# CONVOLUTION OF ORBITAL MEASURES ON SYMMETRIC SPACES OF TYPE $C_p$ AND $D_p$

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### Abstract

We study the absolute continuity of the convolution $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ of two orbital measures on the symmetric spaces $\mathbf{SO}_0(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$, $\mathbf{SU}(p, p)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ and $\mathbf{Sp}(p, p)/\mathbf{Sp}(p) \times \mathbf{Sp}(p)$. We prove sharp conditions on $X, Y \in \mathfrak{a}$ for the existence of the density of the convolution measure. This measure intervenes in the product formula for the spherical functions.

## 1. Introduction

The spaces $G/K = \mathbf{SO}_0(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$ are Riemannian symmetric spaces of noncompact type corresponding to root systems of type $D_p$. The spaces $\mathbf{SU}(p, p)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ and $\mathbf{Sp}(p, p)/\mathbf{Sp}(p) \times \mathbf{Sp}(p)$ correspond to root systems of type $C_p$.

Consider $X, Y \in \mathfrak{a}$ and let $m_K$ denote the Haar measure of the group $K$. We define $\delta^\natural_{e^X} = m_K \star \delta_{e^X} \star m_K$. It is the uniform measure on the orbit $K e^X K$. The problem of the absolute continuity of the convolution

$$m_{X,Y} = \delta^\natural_{e^X} \star \delta^\natural_{e^Y}$$

of two orbital measures that we address in this paper has important applications in harmonic analysis of spherical functions on $G/K$ and in probability theory. The applications in harmonic analysis concern the product formula for the spherical functions on Riemannian symmetric spaces which we now briefly recall. Let $\lambda$ be a complex-valued linear form on $\mathfrak{a}$ and $\phi_\lambda(e^X)$ be the spherical function, which is the spherical Fourier transform of the measure $\delta^\natural_{e^X}$. The product formula for the spherical functions states that

$$\phi_\lambda(e^X) \, \phi_\lambda(e^Y) = \int_\mathfrak{a} \phi_\lambda(e^H) \, d\mu_{X,Y}(H)$$

where $\mu_{X,Y}$ is the projection of the measure $m_{X,Y}$ on $\mathfrak{a}$ via the Cartan decomposition $G = KAK$. The investigation of the product formula was initiated by Helgason [9, Proposition IV.10.13, page 480]. Helgason proposes in [10, page 367] the study of properties of $\mu_{X,Y}$ in relation to the structure of $G$ as an interesting open problem. Questions on the existence and explicit expressions of a density of $\mu_{X,Y}$ are thus of great importance. The investigation of this problem was started by Flensted-Jensen and Koornwinder on hyperbolic spaces [1, 13]. The authors of this paper studied the existence, the explicit form and properties of the density of $\mu_{X,Y}$ in a series of papers listed in [6]. The same questions were investigated by Rösler and other authors (see [16] and references therein), in the hope of a generalization in the Dunkl and hypergroups setting.

The applications in probability concern the arithmetic of probability measures and properties of random walks on semisimple Lie groups. In order to characterize the important class $I_0$ of probability laws without indecomposable factors (in the sense of the convolution product) Ostrovskii [15], Trukhina [18] and Voit [19] use the product formula and some properties of its kernel as their main tools, respectively on $\mathbf{R}^n/\mathbf{O}(n)$, on real hyperbolic spaces and on some hypergroups. We conjecture that our results will allow the characterization of the class $I_0$ on all symmetric spaces $G/K$ as $K$-invariant Gaussian measures.

The property of absolute continuity of sufficiently large convolution powers $(\delta_{e^X}^\natural)^l$ is essential in the study of random walks on groups (see, for example, [11, 12, 14]). The measures, with a certain convolution power allowing an absolutely continuous part, are called 'spread out'. We also believe that our results on the convolution powers of $\delta_{e^X}^\natural$ will be useful in the study of isotropic $K$-invariant random walks on $G/K$; see [3].

It was proven in [4] that as soon as the space $G/K$ is irreducible and one of the elements $X$, $Y$ is regular and the other nonzero, then the convolution $\delta_{e^X}^\natural \star \delta_{e^Y}^\natural$ has a density. The density can, however, still exist when both $X$ and $Y$ are singular. It is a challenging problem to characterize all such pairs $X$, $Y$.

This problem was solved for symmetric spaces of type $A_n$ and for the exceptional space $\mathbf{SL}(3, \mathbf{O})/\mathbf{SU}(3, \mathbf{O})$ of type $E_6$ in [6], and for symmetric spaces of type $B_p$ and $BC_p$ in [7]. In the present paper, we present the solution of the problem for Riemannian symmetric spaces of type $C_p$ and $D_p$.

To understand the methods of this paper, knowledge of the papers [4, 7] is useful; nevertheless, we have made this paper self-contained, briefly recalling arguments from those papers that we use here. Moreover, the cases $C_p$ and $D_p$ require many original ideas that did not appear in the case $B_p$. We refer to Helgason's books [8, 9] for the standard notation and results.

In Section 2 we recall basic information about the Lie group $\mathbf{SO}(p, p)$ and its Lie algebra $\mathfrak{so}(p, p)$. We also provide the notation necessary to describe the configuration of an element of the Cartan subalgebra $\mathfrak{a}$ of $\mathbf{SO}(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$. This configuration notion allows us to 'measure' how singular an element of $\mathfrak{a}$ is and to describe in a precise manner which pairs $X, Y \in \mathfrak{a}$ are 'eligible', the sharp criterion that

we establish in the paper for the absolute continuity of $\mu_{X,Y}$. The following theorem is the main result of the paper:

THEOREM A. *Let $G/K = \mathbf{SO}_0(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$ and $X, Y \in \mathfrak{a}$. The density of the convolution $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ exists if and only if $X$ and $Y$ are eligible (see Definition 2.4).*

In Section 3 a series of definitions and accessory results are given to set the stage for the proof of Theorem A. In Section 4 we show that $(X, Y)$ has to be an eligible pair for the measure $\mu_{X,Y}$ to be absolutely continuous. In Section 5 we then show that the eligibility condition is also sufficient. The proof is based on the criterion given in Equation (3.3). We will proceed by induction by selecting eligible 'predecessors' $X'$, $Y'$ of $X$ and $Y$ of lower dimension. These predecessors are constructed from $X$ and $Y$ in a prescribed manner.

In the final section, as in [6] and [7], we extend our results to the complex and quaternionic cases. Again, the richness of the root structure comes into play: in the table in Remark 6.1, we find that the complex and quaternionic cases have much more in common with the cases $q > p$ than with the real case $\mathbf{SO}(p, p)$.

We conclude the paper with a discussion of the absolute continuity of powers of $\delta^\natural_{e^X}$ for a nonzero $X \in \mathfrak{a}$.

## 2. Preliminaries and definitions

We start by reviewing some useful information on the Lie group $\mathbf{SO}_0(p, p)$, its Lie algebra $\mathfrak{so}(p, p)$ and the corresponding root system. Most of this material was given in [17]. For the convenience of the reader, we gather below the properties we will need in the sequel. In this paper, $E_{ij}$ is a rectangular matrix with 0s everywhere except at the position $(i, j)$ where a 1 appears. Recall that $\mathbf{SO}(p, p)$ is the group of matrices $g \in \mathbf{SL}(2p, \mathbf{R})$ such that $g^T I_{p,p} g = I_{p,p}$, where $I_{p,p} = \begin{bmatrix} -I_p & 0_{p \times p} \\ 0_{p \times p} & I_p \end{bmatrix}$. Unless otherwise specified, all $2 \times 2$ block decompositions in this paper follow the same pattern. The group $\mathbf{SO}_0(p, p)$ is the connected component of $\mathbf{SO}(p, p)$ containing the identity. The Lie algebra $\mathfrak{so}(p, p)$ of $\mathbf{SO}_0(p, p)$ consists of the matrices

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

where $A$ and $D$ are skew-symmetric. A very important element in our investigations is the Cartan decomposition of $\mathfrak{so}(p, p)$ and $\mathbf{SO}(p, p)$. The maximal compact subgroup $K$ is the subgroup of $\mathbf{SO}(p, p)$ consisting of the matrices

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

of size $2p \times 2p$ such that $A, D \in \mathbf{SO}(p)$ (hence $K \simeq \mathbf{SO}(p) \times \mathbf{SO}(p)$). If $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is the set of matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \tag{2.1}$$

TABLE 1. Restricted roots and associated root vectors.

| Root $\alpha$ | Multiplicity | Root vectors $X_\alpha$ |
|---|---|---|
| $\alpha(H) = \pm(h_i - h_j)$ | 1 | $Y_{i,j}^{\pm} = \pm(E_{i,j} - E_{j,i} + E_{p+i,p+j} - E_{p+j,p+i})$ |
| $1 \le i, j \le p, i < j$ | | $\quad + E_{i,p+j} + E_{p+j,i} + E_{j,p+i} + E_{p+i,j}$ |
| $\alpha(H) = \pm(h_i + h_j)$ | 1 | $Z_{i,j}^{\pm} = \pm(E_{i,j} - E_{j,i} - E_{p+i,p+j} + E_{p+j,p+i})$ |
| $1 \le i, j \le p, i < j$ | | $\quad - (E_{i,p+j} + E_{p+j,i}) + E_{j,p+i} + E_{p+i,j}$ |

then the Cartan decomposition is given by $\mathfrak{so}(p, p) = \mathfrak{k} \oplus \mathfrak{p}$ with corresponding Cartan involution $\theta(X) = -X^T$. For convenience of notation, for $X \in \mathfrak{p}$ as in (2.1), we will write $X^s = B$.

The Cartan space $\mathfrak{a} \subset \mathfrak{p}$ is the set of matrices

$$H = \begin{bmatrix} 0_{p \times p} & \mathcal{D}_H \\ \mathcal{D}_H & 0_{p \times p} \end{bmatrix}$$

where $\mathcal{D}_H = \mathrm{diag}[h_1, \ldots, h_p]$. Its canonical basis is given by the matrices

$$A_i := E_{i,p+i} + E_{p+i,i}, \quad 1 \le i \le p.$$

The restricted roots and associated root vectors for the Lie algebra $\mathfrak{so}(p, p)$ with respect to $\mathfrak{a}$ are given in Table 1. The positive roots can be chosen as $\alpha(H) = h_i \pm h_j$, $1 \le i < j \le p$. The simple roots are given by $\alpha_i(H) = h_i - h_{i+1}$, $i = 1, \ldots, p - 1$, and $\alpha_p(H) = h_{p-1} + h_p$. We therefore have the positive Weyl chamber

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : h_1 > h_2 > \cdots > h_{p-1} > |h_p|\}.$$

The elements of the Weyl group $W$ act as permutations of the diagonal entries of $\mathcal{D}_X$ with eventual sign changes of any even number of these entries. The Lie algebra $\mathfrak{k}$ is generated by the vectors $X_\alpha + \theta X_\alpha$. We will use the notation

$$k_{X_\alpha}^t = e^{t(X_\alpha + \theta X_\alpha)}.$$

The linear space $\mathfrak{p}$ has a basis formed by $A_i \in \mathfrak{a}$, $1 \le i \le p$, and by the symmetric matrices $X_\alpha^s := \frac{1}{2}(X_\alpha - \theta X_\alpha)$ which have the form

$$Y_{i,j} := E_{i,p+j} + E_{j,p+i} + E_{p+j,i} + E_{p+i,j}, \quad 1 \le i < j \le p,$$
$$Z_{i,j} := -E_{i,p+j} + E_{j,p+i} - E_{p+j,i} + E_{p+i,j}, \quad 1 \le i < j \le p.$$

Thus, if $X \in \mathfrak{p}$ is as in (2.1), then the vectors $A_i$ generate the diagonal entries of $B = X^s$ and $Y_{i,j}$ and $Z_{i,j}$ the nondiagonal entries.

