigma)/2.$$

If $\pi = \pm i(\alpha^2 + \beta^2 i)$, $\sigma = \mp i(\alpha^2 - \beta^2 i)$, then

$$\alpha^2 = \pm i(\pi - \sigma)/2, \qquad \beta^2 = \pm(\pi + \sigma)/2.$$

The latter can be obtained from (2.10) by interchanging α with β. Thus, α and β are uniquely determined through (2.10).

Note. There are many examples which show that the above theorems 2, 2′ do not necessarily hold for a product of more than two prime numbers.

Ex. 1. $17 \cdot 63113 \cdot 80537 = 542^4 + 103^4 = 514^4 + 359^4$

Ex. 2. $2 \cdot 113 \cdot 4889 \cdot 2953 = 239^4 + 7^4 = 227^4 + 157^4$

Ex. 3. $(1+4i)(7-8i)(3+20i) = (10+3i)^4 + (9-5i)^4$

$$= (5+2i)^4 + 3^4(1+i)^4.$$

§ **3.** It is also a hard problem to determine generally whether a product of several given B type prime numbers has B. P. or not. First let us state a preliminary lemma without proof.

LEMMA. *Let a product $N = p_1 p_2 \cdots p_n$ of different B type rational prime numbers $p_m = a_m^4 + b_m^4$ $(m = 1, 2, \ldots, n)$ be of B type and put*

$$N = a^4 + b^4, \quad (a, b) = 1, \tag{3.1}$$

then the following relation holds;

$$\prod_{m=1}^{n} (a_m + b_m\sqrt{i}) = \pm(\sqrt{i})^k \varepsilon^l (a + b\sqrt{i}), \tag{3.2}$$

where $k = 0, 1, 2$ or 3, $l = 0, \pm 1, \pm 2, \ldots$ and $\varepsilon^l = r + s(1-i)\sqrt{i}$. Without any loss of generality, the following conditions can be added:

$$a > 0, \quad b > 0, \quad 2 | b, \quad a_m > 0, \quad r > 0. \tag{3.3}$$

Under these conditions the right hand side of (3.2) *is written as follows:*

$$\prod_{m=1}^{n} (a_m + b_m\sqrt{i}) = \pm(\sqrt{i})^k \{ra + sb + (rb + sa)\sqrt{i} + sbi - sai\sqrt{i}\}. \tag{3.4}$$

For $N = 2p_1 p_2 \cdots p_n$, we have similarly

$$(1+i)\prod_{m=1}^{n} (a_m + b_m\sqrt{i}) = \pm(\sqrt{i})^k \{ra + sb + (rb + sa)\sqrt{i} + sbi - sai\sqrt{i}\}, \tag{3.5}$$

where ab is to be odd.

THEOREM 3. *Notations being as in the preceding lemma, none of the products $p_1 p_2$, $p_1 p_2 p_3$, $2p_1$, $2p_1 p_2$ and $2p_1 p_2 p_3$ can be of B type.*

Proof. In the case of $N = p_1 p_2$, (3.4) gives

(3.6) $$(a_1+b_1\sqrt{i})(a_2+b_2\sqrt{i}) = \pm(\sqrt{i})^k\{ra+sb+(rb+sa)\sqrt{i}+sbi-sai\sqrt{i}\}.$$

Now let us closely examine four cases of $k=0$, 1, 2 and 3.

(1) If $k=0$, then, since $(1, \sqrt{i}, i, i\sqrt{i})$ is a basis of $R(\sqrt{i})$, the following relations hold ($\rho=\pm 1$):

(3.7) $$\begin{cases} \rho(ra+sb)=a_1a_2 \\ \rho(rb+sa)=a_1b_2+a_2b_1 \\ \rho sb=b_1b_2 \\ -\rho sa=0. \end{cases}$$

From the last formula of (3.7) we have $s=0$, which is contrary to the assumption $b_1 \neq 0$, $b_2 \neq 0$.

(2) If $k=1$, then we have $\rho sb=0$ and $\rho sa=a_1a_2$, which is a contradiction.

(3) If $k=2$, then the following relations hold:

(3.8) $$\begin{cases} \rho(ra+sb)=b_1b_2 \\ \rho(rb+sa)=0 \\ -\rho sb=a_1a_2 \\ \rho sa=a_1b_2+a_2b_1. \end{cases}$$

Here $s=0$ is impossible, for $a_1a_2 \neq 0$. If $|s| \geqq 2$, then it follows from the 3rd and 4th formulas of (3.8) that

(3.9) $$b \leqq \frac{1}{2}a_1a_2, \qquad a \leqq \max(|a_1b_2|, |a_2b_1|).$$

But these can not be true, for $a^4+b^4=(a_1a_2)^4+(a_1b_2)^4+(a_2b_1)^4+(b_1b_2)^4$. Accordingly $|s|=1$, but this is also impossible from the 2nd relation of (3.8).

(4) If $k=3$, the proof is similar to the case (3).