We now recall the useful matrix $S \in \mathbf{SO}(p + q)$ which allows us to diagonalize simultaneously all the elements of $\mathfrak{a}$. Let

$$S = \begin{bmatrix} \dfrac{\sqrt{2}}{2} I_p & \dfrac{\sqrt{2}}{2} J_p \\ \dfrac{\sqrt{2}}{2} I_p & -\dfrac{\sqrt{2}}{2} J_p \end{bmatrix}$$

where $J_p = (\delta_{i,p+1-i})$ is a matrix of size $p \times p$. If $H = \begin{bmatrix} 0 & \mathcal{D}_H \\ \mathcal{D}_H & 0 \end{bmatrix}$ with $\mathcal{D}_H = \text{diag}[h_1, \ldots, h_p]$ then $S^T H S = \text{diag}[h_1, \ldots, h_p, -h_p, \ldots, -h_1]$. The 'group' version of this result is as follows:

$$S^T e^H S = \text{diag}[e^{h_1}, \ldots, e^{h_p}, e^{-h_p}, \ldots, e^{-h_1}].$$

REMARK 2.1. The Cartan projection $a(g)$ on the group $\mathbf{SO}_0(p, p)$, defined as usual by

$$g = k_1 e^{a(g)} k_2, \quad a(g) \in \overline{\mathfrak{a}^+}, k_1, \ k_2 \in K,$$

is related to the singular values of $g \in \mathbf{SO}(p, p)$ in the following way. Recall that the singular values of $g$ are defined as the nonnegative square roots of the eigenvalues of $g^T g$. Let us write $H = a(g)$. We have

$$g^T g = k_2^T e^{2H} k_2 = (k_2^T S)(S^T e^{2H} S)(S^T k_2)$$

where $S^T e^{2H} S$ is a diagonal matrix with nonzero entries $e^{2h_1}, \ldots, e^{2h_p}, e^{-2h_p}, \ldots, e^{-2h_1}$, satisfying $h_1 \geq \cdots \geq h_{p-1} \geq |h_p|$. Let us write $a_j = e^{h_j}$ for $j = 1, \ldots, p-1$ and $a_p = e^{|h_p|}$. Thus the set of $2p$ singular values of $g$ contains the values $a_1 \geq \cdots \geq a_p \geq a_p^{-1} \geq \cdots \geq a_1^{-1}$ with $a_1 \geq \cdots \geq a_p \geq 1$. Then

$$a(g) = \begin{bmatrix} 0 & \mathcal{D}_{a(g)} \\ \mathcal{D}_{a(g)} & 0 \end{bmatrix} \quad \text{with } \mathcal{D}_{a(g)} = \text{diag}[\log a_1, \ldots, \log a_{p-1}, \text{sgn}(h_p) \log a_p].$$

Note that this method does not allow us to distinguish between the situations where $h_p$ is positive or negative.

*Singular elements of* $\mathfrak{a}$. In what follows, we will consider singular elements $X$, $Y \in \partial \mathfrak{a}^+$. We need to control the irregularity of $X$ and $Y$, that is, consider the simple positive roots annihilating $X$ and $Y$. A special role is played by the last simple root $\alpha_p = h_{p-1} + h_p$, different from the simple roots $\alpha_i(H) = h_i - h_{i+1}$, $i = 1, \ldots, p-1$. Note that $\alpha_p(X) = 0$ implies that the last diagonal entry of $\mathcal{D}_X$ is negative or zero.

We introduce the following definition of the configuration of $X \in \overline{\mathfrak{a}^+}$.

DEFINITION 2.2. Let $X \in \overline{\mathfrak{a}^+}$. In what follows, $x_i > 0$, $x_i > x_j$ for $i < j$, $s_i \geq 1$, $u \geq 0$ and $\sum_{i=1}^r s_i + u = p$. Let $\mathbf{s} = (s_1, \ldots, s_r)$. We define the configuration of $X$ by:

$$[\mathbf{s}; u] \text{ if } \mathcal{D}_X = \text{diag}[\overbrace{x_1, \ldots, x_1}^{s_1}, \overbrace{x_2, \ldots, x_2}^{s_2}, \ldots, \overbrace{x_r, \ldots, x_r}^{s_r}, \overbrace{0, \ldots, 0}^{u}] \tag{2.2}$$

$$\mathbf{s} \text{ if } \mathcal{D}_X = \text{diag}[\overbrace{x_1, \ldots, x_1}^{s_1}, \overbrace{x_2, \ldots, x_2}^{s_2}, \ldots, \overbrace{x_{r-1}, \ldots, x_{r-1}}^{s_{r-1}}, \overbrace{-x_r}^{s_r=1}] \tag{2.3}$$

$$\mathbf{s}^- \text{ if } \mathcal{D}_X = \text{diag}[\overbrace{x_1, \ldots, x_1}^{s_1}, \overbrace{x_2, \ldots, x_2}^{s_2}, \ldots, \overbrace{x_r, \ldots, x_r}^{s_r}, -x_r]. \tag{2.4}$$

We extend the definition of configuration naturally to any $X \in \mathfrak{a}$, whose configuration is defined as that of the projection $\pi(X)$ of $X$ on $\overline{\mathfrak{a}^+}$.

REMARK 2.3. We will often write $X = X[\cdots]$ when $[\cdots]$ is a configuration of $X$. For the configuration (2.4), we write the $-$ superscript in $X[\mathbf{s}]^-$ to indicate that $X$ contains nonzero opposite entries. We omit the $-$ superscript in (2.3) and write $X[\mathbf{s}]$ because we are essentially in the same case as in (2.2) with $u = 0$ and $s_r = 1$.

If the number of zero entries $u = 0$, it may be omitted in (2.2). In particular, in the configurations (2.3) and (2.4), $u = 0$. Note that $X = 0$ has configuration $[0; p]$. A regular $X \in \mathfrak{a}^+$ has the configuration $[1^p; 0]$ or $[1^{p-1}; 1]$, where $1^k = (1, \ldots, 1)$ with 1 repeated $k$ times.

In what follows, we will write $\max \mathbf{s} = \max_i s_i$ and $\max(\mathbf{s}, u) = \max(\max \mathbf{s}, u)$. We will show that in the case of the symmetric spaces $\mathbf{SO}_0(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$, the criterion for the existence of the density of the convolution $\delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$ is given by the following definition of an eligible pair $X$ and $Y$.

DEFINITION 2.4. Let $X$ and $Y$ be two elements of $\mathfrak{a}$ with configurations $[\mathbf{s}; u]$ or $[\mathbf{s}]^-$ and $[\mathbf{t}; v]$ or $[\mathbf{t}]^-$, respectively. We say that $X$ and $Y$ are eligible if one of the two following cases holds

$$u \leq 1, v \leq 1 \quad \text{and} \quad \max(\mathbf{s}) + \max(\mathbf{t}) \leq 2p - 2, \tag{2.5}$$

$$u \geq 2 \text{ or } v \geq 2 \quad \text{and} \quad \max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) \leq 2p \tag{2.6}$$

and, for $p = 4$, $\{\mathcal{D}_X, \mathcal{D}_Y\} \neq \{\text{diag}[a, a, a, a], \text{diag}[b, b, c, c]\}$ nor

$$\{\text{diag}[a, a, a, -a], \text{diag}[b, b, c, -c]\}, \tag{2.7}$$

for any $a \neq 0$, $b \neq 0$, $|b| \neq |c|$.

Observe that if $X$ and $Y$ are eligible, then $X \neq 0$ and $Y \neq 0$. The noneligible pairs given by (2.7) are $\{[4], [(2, 2)]\}$, $\{[4]^-, [(2, 2)]^-\}$ with $u = v = 0$ or $\{[4], [2; 2]\}$ and $\{[4]^-, [2; 2]\}$ with $u = 0$ and $v = 2$.

REMARK 2.5. It is interesting to note that the definition of eligible pairs is more complicated for the space $\mathbf{SO}(p, p)$ than for the spaces $\mathbf{SO}(p, q)$ with $p < q$ (recall that the latter spaces have a much richer root structure). As for the spaces $\mathbf{SO}(p, q)$ with $p < q$, the number of zeros on the diagonal of $\mathcal{D}_X$ is important. Unlike in the case $\mathbf{SO}(p, q)$ with $p < q$, this only becomes a factor when the number of zeros is greater than one (as opposed to greater than zero).

In [7], we showed that if $p < q$ and $X$ and $Y \in \mathfrak{a}$ were such that the $\mathcal{D}_X$ and $\mathcal{D}_Y$ have no zero diagonal elements on the diagonal then $\mu_{X,Y}$ was absolutely continuous. This is no longer the case when $p = q$ and this is one of main differences between the $\mathbf{SO}(p, p)$ and $\mathbf{SO}(p, q)$ cases. Another difference is the anomalous case when $p = 4$ seen in (2.7) and the fact that lower-dimensional cases require different proofs. On the other hand, when the number of zeros on the diagonal of either $\mathcal{D}_X$ or $\mathcal{D}_Y$ is at least two, then the proof of Theorem A is similar to that found in [7] but requires separate consideration of a low-dimension case $X[5], Y[3; 2]$.

## 3. Basic tools and reductions

DEFINITION 3.1. For $Z \in \mathfrak{a}$, let $V_Z$ be the subspace of $\mathfrak{p}$ defined by

$$V_Z = \text{span}\{X_\alpha - \theta(X_\alpha) \mid \alpha(Z) \neq 0\} \subset \mathfrak{p}.$$

We denote by $|V_Z|$ the dimension of $V_Z$. It equals the number of positive roots $\alpha$ such that $\alpha(Z) \neq 0$.

The following definition and lemmas will help reduce the number of cases of configurations of $(X, Y)$ to verify.

DEFINITION 3.2. We will say that $X$ and $X' \in \mathfrak{a}$ are *relatives* if exactly one of the diagonal entries of $\mathcal{D}_X$ and $\mathcal{D}_{X'}$ differs by sign. If $X$ is a relative of $X'$ and $Y$ is a relative of $Y'$ then we will say that $(X, Y)$ is a relative pair of $(X', Y')$.

The properties listed in the following lemma are straightforward.

LEMMA 3.3.

(1) $X$ is a relative of $X$ if and only if $\mathcal{D}_X$ has at least one diagonal entry equal to zero, that is, $u \geq 1$ in $X[\mathbf{s}; u]$. Thus $X \in \overline{\mathfrak{a}^+}$ has no other relatives in $\overline{\mathfrak{a}^+}$.
(2) If $X$ and $X'$ are relatives then $|V_X| = |V_{X'}|$.
(3) If $(X, Y)$ and $(X', Y')$ are relative pairs then $(X, Y)$ is an eligible pair if and only if $(X', Y')$ is an eligible pair.
(4) If $X = X[\mathbf{s}; u] \in \overline{\mathfrak{a}^+}$ with $u > 0$ then all $X_i$ are nonnegative.
(5) If $X = X[\mathbf{s}; u]$, $Y[\mathbf{t}; v] \in \overline{\mathfrak{a}^+}$ are such that $u > 0$ or $v > 0$ then either $\mathcal{D}_X$, $\mathcal{D}_Y$ have no negative entries or we can choose a relative pair $X'$, $Y' \in \overline{\mathfrak{a}^+}$ in such a way that $\mathcal{D}_{X'}$, $\mathcal{D}_{Y'}$ have no negative entries.
(6) If $X$ is a relative of $X'$ and $X'$ is a relative of $X''$ then $X$ and $X''$ are in the same Weyl-group orbit.

LEMMA 3.4. If $(X, Y)$ and $(X', Y')$ are relative pairs then either both sets $Ke^X K e^Y K$ and $Ke^{X'} K e^{Y'} K$ have nonempty interiors or neither has.

PROOF. Let $J_0 = \text{diag}\{\overbrace{1, \ldots, 1}^{2p-1}, -1\}$ and note that $J_0$ is orthogonal and that $J_0^2 = I \in \mathbf{SO}(p) \times \mathbf{SO}(p)$. Suppose that $X$, $Y$, $X'$ and $Y'$ are as in the statement of the lemma. Suppose that $w_1, w_2 \in W \subset K$ are such that the element of $\mathcal{D}_{w_1 \cdot X'}$ (respectively, $\mathcal{D}_{w_2 \cdot Y'}$) that differs by sign from the corresponding element of $\mathcal{D}_X$ (respectively, $\mathcal{D}_Y$) is placed at the end. Then

$$\begin{aligned}
K e^X K e^Y K &= J_0 K J_0 w_1 e^X w_1^{-1} J_0 K J_0 w_2 e^Y w_2^{-1} J_0 K J_0 \\
&= J_0 K (J_0 w_1 e^X w_1^{-1} J_0) K (J_0 w_2 e^Y w_2^{-1} J_0) K J_0 \\
&= J_0 K e^{X'} K e^{Y'} K J_0
\end{aligned}$$

which allows us to conclude. □

In the sequel we strengthen and complete some ideas, results and notation of [6, Section 3]. These lead to Corollary 3.9 which will be our main tool.

PROPOSITION 3.5.

(i)    *The density of the measure $m_{X,Y}$ exists if and only if its support $Ke^X Ke^Y K$ has nonempty interior.*

(ii)   *Consider the analytic map $T : K \times K \times K \to G$ defined by*

$$T(k_1, k_2, k_3) = k_1 e^X k_2 e^Y k_3.$$

*The set $T(K \times K \times K) = Ke^X Ke^Y K$ contains an open set if and only if the derivative of $T$ is surjective for some choice of $\mathbf{k} = (k_1, k_2, k_3)$.*

PROOF. We first prove that the absolute continuity of the measure $m_{X,Y}$ is equivalent to the surjectivity of the derivative of $T$ for some $\mathbf{k}$. The main tool of the proof is Sard's theorem (see, for example, [9, page 479] and the reference therein). This states that if $T$ is analytic and $C$ is the set of its critical points, where $dT$ is not surjective, then the invariant measure of $T(X)$ is zero. Thus, if $dT$ is never surjective, then the support of $m_{X,Y}$, equal to $T(K \times K \times K)$, is of measure zero on $G$ and, consequently, $m_{X,Y}$ has no density. We have proved that the existence of a density of $m_{X,Y}$ implies the surjectivity of the derivative of $T$ for some $\mathbf{k} = (k_1, k_2, k_3)$. The converse, also based on classical differential calculus, was proved in [4, Proposition 2.8] on the level of the map $F(k) = a(e^X k e^Y)$ and remains true for the map $T$. Part (ii) is justified, for example, by Helgason [9, page 479] and part (i) follows.                                            □

COROLLARY 3.6. *Let $(X, Y)$ and $(X', Y')$ be relative pairs. The measure $m_{X,Y}$ is absolutely continuous if and only if the measure $m_{X',Y'}$ is absolutely continuous.*

PROOF. We use Proposition 3.5(i) and Lemma 3.4.                                            □

PROPOSITION 3.7. *Let $U_Z = \mathfrak{k} + \mathrm{Ad}(e^Z)\mathfrak{k}$. The measure $m_{X,Y}$ is absolutely continuous if and only if there exists $k \in K$ such that*

$$U_{-X} + \mathrm{Ad}(k)U_Y = \mathfrak{g}. \tag{3.1}$$

PROOF. We want to show that this condition is equivalent to the derivative of $T$ at $\mathbf{k}$ being surjective. We have

$$
\begin{aligned}
dT_{\mathbf{k}}(A, B, C) &= \frac{d}{dt}\bigg|_{t=0} e^{tA} k_1 e^X e^{tB} k_2 e^Y e^{tC} k_3 \\
&= A\,k_1 e^X k_2 e^Y k_3 + k_1 e^X B\,k_2 e^Y k_3 + k_1 e^X k_2 e^Y C\,k_3 \tag{3.2}
\end{aligned}
$$

We now transform the space of all matrices of the form (3.2) without modifying its dimension:

$$
\begin{aligned}
\dim\{&A\,k_1 e^X k_2 e^Y k_3 + k_1 e^X B\,k_2 e^Y k_3 + k_1 e^X k_2 e^Y C\,k_3 : A, B, C \in \mathfrak{k}\} \\
&= \dim\{k_1^{-1} A\,k_1 e^X k_2 e^Y + e^X B\,k_2 e^Y + e^X k_2 e^Y C : A, B, C \in \mathfrak{k}\} \\
&= \dim\{A\,e^X k_2 e^Y + e^X B\,k_2 e^Y + e^X k_2 e^Y C : A, B, C \in \mathfrak{k}\} \\
&= \dim\{e^{-X} A\,e^X + B + k_2 e^Y C\,e^{-Y}k_2^{-1} : A, B, C \in \mathfrak{k}\}.
\end{aligned}
$$

The space in the last line equals $\mathfrak{k} + \mathrm{Ad}(e^{-X})(\mathfrak{k}) + \mathrm{Ad}(k_2)(\mathrm{Ad}(e^Y)(\mathfrak{k})) = U_{-X} + \mathrm{Ad}(k_2)U_Y$. □

In order to apply condition (3.1), we will consider convenient symmetrized root vectors and the spaces $V_Z$ generated by them.

LEMMA 3.8. *Let $Z \in \mathfrak{a}$. Then $U_Z = \mathfrak{k} + V_Z = U_{-Z}$.*

PROOF. Clearly $V_Z = V_{-Z}$. We show that $V_Z \subset U_Z$ and therefore that $\mathfrak{k} + V_Z \subset U_Z$. Let $\alpha$ be a root such that $\alpha(Z) \neq 0$. Note that $[Z, X_\alpha] = \alpha(Z)X_\alpha$ and $[Z, \theta(X_\alpha)] = -\alpha(Z)\theta(X_\alpha)$. Let $U = X_\alpha + \theta(X_\alpha) \in \mathfrak{k}$. Now

$$\mathrm{Ad}(e^Z)U = e^{\mathrm{ad}\,Z}(X_\alpha + \theta(X_\alpha)) = e^{\alpha(Z)}X_\alpha + e^{-\alpha(Z)}\theta(X_\alpha).$$

Therefore $X_\alpha = (e^{\alpha(Z)} - e^{-\alpha(Z)})^{-1}(-e^{-\alpha(Z)}U + \mathrm{Ad}(e^Z)U) \in \mathfrak{k} + \mathrm{Ad}(e^Z)(\mathfrak{k}) = U_Z$. The vector $\theta X_\alpha$ is a root vector for the root $-\alpha$, so we also have $\theta X_\alpha \in U_Z$.

It remains to show that $U_Z \subset \mathfrak{k} + V_Z$. It suffices to show that $\mathrm{Ad}(e^Z)\mathfrak{k} \subset \mathfrak{k} + V_Z$: for every $\alpha$, $\mathrm{Ad}(e^Z)(X_\alpha + \theta(X_\alpha)) = e^{\alpha(Z)}X_\alpha + e^{-\alpha(Z)}\theta(X_\alpha) = ((e^{\alpha(Z)} + e^{-\alpha(Z)})/2)(X_\alpha + \theta(X_\alpha)) + ((e^{\alpha(Z)} - e^{-\alpha(Z)})/2)(X_\alpha - \theta(X_\alpha)) \in \mathfrak{k} + V_Z$. □

The following corollary is then straightforward.

COROLLARY 3.9. *The measure $m_{X,Y}$ is absolutely continuous if and only if there exists $k \in K$ such that*

$$V_X + \mathrm{Ad}(k)V_Y = \mathfrak{p}. \tag{3.3}$$

COROLLARY 3.10. *The measure $m_{X,Y}$ is absolutely continuous if and only if there exists a dense open subset $U \subset K$ such that, for every $k \in U$:*

(1) $V_{w_1 \cdot X} + \mathrm{Ad}(k)V_{w_2 \cdot Y} = \mathfrak{p}$, *for every $w_1, w_2 \in W$;*

(2) *for every $r < 2p$, the matrix obtained by removing the first $r$ rows and $r$ columns of $k$ is nonsingular.*

PROOF. First we note that condition (3.3) for some $k$ is actually equivalent to the existence of a dense open subset $U \subset K$ such that (3.3) holds for every $k \in U$. Indeed, since equality (3.3) can be expressed in terms of nonzero determinants, if it is satisfied for one value of $k$, it will be satisfied for every $k$ in a dense open subset of $K$.

In addition, (3.3) is equivalent to the fact that $a(e^X K e^Y)$ has nonempty interior which, in turn, implies that $a(e^{w_1 \cdot X} K e^{w_2 \cdot Y})$ has nonempty interior for any given $w_1$, $w_2 \in W$ and for every $k \in U_{w_1,w_2}$ where $U_{w_1,w_2}$ is open and dense. Hence, for any given $w_1, w_2 \in W$, there is a dense open set $U_{w_1,w_2}$ with $V_{w_1 \cdot X} + \mathrm{Ad}(k)V_{w_2 \cdot Y} = \mathfrak{p}$. For similar reasons, there exists a dense open subset of $K$ such that the second condition is satisfied (the condition being satisfied by the identity matrix). Given that a finite intersection of dense open sets is a dense open set, the statement follows. □

REMARK 3.11. Corollary 3.9 and the fact that $V_{w \cdot X} = \mathrm{Ad}(w)V_X$ for $w \in W$ and $X \in \mathfrak{a}$ (given that $V_X = \bigoplus_{\alpha(X)\neq 0} \mathfrak{g}_\alpha$) imply that in the proof of Theorem A one can assume that $X$ has a configuration $[\mathbf{s}; u]$ with $s_1 \geq s_2 \geq \cdots \geq s_r$ or a configuration $[\mathbf{s}]^-$ with $s_1 \geq s_2 \geq \cdots \geq s_{r-1}$. The same remark applies to the configuration of $Y$.