In the case of $N=p_1p_2p_3$, (3.4) yields

(3.10) $$\prod_{m=1}^{3}(a_m+b_m\sqrt{i}) = \pm(\sqrt{i})^k\{ra+sb+(rb+sa)\sqrt{i}+sbi-sai\sqrt{i}\}.$$

Put, for convenience,

(3.11) $$\begin{cases} A=a_1a_2a_3 \\ B=b_1a_2a_3+b_2a_3a_1+b_3a_1a_2 \\ C=a_1b_2b_3+a_2b_3b_1+a_3b_1b_2 \\ D=b_1b_2b_3. \end{cases}$$

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Then one and only one of A, B, C and D is odd, because a_1b_1, a_2b_2, a_3b_3 are all even. Now let us examine four cases of $k=0$, 1, 2 *and* 3.

(1) If $k=0$, then it follows from (3.10) and (3.11) ,that

$$(3.12)\qquad \begin{cases} \rho(ra+sb)=A \\ \rho(rb+sa)=B \\ \rho sb=C \\ -\rho sa=D. \end{cases}$$

If s is odd in (3.12), then A and D are odd, since b is even. This is, however, contrary to the fact mentioned above. Hence s must be even. If $s=0$, then $D=b_1b_2b_3=0$, which can not hold. If $|s|\geqq 4$, then the 3rd and 4th formulas of (3.12) give

$$a\leqq\frac{1}{4}|b_1b_2b_3|,\quad b\leqq\frac{3}{4}\max(|a_1b_2b_3|,\quad |a_2b_3b_1|,\quad |a_3b_1b_2|),$$

which can not hold by a quite similar reason as (3.9) did not. Finally suppose $|s|=2$, then the 1st formula of (3.12) implies $2|a_1a_2a_3$, $8|D$, $8|sa$ and $4|a$, which contradicts the assumption $2\nmid a$.

(2) If $k=1$, we have $\rho sb=D$ and $\rho sa=A$, which is impossible.

(3) If $k=2$, then (3.10) gives

$$(3.13)\qquad \begin{cases} \rho(ra+sb)=C \\ \rho(rb+sa)=D \\ -\rho sb=A \\ \rho sa=B. \end{cases}$$

Here s must be even from a similar reason to the case of $k=0$. But neither $s=0$ nor $|s|\geqq 4$ can hold. Hence $|s|=2$ $(r=3)$. Now by eliminating a, b, a_3 and b_3 from (3.13), we obtain the relation

$$(r^2-s^2)a_1^2a_2^2-rs(a_1b_2+a_2b_1)(a_1a_2+b_1b_2)+s^2\{b_1^2b_2^2+(a_1b_2+a_2b_1)^2\}=0.$$

Put $s=2\rho_1$ $(\rho_1=\pm1)$ and $r=3$, and further put $(a_1/b_1)=t_1$ and $(a_2/b_2)=t$. Then by an easy computation the above relation turns out

$$(3.14)\qquad (5t_2^2+6\rho_1t_2+4)t_1^2+2\rho_1(3t_2^2+4\rho_1t_2+3)t_1+4t_2^2+6\rho_1t_2+4=0.$$

Now we can easily prove that (3.14) can not hold for any real values of t_1 and t_2. For, first of all, $5t_2^2+6\rho_1t_2+4>0$, and, D being the discriminant with respect to t_1 of the left hand side of (3.14), we have

$$D=(3t_2^2+4\rho_1t_2+3)^2-(5t_2^2+6\rho_1t_2+4)(4t_2^2+6\rho_1t_2+4),$$

where $5t_2^2+6\rho_1t_2+4>3t_2^2+4\rho_1t_2+3$ and $4t_2^2+6\rho_1t_2+4>3t_2^2+4\rho_1t_2+3$. Hence $D<0$.

(4) If $k=3$, then a similar relation to (3.13) leads to a similar conclusion.

By means of (3.5) we can accomplish an almost same proof in the case of $N=2p$, or $2p_1p_2$ and a comparatively complicated but analogous one in the case of $N=2p_1p_2p_3$.

Here we want to add a supplementary corollary and theorem derived almost immediately from theorem 3 and theorem 1′.

Corollary. *p_1^2, p_1^3 and $p_1^2p_2$ cannot be of B type.*

Proof. Clear, because the lemma is valid for $n\leqq 3$ even if p_m are not necessarily different.

Note. This corollary can, however, not be extended in general, for example, if $p_1=a_1^4+b_1^4$, then $p_1p_2^4=(a_1p_2)^4+(b_1p_2)^4$.

THEOREM 3′. *A product $\nu=\pi_1\pi_2\cdots\pi_n$ of B type prime numbers π_m ($m=1, 2, \ldots, n$) in $R(\sqrt{2})$ has B. P. in $R(\sqrt{2})$, if and only if $n=4h+1$.*

This theorem is well comprehended without proof, because it is easily seen that ν must be of the form $5^n\varepsilon^{4k}$ from theorem 1′ and factors $2+i$ and $2-i$ of 5 are prime in $R(\sqrt{i})$.

Note. We can imagine that theorem 3 may not be extended in general, but for the product of four B type rational primes the theorem seems also to be true.

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