The following necessary criterion for the existence of the density will be very useful.

COROLLARY 3.12. *If $m_{X,Y}$ is absolutely continuous then $|V_X| + |V_Y| \geq \dim \mathfrak{p} = p^2$.*

The following definition and results will be helpful in resolving the exceptional case indicated in (2.7).

DEFINITION 3.13. For $n \geq 1$, let $\mathcal{Z}(n)$ be the group formed by the matrices of the form

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \\ & & \ddots \\ & & & \cos\theta_r & -\sin\theta_r \\ & & & \sin\theta_r & \cos\theta_r \end{bmatrix}$$

where the last block is replaced by 1 if $n$ is odd.

REMARK 3.14. Note that $\dim \mathcal{Z}(n) \leq n/2$ and that each element in $\mathcal{Z}(n)$ has a square root in $\mathcal{Z}(n)$.

LEMMA 3.15. *Consider $n \geq 2$ and $k \in \mathbf{SO}(n)$. Then there exists $A \in \mathbf{SO}(n)$ such that $A^{-1}kA \in \mathcal{Z}(n)$.*

PROOF. Consult, for example, [2]. Recall that the eigenvalues of $k$ are $e^{\pm i\theta_j}, \theta_j \in \mathbf{R}$. □

COROLLARY 3.16. *Every matrix $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbf{SO}(p) \times \mathbf{SO}(p)$ can be written in the form*

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$$

*with $A, C \in \mathbf{SO}(p)$ and $B \in \mathcal{Z}(p)$.*

PROOF. According to Lemma 3.15, there exists a matrix $A \in \mathbf{SO}(p)$ such that $B' = A^{-1}A_1 A_2^{-1} A \in \mathcal{Z}(p)$. Pick $B \in \mathcal{Z}(p)$ such that $B^2 = B'$ and let $C = B^{-1} A^{-1} A_1$. Then $ABC = A_1$ and

$$A B^{-1} C = A B^{-1} (B^{-1} A^{-1} A_1) = A (B^2)^{-1} A^{-1} A_1$$
$$= A (B')^{-1} A^{-1} A_1 = A (A^{-1} A_1 A_2^{-1} A)^{-1} A^{-1} A_1 = A_2$$

which proves the lemma.                                                                                     □

REMARK 3.17. The matrices $\begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix}$ in the last corollary can be written as $\prod_{i=1}^{[p/2]} k_{Z_{2i-1,2i}^+}^{t_i}$ for an appropriate choice of $t_i$s.

In the proof of the necessity of the eligibility condition, we will use the following result stated in [5, Step 1, page 1767]. Let the Cartan decomposition of $\mathbf{SL}(N, \mathbf{F})$ be written as $g = k_1 e^{\tilde{a}(g)} k_2$.

LEMMA 3.18. *Let $U = \operatorname{diag}([\overbrace{u_0, \ldots, u_0}^{r}, u_1, \ldots, u_{N-r}]$ and $V = \operatorname{diag}([\overbrace{v_0, \ldots, v_0}^{s}, v_1, \ldots, v_{N-s}])$ where $r < N$, $s < N$, $r + s > N$, and the $u_i$ and $v_j$ are arbitrary. Then each element of the diagonal of $\tilde{a}(e^U \operatorname{SU}(N, \mathbf{F}) e^V)$ has at least $r + s - N$ entries equal to $u_0 + v_0$.*

We will use Lemma 3.18 with $N = p + q$ in the proofs of Proposition 4.5 and Theorem 6.7.

In the proof of Theorem A we will need the following technical lemma [7, Lemma 4.7]. Its proof involves a careful evaluation of $\operatorname{Ad}(e^{t(Z+\theta Z)})(W)$ for appropriate root vectors $Z$. One also uses the well-known properties $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $[X_\alpha, \theta X_\alpha] \in \mathfrak{a}$.

LEMMA 3.19.

(1) *For the root vectors $Z_{i,j}^+$ and $Y_{i,j}^+$,*

$$\operatorname{Ad}(e^{t(Y_{i,j}^+ + \theta(Y_{i,j}^+))})(Y_{i,j}) = \cos(4t)Y_{i,j} + 2\sin(4t)(A_i - A_j),$$
$$\operatorname{Ad}(e^{t(Z_{i,j}^+ + \theta(Z_{i,j}^+))})(Z_{i,j}) = \cos(4t)Z_{i,j} + 2\sin(4t)(A_i + A_j).$$

(2) *The operators $\operatorname{Ad}(e^{t(Y_{i,j}^+ + \theta(Y_{i,j}^+))})$ and $\operatorname{Ad}(e^{t(Z_{i,j}^+ + \theta(Z_{i,j}^+))})$ applied to the other symmetrized root vectors do not produce any components in $\mathfrak{a}$.*

In the proof of the existence of the density for the pairs $X[4], Y[2, 2]^-$ and $X[5], Y[3; 2]$ without predecessors, we will need the following elementary lemma. Recall that $Z_{k,l} = \frac{1}{2}(Z_{k,l}^+ - \theta(Z_{k,l}^+)) \in \mathfrak{p}$.

LEMMA 3.20. *Let $1 \le i < j \le p$ and $1 \le k < l \le p$. Then*

$$[Z_{i,j}^+ + \theta(Z_{i,j}^+), Z_{k,l}] = \begin{cases} 0 & \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \\ 4(A_i + A_j) & \text{if } \{i, j\} = \{k, l\}, \\ 2Y_{\min(j,l),\max(j,l)} & \text{if } i = k, \ j \ne l, \\ 2Y_{\min(i,k),\max(i,k)} & \text{if } i \ne k, \ j = l, \\ -2Y_{i,l} & \text{if } j = k, \\ -2Y_{k,j} & \text{if } i = l. \end{cases}$$

PROOF. We apply the well-known fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ when $\alpha + \beta$ is a root and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ otherwise. For the computation of exact coefficients in the formulas, we use Table 1. □

## 4. Necessity of the eligibility condition

PROPOSITION 4.1. *If $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v] \in \mathfrak{a}$ with $u \le 1$ and $v \le 1$ are such that $\max \mathbf{s} + \max \mathbf{t} > 2p - 2$ then $|V_X| + |V_Y| < p^2$.*

PROOF. Assume that $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v] \in \bar{\mathfrak{a}}^+$. Without loss of generality, assume that $\max \mathbf{s} \geq \max \mathbf{t}$. We then have $\max \mathbf{s} = p$ and $\max \mathbf{t} \geq p - 1$. The only possible pairs are

$$
\begin{aligned}
&X[p], Y[p] \text{ and the relative pair } X'[p]^-, Y'[p]^-, \\
&X[p], Y[p-1, 1] \text{ and the relative pair } X'[p]^-, Y'[1, p-1]^-, \\
&\quad X[p], Y[p]^- \text{ and the relative pair } (X', Y') = (Y, X), \\
&X[p], Y[1, p-1]^- \text{ and the relative pair } X'[p]^-, Y'[p-1, 1].
\end{aligned}
\tag{4.1}
$$

By Remark 3.11 we do not need to consider the configuration $[1, p-1]$. We have $|V_X| = p(p-1)/2$ and $|V_{Y[p-1,1]}| = p(p-1)/2 + p - 1$. We apply Lemma 3.3 and find by examination that $|V_X| + |V_Y| \leq p^2 - 1$ in all cases. □

COROLLARY 4.2. *Let $p \geq 2$. Consider a pair $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$ with $u \leq 1$ and $v \leq 1$. Then $|V_X| + |V_Y| \geq p^2$ if and only if*

$$
\max(\mathbf{s}) + \max(\mathbf{t}) \leq 2p - 2.
\tag{4.2}
$$

PROOF. By Proposition 4.1 only the sufficiency of condition (4.2) needs to be proven. Suppose that $\max(\mathbf{s}) + \max(\mathbf{t}) \leq 2p - 2$ and that $\max \mathbf{s} \geq \max \mathbf{t}$. If $\max \mathbf{s} = p$ then $|V_X| = p(p-1)/2$ and $\max \mathbf{t} \leq p - 2$ implies $|V_Y| \geq p(p-1)/2 + 2(p-2)$. If both $\max \mathbf{s} \leq p - 1$ and $\max \mathbf{t} \leq p - 1$ then $|V_X| \geq p(p-1)/2 + p - 1$ and $|V_Y| \geq p(p-1)/2 + p - 1$. In both cases, the result follows. □

DEFINITION 4.3. We will call the set of configurations listed in *(4.1)* exceptional and denote it by $\mathcal{E}$.

PROPOSITION 4.4. *Let $X, Y \in \mathfrak{a}$ be such that $\mathcal{D}_X = \mathrm{diag}[b, b, c, c]$ with $b > c > 0$ and $\mathcal{D}_Y = \mathrm{diag}[a, a, a, a]$ with $a > 0$. Then $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ has no density.*

PROOF. According to Corollary 3.16, we can write

$$
\begin{aligned}
K e^X K e^Y K &= K e^X \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} k^{t_1}_{Z^+_{1,2}} k^{t_2}_{Z^+_{3,4}} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} e^Y K \\
&= K e^X \exp\left( \begin{bmatrix} 0 & R & 0 & 0 \\ -R & 0 & 0 & 0 \\ 0 & 0 & 0 & R \\ 0 & 0 & -R & 0 \end{bmatrix} \right) k^{t_1}_{Z^+_{1,2}} k^{t_2}_{Z^+_{3,4}} e^Y K \\
&= K e^X k^{r_1}_{Y^+_{1,3}} k^{r_2}_{Y^+_{2,4}} k^{t_1}_{Z^+_{1,2}} k^{t_2}_{Z^+_{3,4}} e^Y K
\end{aligned}
$$

(here $Z^+_{1,2}, Z^+_{3,4}, Y^+_{1,3}, Y^+_{2,4}$ are exactly as in Table 1). We have used the fact that $e^Y$ and $\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$ commute, the Cartan decomposition

$$
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \exp\left( \begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix} \right) \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}
$$

$(R = \mathrm{diag}[r_1, r_2]$, $A_i$, $C_i \in \mathbf{SO}(2))$, and the facts that $e^X$, $\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}$ commute and

$\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix}$ commutes with $k_{Z_{1,2}^+}^{t_1} k_{Z_{3,4}^+}^{t_2}$ and with $e^Y$.

Now it is easy to see by considering the proof of Proposition 3.5(ii) that for these particular $X$ and $Y$, the condition $V_X + \mathrm{Ad}(k)V_Y = \mathfrak{p}$ must be satisfied by $k$ of the form $k_0 = k_{Y_{1,3}^+}^{r_1} k_{Y_{2,4}^+}^{r_2} k_{Z_{1,2}^+}^{t_1} k_{Z_{3,4}^+}^{t_2}$. On the other hand, $V_Y = \langle Z_{i,j}, \ i < j \rangle$ and $\mathrm{Ad}(k_0)(Z_{i,j})$, $i < j$, can only produce diagonal elements which satisfy $h_1 + h_3 = h_2 + h_4$, as can be checked by Lemma 3.19 and using the fact that $\mathrm{Ad}(k_0) = \mathrm{Ad}(k_{Y_{1,3}^+}^{r_1}) \, \mathrm{Ad}(k_{Y_{2,4}^+}^{r_2}) \, \mathrm{Ad}(k_{Z_{1,2}^+}^{t_1}) \, \mathrm{Ad}(k_{Z_{3,4}^+}^{t_2})$. Consequently, $\mathfrak{a} \not\subset V_X + \mathrm{Ad}(k_0)V_Y$ and the density cannot exist. □

PROPOSITION 4.5. *If $X$ and $Y$ are not eligible then the density does not exists.*

PROOF. Let the configuration of $X$ be $[\mathbf{s}; u]$ or $[\mathbf{s}]^-$ and the configuration of $Y$ be $[\mathbf{t}; v]$ or $[\mathbf{t}]^-$. Propositions 4.1, 4.4 and Corollary 3.6 imply that the density does not exist when $X$ and $Y$ are not eligible and $u \le 1$ and $v \le 1$.

Suppose then that $u \ge 2$ or $v \ge 2$ and $\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) > 2p$ and consider the matrices $a(e^X k e^Y)$, $k \in \mathbf{SO}(p) \times \mathbf{SO}(p)$. Using Lemmas 3.3(5) and 3.4, we may assume that the diagonal entries of $\mathcal{D}_X$ and $\mathcal{D}_Y$ are nonnegative. Applying Remark 2.1, we have

$$\tilde{a}(e^X k e^Y) = \tilde{a}((\overbrace{S^T e^X S}^{e^{S^T X S}}) \, (\overbrace{S^T k S}^{\in \mathbf{SO}(p+q)}) \, (\overbrace{S^T e^Y S}^{e^{S^T Y S}}))$$

where $\tilde{a}(g)$ is the diagonal matrix with the singular values of $g$ on the diagonal, in decreasing order (see the explanation before Lemma 3.18).

If $u + v > p$ then there are $r + s - N = 2u + 2v - 2p = 2(u + v - p)$ repetitions of $0 + 0 = 0$ in coefficients of $\tilde{a}(e^X k e^Y)$. Therefore zero occurs at least $u + v - p > 0$ times as a diagonal entry of $\mathcal{D}_H$ for every $H \in a(e^X K e^Y)$, which implies that $a(e^X K e^Y)$ has empty interior. If $2u + \max(\mathbf{t}) > 2p$, denote $t = \max(\mathbf{t})$. Let $Y_i \ne 0$ be repeated $t$ times in $\mathcal{D}_Y$ (or, if $t = t_r$ and $Y = Y[t]^-$, we have $t - 1$ times $Y_r$ and once $-Y_r$ in $\mathcal{D}_Y$). Then there are $r + s - N = 2u + t - 2p$ repetitions of $Y_i + 0$ in coefficients of $\tilde{a}(e^X k e^Y)$. Therefore $Y_i$ occurs at least $2u + t - 2p > 0$ times as a diagonal entry of $\mathcal{D}_H$ for every $H \in a(e^X K e^Y)$, which implies that $a(e^X K e^Y)$ has empty interior. □

# 5. Sufficiency of the eligibility condition

## 5.1. Case $u \le 1$ and $v \le 1$.

REMARK 5.1. In our proof, the case $u \le 1$ and $v \le 1$ is equivalent to the case $u = v = 0$. Indeed, for $H \in \overline{\mathfrak{a}^+}$, if the sole diagonal entry zero in $\mathcal{D}_H$ is replaced by a positive entry

different from the existing diagonal entries of $\mathcal{D}_H$, then $V_H$ is unchanged. We will therefore assume in this section that $u = 0$ and $v = 0$.

DEFINITION 5.2. Let $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be two partitions of $p$ ($\sum_i s_i = p = \sum_i t_j$). We will say that $\mathbf{s}$ is finer than $\mathbf{t}$ if the $t_i$ are sums of disjoint subsets of the $s_j$ (for example, $\mathbf{s} = [3, 2, 2, 1, 1, 1]$ is finer than $\mathbf{t} = [5, 3, 2] = [3 + 2, 2 + 1, 1 + 1]$).

REMARK 5.3. If $X = X[\mathbf{s}]$ and $Y = Y[\mathbf{t}]$ and $\mathbf{s}$ is finer than $\mathbf{t}$ then $V_Y \subset V_X$.

In the following lemma, we significantly reduce the number of elements for which we must prove the existence of the density.

LEMMA 5.4. *For $p \geq 5$, it is sufficient to prove the existence of the density in the following cases:*

$S_1$:    $X[p], Y[p - k, k]$ *for* $p - k \geq k \geq 2$;
$S_2$:    $X[p]^-, Y[p - k, k]$ *for* $p - k \geq k \geq 2$;
$S_3$:    $X[1, p - 1]^-, Y[p - 1, 1]$;
$S_4$:    $X[p - 1, 1], Y[p - 1, 1]$.

*For $p = 4$ the same is true provided the case $S_1$ is replaced by the cases $X[4], Y[2, 1, 1]$ and $X[3, 1], Y[2, 2]$.*

PROOF. Suppose that $p \geq 5$. Let us call $A_0$ the configurations of the form $[\mathbf{s}]$ and $A_1$ all the others, that is, the configurations of the form $[\mathbf{s}]^-$.

(a) We first observe that if the density exists for $S_1$ then it follows that it exists for all pairs $\{X, Y\}$ such that $X, Y \in A_0$, except when $X$ or $Y$ have configurations $[p]$ or $[p - 1, 1]$. This comes from the fact that all these $X, Y$ have structures that are finer and, consequently, the corresponding $V_X$ and $V_Y$ are larger. Thus, existence of the density in the cases $S_1$ together with $S_4$ will imply the existence of the density in all the cases when $X, Y \in A_0$, except when $\{X, Y\} \in \mathcal{E}$.

(b) By switching to relatives and changing the order of $X$ and $Y$, we see that it implies the existence of the density in all the cases when $X, Y \in A_1$, except when $\{X, Y\} \in \mathcal{E}$.

(c) It remains to show that the cases $S_2$ and $S_3$ imply the existence of the density in all the cases when $X \in A_1$ and $Y \in A_0$, except when $\{X, Y\} \in \mathcal{E}$. Note first that if $X = [\mathbf{s}]^-$ then either $[\mathbf{s}]^- = [p]^-$ or $V_{X'} \subset V_X$ with $X' = X'[1, p - 1]^-$. In the first case, we observe that the case $S_2$ implies the cases $X[p]^-$ and $Y \in A_0 \setminus \{[p], [p - 1, 1]\}$ for the same reason as in (a). The only cases that remain with $Y[p - 1, 1]$ are covered by $S_3$. Finally, switching to relatives, we get the pairs $X \in A_1, Y[p]$ which are not in $\mathcal{E}$.

We illustrate the proof of the lemma in the case $p = 5$.

| $[2,1^3]$ | $[2,2,1]$ | $[3,1,1]$ | $[3,2]$ | $[4,1]$ | $[5]$ | $[1^3,2]^-$ | $[2,1,2]^-$ | $[1,1,3]^-$ | $[3,2]^-$ | $[2,3]^-$ | $[1,4]^-$ | $[5]^-$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | $[2,1^3]$ |
| | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | $[2,2,1]$ |
| | | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | $[3,1,1]$ |
| | | | √ | √ | $s_1$ | √ | √ | √ | √ | √ | √ | $s_2$ | $[3,2]$ |
| | | | | $s_4$ | x | √ | √ | √ | √ | √ | $s_3$ | x | $[4,1]$ |
| | | | | | x | √ | √ | √ | √ | √ | x | x | $[5]$ |
| | | | | | | √ | √ | √ | √ | √ | √ | √ | $[1^3,2]^-$ |
| | | | | | | | √ | √ | √ | √ | √ | √ | $[2,1,2]^-$ |
| | | | | | | | | √ | √ | √ | √ | √ | $[1,1,3]^-$ |
| | | | | | | | | | √ | √ | √ | √ | $[3,2]^-$ |
| | | | | | | | | | | √ | √ | √ | $[2,3]^-$ |
| | | | | | | | | | | | √ | x | $[1,4]^-$ |
| | | | | | | | | | | | | x | $[5]^-$ |

In the above table, √ indicates that the pair is eligible, X indicates that the pair is not eligible and therefore that the density does not exist (the cases identified in (4.1)), and the $S_i$ correspond to the notation above (the pair is eligible where the $S_i$ appear). We use the reduction from Remark 3.11.                                    □

THEOREM 5.5. *Let* $p \geq 2$ *and suppose that* $X = X[\mathbf{s}; u]$ *or* $[\mathbf{s}]^-$ *and* $Y = Y[\mathbf{t}; v]$ *or* $[\mathbf{t}]^-$ *are such that* $u \leq 1$ *and* $v \leq 1$. *If the pair* $\{X, Y\}$ *does not belong to the set* $\mathcal{E} \cup \{[4], [2, 2]\} \cup \{[4]^-, [2, 2]^-\}$ *then the density exists.*

PROOF. The proof proceeds by induction. The induction principle was applied similarly in [7], but in the case $D_p$ considered here we need a different 'asymmetric' technique for executing Steps 2 and 3. Also, for smaller values of $p$ separate proofs are required, due to the lack of available good predecessors. With some exceptions in the starting phase of the induction, and in the case $S_3$ of Lemma 5.4, the elements $X$ and $Y$ will be in $\overline{\mathfrak{a}^+}$ and their 'usual' predecessors will be obtained by skipping the first diagonal terms of $\mathcal{D}_X$ and $\mathcal{D}_Y$.

The only case of existence of the density for $p = 2$ is for regular $X[1, 1]$ and $Y[1, 1]$ (according to (4.1), only the pair $X[1, 1], Y[1, 1]$ of two regular elements satisfies $|V_X| + |V_Y| \geq p^2$).

For $p = 3$, we have four possible configurations of nonzero singular elements: $[2, 1]$, $[3], [1, 2]^-, [3]^-$.

All exceptional cases listed in (4.1) appear and only three pairs of singular configurations, namely $(X[2, 1], Y[2, 1])$, $(X[2, 1], Y[1, 2]^-)$ and $(X[1, 2]^-, Y[1, 2]^-)$, satisfy $|V_X| + |V_Y| \geq p^2$. Given that the pairs $(X[2, 1], Y[2, 1])$ and $(X[1, 2]^-, Y[1, 2]^-)$ are relatives, we only have to check the cases $(X[2, 1], Y[2, 1])$ and $(X[2, 1], Y[1, 2]^-)$.

In the case $X[2, 1], Y[2, 1]$, we write $\mathcal{D}_X = \text{diag}[a, a, b]$, $\mathcal{D}_Y = \text{diag}[c, c, d]$ and the predecessors $\mathcal{D}_{X'} = \text{diag}[a, b]$, $\mathcal{D}_{Y'} = \text{diag}[c, d]$, obtained by skipping the first coordinates, are regular. In the case $X[2, 1], Y[1, 2]^-$ we consider $\mathcal{D}_X = \text{diag}[a, a, b]$,

$\mathcal{D}_{w \cdot Y} = \text{diag}[-d, c, d]$ and only now go to regular predecessors $\mathcal{D}_{X'} = \text{diag}[a, b]$, $\mathcal{D}_{(w \cdot Y)'} = \text{diag}[c, d]$. The general proof given below applies in these cases.

When $p = 4$, by Lemma 5.4, we must show the existence of the density for:

(1)  $X[4], Y[2, 1, 1]$ and $X[2, 2], Y[3, 1]$;
(2)  $X[4]^-, Y[2, 2]$ or equivalently the relative pair $X[4], Y[2, 2]^-$;
(3)  $X[1, 3]^-, Y[3, 1]$;
(4)  $X[3, 1], Y[3, 1]$.

In cases (1), (3) and (4), the usual predecessors have density when $p = 3$. The general proof given below applies in these cases.

For case (2), observe that when $p = 3$, the configuration $X'[3]$ never gives the existence of density when $Y'$ is singular. That is why the second case $X[4], Y[2, 2]^-$ has no good predecessors and this case must be proved separately. We will deal with it after the general proof.

Starting from $p = 5$, the *general proof* by induction applies, the exceptions due to small values of $p$ being taken care of. We present this proof now.

*Step 1.* Let $Y \in \overline{\mathfrak{a}^+}$ be such that $\mathcal{D}_Y = \text{diag}[\overbrace{a, \dots, a}^{p-k}, \overbrace{b, \dots, b}^{k}]$ and let its predecessor $Y'$ be such that $\mathcal{D}_{Y'} = \text{diag}[\overbrace{a, \dots, a}^{p-k-1}, \overbrace{b, \dots, b}^{k}]$. The space $V_Y$ is generated by completing a basis of $V_{Y'}$ with

$$N_Y = \{Y_{1, p-k+1}, \dots, Y_{1, p}, Z_{1, 2}, \dots, Z_{1, p}\}.$$

We choose the predecessor $X'$ of $X$ in the same manner, except in the case $S_3$, where we first write $\mathcal{D}_X = \text{diag}[b, b, \dots, b, a, -b]$ where $a > b > 0$ and skip the first term $b$ in $\mathcal{D}_X$.

It is easy to see that the predecessors of $X$ and $Y$ are in the corresponding classes $S_i$ for $p - 1$, so with density, except for $X[5], Y[3, 2]$, due to the noneligible case $X[4], Y[2, 2]$. In this last case we arrange $\mathcal{D}_X = \text{diag}[a, a, a, a, a]$ and $\mathcal{D}_{wY} = \text{diag}[-b, b, b, c, -c]$ and go down to good predecessors $X'[4], (wY)'[2, 2]^-$. The proof described below leads to the existence of the density.

By the induction hypothesis and considering Corollary 3.10, there exists an open dense subset $D'$ of $\mathbf{SO}(p-1) \times \mathbf{SO}(p-1)$ such that for all $w' \in W'$ and $k_0 \in D'$,

$$V_{w' \cdot X'} + \text{Ad}(k_0) V_{Y'} = \mathfrak{p}' \tag{5.1}$$

and $k_0$ satisfies condition (2) of Corollary 3.10.

We embed $K' = \mathbf{SO}(p-1) \times \mathbf{SO}(p-1)$ in $\mathbf{SO}(p) \times \mathbf{SO}(p)$ in the following manner:

$$K' \ni k' = \begin{bmatrix} 1 & & \\ & k_{0,1} & \\ & & 1 \\ & & & k_{0,2} \end{bmatrix} \in \begin{bmatrix} \mathbf{SO}(p) & \\ & \mathbf{SO}(p) \end{bmatrix}, \quad k_{0,1}, k_{0,2} \in \mathbf{SO}(p-1).$$

Hence, we have (identifying $\mathfrak{p}'$ with its natural embedding into $\mathfrak{p}$)

$$V_1 := V_{w' \cdot X'} + \mathrm{Ad}(k_0)V_{Y'} = \mathfrak{p}' = \begin{bmatrix} 0 & B' \\ B'^T & 0 \end{bmatrix}$$

for any $w' \in W'$, where

$$B' = \left[ \begin{array}{c|c} 0_{1 \times 1} & 0_{1 \times (p-1)} \\ \hline 0_{(p-1) \times 1} & B''_{(p-1) \times (p-1)} \end{array} \right],$$

and the matrix $B''$ is arbitrary (note that $\mathfrak{p}'$ is of dimension $(p-1)^2$). We must show that for some $k \in K$, the space $V_X + \mathrm{Ad}(k)V_Y = \mathfrak{p}$.

*Step 2.* The element $Y$ is always of the same form, so the next step of the proof is common to all four cases. We prove that for $k_0 \in D' \subset \mathbf{SO}(p-1) \times \mathbf{SO}(p-1)$ the space $\mathrm{Ad}(k_0)\mathrm{span}(N_Y)$ is of dimension $p+k-1$ and its elements can be written in the form

$$\begin{bmatrix} \begin{array}{c|ccc} 0 & a_1 & \ldots & a_{p-1} \\ \hline \tau_1 & & & \\ \vdots & & & \\ \tau_{p-1-k} & & 0 & \\ a_{p+k-1} & & & \\ \vdots & & & \\ a_p & & & \end{array} \end{bmatrix}^s$$

with $a_i \in \mathbf{R}$ arbitrary and $\tau_j = \tau_j(a_1, \ldots, a_{p+k-1})$. We will not need to write the functions $\tau_j$ explicitly.

The proof of Step 2 proceeds similarly to Step 2 of the proof of Theorem 4.8, case (i), in [7]. For the sake of completeness, we give this proof here.

Step 2 comes from the fact that the action of $\mathrm{Ad}(k_0)$ on the elements of $N_Y$ gives the linearly independent matrices

$$\begin{aligned} \mathrm{Ad}(k_0)Y_{1,i} &= \left[ \begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline \alpha_{i-1} & 0 \end{array} \right]^s, \quad i = p-k+1, \ldots, p, \\ \mathrm{Ad}(k_0)Z_{1,i} &= \left[ \begin{array}{c|c} 0 & -\beta_{i-1}^T \\ \hline \alpha_{i-1} & 0 \end{array} \right]^s, \quad i = 2, \ldots, p, \end{aligned} \tag{5.2}$$

where the $\alpha_i$ are the columns of $k_{0,1}$ and the $\beta_i$ are the columns of $k_{0,2}$. Let us write $\alpha'_i$ for a column $\alpha_i$ with the first $p-1-k$ entries omitted. In order to prove the statement of Step 2, we must show that the matrices obtained by replacing $\alpha_i$ by $\alpha'_i$ in (5.2) are still linearly independent. This is equivalent to the linear independence of the matrices

$$\left[ \begin{array}{c|c} 0 & -\beta_i^T \\ \hline \alpha'_i & 0 \end{array} \right]^s, \quad i = 1, \ldots, p-k-1, \quad \left[ \begin{array}{c|c} 0 & \beta_i^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = p-k, \ldots, p-1,$$

$$\left[ \begin{array}{c|c} 0 & 0 \\ \hline \alpha'_i & 0 \end{array} \right]^s, \quad i = p-k, \ldots, p-1. \tag{5.3}$$

The matrices in (5.3) are linearly independent given that the matrix $k_0$ was assumed to satisfy condition (2) of Corollary 3.10.

Observe that, contrary to [7], we have filled the zero margins of the matrix $B'$ asymmetrically, which is why we call this method 'asymmetric'. The reason for doing this will be clear from the structure of the set $N_X$ that we study now.

*Step 3.* Let us write the set $N_X$ in the four cases $S_i$:

(1)  $S_1$: $N_X = \{Z_{1,2}, \ldots, Z_{1,p}\}$;

(2)  $S_2$: $N_X = \{Z_{1,2}, \ldots, Z_{1,p-1}, Y_{1,p}\}$;

(3)  $S_3$: $N_X = \{Z_{1,2}, \ldots, Z_{1,p-1}, Y_{1,p-1}, Y_{1,p}\}$;

(4)  $S_4$: $N_X = \{Z_{1,2}, \ldots, Z_{1,p}, Y_{1,p}\}$.

We will now use the elements of $N_X$ in order to generate the missing $p - k - 1$ dimensions $\tau_j$ in the margins of $B'$. We use for this the vectors $Z_{1,2}, \ldots, Z_{1,p-k}$ available in all four cases for $k \geq 1$. We proceed as follows.

If $\tau_1(1, 0, \ldots, 0) = -1$, the vector $Z_{1,2} \in N_X$ is unhelpful. We change $X'$ into $X''$ by putting the sign $-$ before the second and the last term of $X'$. We obtain $X'' = w'' \cdot X'$ such that $N_X$ contains $Y_{1,2}$ instead of $Z_{1,2}$ and $w'' \in W'$ changes two signs of $X'$. This manipulation is justified by the fact that (5.1) holds for any $w'' \in W'$. We repeat this procedure, if needed, whenever $\tau_j(\mathbf{e}_j) = -1$ and obtain, from elements of $N_{w \cdot X}$ and $\mathrm{Ad}(k_0)(N_Y)$,

$$\left[ \begin{array}{c|ccc} 0 & a_1 & \ldots & a_{p-1} \\ \hline a_{2p-2} & & & \\ \vdots & & & \\ a_{p+k-1} & & 0 & \\ \vdots & & & \\ a_p & & & \end{array} \right]^s \tag{5.4}$$

for $w \cdot X$ with $w \in W$ and where the $a_i$ are arbitrary.

*Step 4.* Noting that we have at least one element of $N_X$ that has not been used, either $Y_{1,p}$ or $Z_{1,p}$, combining (5.1) and (5.4), we have

$$V_0 := \widetilde{V_{w \cdot X}} + \mathrm{Ad}(k_0)(V_Y) = \left[ \begin{array}{c|ccc} 0 & * & \ldots & * \\ \hline * & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & * \end{array} \right]$$

where $\widetilde{V_{w \cdot X}}$ corresponds to all of $V_X$ without using the remaining $Y_{1,p}$ or $Z_{1,p}$. To fix things, let us assume that the unused element is $Z_{1,p}$, the reasoning being similar if it is $Y_{1,p}$ instead.

The end of the proof is similar to the final step of the proof in [7], but we explain it here for the sake of completeness. Refer to Lemma 3.19 and note that for $t$ small, $\mathrm{Ad}(e^{t(Z_{i,j}^+ + \theta Z_{i,j}^+)})(\widetilde{V_{w \cdot X}}) + \mathrm{Ad}(k_0)(V_Y) = V_0$. Indeed, the lemma shows that no new element is introduced and, for $t$ small, the dimension is unchanged. On the other

hand, $V_0$ is strictly included in $\mathrm{Ad}(e^{t(Z_{i,j}^+ + \theta Z_{i,j}^+)})(\langle Z_{1,p}\rangle \cup \widetilde{V_{w\cdot X}}) + \mathrm{Ad}(k_0)(V_Y)$ since, still by Lemma 3.19, a new diagonal element is introduced. We conclude that for $t$ small enough, $\mathrm{Ad}(e^{t(Z_{i,j}^+ + \theta Z_{i,j}^+)})(\langle Z_{1,p}\rangle \cup \widetilde{V_{w\cdot X}}) + \mathrm{Ad}(k_0)(V_Y) = \mathfrak{p}$. Finally,

$$V_{w\cdot X} + \mathrm{Ad}(e^{-t(Z_{i,j}^+ + \theta Z_{i,j}^+)} k_0)(V_Y) = \mathfrak{p}.$$

*The case* $X[2,2]^-$, $Y[4]$: This case is awkward since $(X, Y)$ do not have eligible predecessors. We select $X$ and $Y$ such that $\mathcal{D}_X = \mathrm{diag}[a, a, b, -b]$ and $\mathcal{D}_Y = \mathrm{diag}[c, c, c, c]$ (assuming $a$, $b$, $c \neq 0$ and $a \neq b$). Then $V_Y$ is generated by the basis $B_Y$ composed of all the six vectors $Z_{i,j}$, $i < j$, while the basis $B_X$ of $V_X$ contains the vectors $Y_{1,3}$, $Y_{1,4}$, $Y_{2,3}$, $Y_{2,4}$, $Y_{3,4}$ and all the vectors $Z_{i,j}$ except $Z_{3,4}$. Note that $|V_X| = 10$.

For a root vector $Z_{i,j}^+$ denote $Z_{i,j}^{\mathfrak{t}} = Z_{i,j}^+ + \theta(Z_{i,j}^+) \in \mathfrak{t}$. Define $Z_0 = Z_{1,2}^{\mathfrak{t}} + Z_{2,3}^{\mathfrak{t}} + Z_{1,4}^{\mathfrak{t}} + Z_{2,4}^{\mathfrak{t}} \in \mathfrak{t}$. We denote

$$F_t = V_X + \mathrm{Ad}(e^{tZ_0})V_Y = V_X + e^{t\,\mathrm{ad}\,Z_0}(V_Y),$$
$$E_t = V_X + \langle\{v + t[Z_0, v] : v \in V_Y\}\rangle.$$

We will write

$$f_t = \det(B_X, \mathrm{Ad}(e^{tZ_0})B_Y)$$

where the elements of $\mathfrak{p}$ are seen as column vectors in $\mathbf{R}^{p^2}$. Analogously, we denote by $e_t$ the determinant constructed in a similar way from the vectors of $B_X$ and the vectors $v + t[Z_0, v]$, $v \in B_Y$, belonging to $E_t$. We write $f_t = e_t + r_t$ and we now analyse $e_t$ and $r_t$ in order to show that $f_t \neq 0$ for some small nonzero $t$.

Using Lemma 3.20, we check that $e_t = ct^5$ with $c = \det(B_X, Z_{3,4}, [Z_0, Z_{1,2}], \ldots, [Z_0, Z_{2,4}]) \neq 0$. The coefficient of $t^6$ in $e_t$ equals zero since $\det(B_X, [Z_0, B_Y]) = 0$. On the other hand, it is easy to see in a similar way that the remainder $r_t$ in the analytic expansion $f_t = e_t + r_t$ does not have terms in $t^n$ for $n < 6$. We conclude that $f_t \neq 0$ for small nonzero $t$.                                                                                      □

## 5.2. Case $u \geq 2$ or $v \geq 2$.

**THEOREM 5.6.** *Let $p \geq 2$ and suppose that $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$ are such that $u \geq 2$ or $v \geq 2$. If the pair $\{X, Y\}$ satisfies condition (2.6),*

$$\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) \leq 2p,$$

*then the density exists.*

PROOF. The proof proceeds by induction and is similar to the proof of Theorem 5.5 and to the proof of the case $u > 0$ or $v > 0$ in [7].

The basis for induction is the previous case ($u \leq 1$ and $v \leq 1$). For $p = 3$, we only need to consider the pair $X[1; 2]$, $Y[2, 1]$ which has regular predecessors. Similarly, for $p = 4$, we see that all eligible pairs with $u \geq 2$ or $v \geq 2$ have eligible predecessors.

In the case $p = 5$, because of (2.7), there are eligible pairs with no eligible predecessors. It suffices to consider the pair $X[5]$, $Y[3; 2]$. In order to show that the

density exists in this case, we use the same technique as for the case $X[2,2]^-$, $Y[4]$. We take $Z_0 = Z_{1,2}^{\mathfrak{t}} + Z_{2,3}^{\mathfrak{t}} + Z_{3,4}^{\mathfrak{t}} + Z_{1,5}^{\mathfrak{t}} + Z_{2,5}^{\mathfrak{t}}$. In order to prove that $e_t = ct^9$ with $c \neq 0$, we check, using Lemma 3.20, that the nine vectors $[Z_0, Z_{1,2}], \ldots, [Z_0, Z_{3,5}]$ produce the missing vectors $Y_{1,2}$, $Y_{1,3}$, $Y_{2,3}$, $Y_{4,5}$ and the diagonal.

Starting from $p = 6$, the induction proof works as in Theorem 5.5 and can be done as in the proof of the case $u > 0$ or $v > 0$ in [7]; we may, however, apply the 'asymmetric' method of the proof of Theorem 5.5. The fact that the roots $\alpha_i$ defined by $\alpha_i(X) = X_i$ are absent in the case $\mathbf{SO}(p, p)$ does not influence the proof from [7], where the roots $\alpha_i$ were not used in Step 4 of the proof. The proof is sufficiently similar to that of Theorem 5.5 and the case $u > 0$ or $v > 0$ in [7], that the details are omitted here.    □

Theorems 5.5 and 5.6 give the sufficiency of the eligibility property in Theorem A.

## 6. Applications

We now extend our results to the symmetric spaces of type $C_p$, that is, to the complex and quaternion cases. Recall that $\mathbf{SU}(p, p)$ is the subgroup of $\mathbf{SL}(2p, \mathbf{C})$ such that $g^* I_{p,p} g = I_{p,p}$, while $\mathbf{Sp}(p, p)$ is the subgroup of $\mathbf{SL}(2p, \mathbf{H})$ such that $g^* I_{p,p} g = I_{p,p}$. Their respective maximal compact subgroups are $\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ and $\mathbf{Sp}(p) \times \mathbf{Sp}(p) \equiv \mathbf{SU}(p, \mathbf{H}) \times \mathbf{SU}(p, \mathbf{H})$. Their subspaces $\mathfrak{p}$ can be described as $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ where $B$ is an arbitrary complex (respectively, quaternionic) matrix of size $p \times p$. The Cartan subalgebra $\mathfrak{a}$ is chosen in the same way as for $\mathfrak{so}(p, p)$, with real entries in the diagonal.

REMARK 6.1. The following table is helpful in showing the differences and similarities between $\mathbf{SO}(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$, $\mathbf{SU}(p, p)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ and $\mathbf{Sp}(p, p)/\mathbf{Sp}(p) \times \mathbf{Sp}(p)$ (the real, complex and quaternionic cases).

| | $\mathbf{SO}(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$ | $\mathbf{SU}(p, p)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ | $\mathbf{Sp}(p, p)/\mathbf{Sp}(p) \times \mathbf{Sp}(p)$ |
|---|---|---|---|
| Root system | $D_p$ | $C_p$ | $C_p$ |
| $m_\alpha$ where $\alpha(X) = X_i - X_j$, $\quad i < j$ | 1 | 2 | 4 |
| $m_\alpha$ where $\alpha(X) = X_i + X_j$, $\quad i < j$ | 1 | 2 | 4 |
| $m_\alpha$ where $\alpha(X) = 2X_i$, $\quad i = 1, \ldots, p$ | 0 | 1 | 3 |
| Dimension of $\mathfrak{p}$ | $p^2$ | $2p^2$ | $4p^2$ |
| Action of the Weyl group on $X \in \mathfrak{a}$ | Permutes the diagonal entries of $\mathcal{D}(X)$ and changes any pair of signs | Permutes the diagonal entries of $\mathcal{D}(X)$ and changes any sign | Permutes the diagonal entries of $\mathcal{D}(X)$ and changes any sign |

THEOREM 6.2. *Consider the symmetric spaces* $\mathbf{SU}(p, p)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ *and* $\mathbf{Sp}(p, p)/\mathbf{Sp}(p) \times \mathbf{Sp}(p)$. *Let* $X = X[\mathbf{s}; u]$ *and* $Y = Y[\mathbf{t}; v] \in \mathfrak{a}$. *Then the measure* $\delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$ *is absolutely continuous if and only if* $\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) \leq 2p$.

PROOF. Let $X, Y \in \mathfrak{a}$. Note that since

$$a(e^X \, \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p)) \, e^Y) \subset a(e^X \, (\mathbf{Sp}(p) \times \mathbf{Sp}(p)) \, e^Y),$$

if the density exists in the complex case, it also exists in the quaternionic case. On the other hand, given Lemma 3.18, one can reproduce Proposition 4.5 using $\mathbf{F} = \mathbf{C}$ and $\mathbf{F} = \mathbf{H}$ to show that the condition is necessary in the complex and quaternionic cases.

However, the root structure is richer in the complex and quaternionic cases compared to the real cases. The existence of the roots $\alpha(X) = 2 X_k$ makes the complex and quaternionic cases very similar to the case $q > p$.

It clearly suffices to prove the result in the complex case. The involution $\theta$ is given by $\theta(X) = -X^*$ and the positive root vectors are generated by

$$X_k^+ = \left[ \begin{array}{c|c} -iE_{k,k} & iE_{k,k} \\ \hline -iE_{k,k} & iE_{k,k} \end{array} \right]$$

for the root $\alpha(H) = 2h_k$, by

$$Y_{r,s}^+ = \left[ \begin{array}{c|c} E_{r,s} - E_{s,r} & E_{r,s} + E_{s,r} \\ \hline E_{r,s} + E_{s,r} & E_{r,s} - E_{s,r} \end{array} \right], \quad Y_{r,s,\mathbf{C}}^+ = \left[ \begin{array}{c|c} i(E_{r,s} + E_{s,r}) & i(E_{r,s} - E_{s,r}) \\ \hline i(E_{r,s} - E_{s,r}) & i(E_{r,s} + E_{s,r}) \end{array} \right]$$

for the root $\alpha(H) = h_r - h_s$, and by

$$Z_{r,s}^+ = \left[ \begin{array}{c|c} E_{r,s} - E_{s,r} & E_{s,r} - E_{r,s} \\ \hline E_{r,s} - E_{s,r} & E_{s,r} - E_{r,s} \end{array} \right], \quad Z_{r,s,\mathbf{C}}^+ = \left[ \begin{array}{c|c} -i(E_{r,s} + E_{s,r}) & i(E_{r,s} + E_{s,r}) \\ \hline -i(E_{r,s} - E_{s,r}) & i(E_{r,s} + E_{s,r}) \end{array} \right]$$

for the root $\alpha(H) = h_r + h_s$ (here the matrices $E_{r,s}$ are of size $p \times p$).

Taking into account the fact that if the density exists in the real case, it also exists in the complex case, we only have a few cases to verify. Given that changing any sign of a diagonal element of $\mathcal{D}_X$, $X \in \mathfrak{a}$, is a Weyl group action, we can always assume that all entries of $\mathcal{D}_X$ are nonnegative. The configuration $[s]^-$ thus disappears.

We will need to show that the cases $(X[p], Y[p])$, $(X[p], Y[p - 1; 1])$ and $(X[4], Y[2; 2])$ all have a density. We will use the case $p = 1$ which is of rank one as the inductive step (there is nothing to prove for that case).

For the case $X[p]$, $Y[p]$, $p > 1$, we proceed much as in [7], but for the sake of completeness we sketch the proof here. The structure of the induction proof is identical to that in Theorem 5.5, with Steps 2 and 3 executed together. We choose predecessors $X'[p - 1]$, $Y'[p - 1]$ and arrange $X, X', Y, Y'$ in the same way as we did in the proof of Theorem 5.5 with $Y$ and $Y'$. In that case, $N_X = N_Y = \{X_1, Z_{12}, \ldots, Z_{1p}\}$.

If $k_0 \in \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ with $k_0 = \begin{bmatrix} 1 & k_{0,1} \\ & 1 \\ & & k_{0,2} \end{bmatrix} \in \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ then

$$\mathrm{Ad}(k_0)(Z_{1,k}) = \left[ \begin{array}{c|c} 0 & -\beta_{k-1}^* \\ \hline \alpha_{k-1} & 0 \end{array} \right]^s,$$

$$\mathrm{Ad}(k_0)(Z_{1,k,\mathbf{C}}) = \left[ \begin{array}{c|c} 0 & i\beta_{k-1}^* \\ \hline i\alpha_{k-1} & 0 \end{array} \right]^s, \quad k = 2, \ldots, p, \quad \text{and} \qquad (6.1)$$

$$\mathrm{Ad}(k_0)(X_1) = \left[ \begin{array}{c|c} i & 0 \\ \hline 0 & 0 \end{array} \right]^s.$$

Given that

$$Z_{1,j} = \left[ \begin{array}{c|c} 0 & -\mathbf{e}_{j-1}^T \\ \hline \mathbf{e}_{j-1} & 0 \end{array} \right]^s, \quad Z_{1,j,\mathbf{C}} = \left[ \begin{array}{c|c} 0 & i\mathbf{e}_{j-1}^T \\ \hline i\mathbf{e}_{j-1} & 0 \end{array} \right]^s, \quad j = 2, \ldots, p. \qquad (6.2)$$

we want to show that the matrices in (6.1) together with those in (6.2) are linearly independent for a $k_0 \in \mathbf{S}(\mathbf{U}(p-1) \times \mathbf{U}(p-1))$ for which equality (5.1) holds. Note that if $k_{0,1} = iI_{p-1}$, $k_{0,2} = -iI_{p-1}$ then the matrices in (6.1) and (6.2) are linearly independent. Since the linear independence is based on a determinant being nonzero, this implies that the set of matrices $k_0$ for which this is true is open and dense in $\mathbf{S}(\mathbf{U}(p-1) \times \mathbf{U}(p-1))$. We conclude that if $N_X' = N_X \setminus \{X_1\}$ then $\mathrm{span}(N_X' + V_{X'}) + \mathrm{Ad}(k_0')V_Y$ has the form

$$\left[ \begin{array}{c|cccc} ia & * & \ldots & * \\ \hline * & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & * \end{array} \right]$$

where the $*$ represent arbitrary complex numbers and $a$ is an arbitrary real number. In order to finish the proof, we reproduce Step 4 of Theorem 5.5 using the vector $X_1^+$.

The case $(X[p], Y[p-1; 1])$ has eligible predecessors $(X'[p-1], Y'[p-1])$. We then have $N_X = \{Z_{1,k}, Z_{1,k,\mathbf{C}}, X_1\}$ and $N_Y = \{Z_{1,k}, Z_{1,k,\mathbf{C}}, Y_{1,k}, Y_{1,k,\mathbf{C}}\}$. The rest follows easily.

Finally, the case $(X[4], Y[2; 2])$ has predecessors $(X'[3], Y'[2; 1])$ which are eligible. □

We conclude this paper with two further applications.

PROPOSITION 6.3. *Let $X$ and $Y \in \mathfrak{a}$ be such that $(\delta_{e^X}^{\natural})^{*2}$ and $(\delta_{e^Y}^{\natural})^{*2}$ are absolutely continuous. Then $\delta_{e^X}^{\natural} * \delta_{e^Y}^{\natural}$ is absolutely continuous.*

PROOF. If $X$ and $Y$ satisfy condition (2.5), then $2\max(\mathbf{s}) \leq 2p - 2$ and $2\max(\mathbf{t}) \leq 2p - 2$ so $\max(\mathbf{s}) + \max(\mathbf{t}) \leq 2p - 2$ and $X$ and $Y$ are eligible. The reasoning is the same if $X$ and $Y$ satisfy condition (2.6). This proof is very similar to that of [7, Proposition 5.2]. □

In previous papers, we have studied a related question: if $X \in \mathfrak{a}$ and $X \neq 0$, for what convolution powers $l$ is the measure $(\delta_{e^X}^{\natural})^l$ absolutely continuous? This problem is equivalent to the study of the absolute continuity of convolution powers of uniform orbital measures $\delta_g^{\natural} = m_K * \delta_g * m_K$ for $g \notin K$. It was proved in [5, Corollary 7] that it is always the case for $l \geq r + 1$, where $r$ is the rank of the symmetric space $G/K$. It was also shown in [7] that $r + 1$ is optimal for this property for symmetric spaces of type $A_n$ [5, Corollary 18] but this is not the case for the symmetric spaces of type $B_p$ where $r$ was shown to be sufficient in [7].

PROPOSITION 6.4. *If $p = 3$ and $\mathcal{D}_X = \mathrm{diag}[a, a, a]$, $a > 0$, then $(\delta_{e^X}^{\natural})^3$ is not absolutely continuous in $\mathbf{SO}(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$ while $(\delta_{e^X}^{\natural})^4$ is absolutely continuous.*

PROOF. Computing the derivative of the map $T(k_1, k_2, k_3, k_4) = k_1 \, e^X \, k_2 \, e^X \, k_3 \, e^X \, k_4$ at $(k_1, k_2, k_3, k_4)$ as in (3.2), we obtain

$$k_1 \, e^X \, k_2 \, (\mathrm{Ad}(k_2^{-1}) \, U_{-X} + \mathrm{Ad}(e^X) \, U_X) \, e^X \, k_3 \, k_4$$
$$= k_1 \, e^X \, k_2 \, (\mathfrak{k} + \mathrm{Ad}(k_2^{-1}) \, V_{-X} + \mathrm{Ad}(e^X) \, V_X) \, e^X \, k_3 \, e^X \, k_4.$$

The dimension of this space is at most $|\mathfrak{k}| + |V_{-X}| + |V_X| = |\mathfrak{k}| + 3 + 3 < |\mathfrak{k}| + |\mathfrak{p}| = |\mathfrak{g}|$ so the map cannot be surjective.

On the other hand, $X'$ such $\mathcal{D}_{X'} = \mathrm{diag}[2a, a, a]$ belongs to $a(e^X \, K \, e^X)$ from the foregoing (taking $x = a$) and the pair $(X', X')$ is eligible. Thence we conclude that $a(e^X \, K \, e^X \, K \, e^X \, K \, e^X)$ has nonempty interior. $\square$

PROPOSITION 6.5. *If $p = 4$ and $\mathcal{D}_X = \mathrm{diag}[a, a, a, a]$, $a > 0$, then $(\delta_{e^X}^{\natural})^3$ is not absolutely continuous in $\mathbf{SO}(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$, while $(\delta_{e^X}^{\natural})^4$ is absolutely continuous. Consequently, if $X = X[\mathbf{s}; u]$ with $u \geq 1$ or $X = X[\mathbf{s}]^-$ then $(\delta_{e^X}^{\natural})^4$ is absolutely continuous.*

PROOF. We know that the elements of $e^X \, K \, e^X$ have the form $k_a \, e^Z \, k_b$ where $\mathcal{D}_Z = \mathrm{diag}[c, c, d, d]$, $c \geq d$. From the end of the proof of Proposition 4.4, we know that for all $H \in a(e^X \, K \, e^X \, K \, e^X)$, $\mathcal{D}_H = \mathrm{diag}[h_1, h_2, h_3, h_4]$ will satisfy $h_1 + h_3 = h_2 + h_4$. We conclude therefore that $a(e^X \, K \, e^X \, K \, e^X)$ has empty interior. On the other hand, since there exists $Z \in a(e^X \, K \, e^X)$ with $\mathcal{D}_Z = \mathrm{diag}[c, c, d, d]$ with $c > d > 0$ and $(Z, Z)$ forms an eligible pair, it follows that $a(e^X \, K \, e^X \, K \, e^X \, K \, e^X)$ has nonempty interior since it contains $a(e^Z \, K \, e^Z)$. $\square$

PROPOSITION 6.6. *If $p \geq 5$ and $\mathcal{D}_X = \mathrm{diag}[a, \ldots, a]$, $a > 0$, then $(\delta_{e^X}^{\natural})^3$ is absolutely continuous in $\mathbf{SO}(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$. Consequently, if $X = X[\mathbf{s}; u]$ with $u \leq 1$ or $X = X[\mathbf{s}]^-$ then $(\delta_{e^X}^{\natural})^3$ is absolutely continuous.*

PROOF. Note that for $t > 0$ small enough, $Z = a(e^X \, k_t^{Z_{p-1,1}^+} \, e^X) \in a(e^X \, K \, e^X)$ is such that $\mathcal{D}_Z = \mathrm{diag}[\overbrace{a, \ldots, a}^{p-2}, x, x]$ with $a > x > 0$. Given that $(Z, X)$ form an eligible pair and that $a(e^Z \, K \, e^X) \subset a(e^X \, K \, e^X \, K \, e^X)$, the result follows. $\square$

THEOREM 6.7. *On symmetric spaces* $\mathbf{SO}_0(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p)$ $(p \geq 4)$, $\mathbf{SU}(p, p)/$ $\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(p))$ *and* $\mathbf{Sp}(p, p)/\mathbf{Sp}(p) \times \mathbf{Sp}(p)$ $(p \geq 2)$, *for every nonzero* $X \in \mathfrak{a}$, *the measure* $(\delta_{e^X}^\natural)^p$ *is absolutely continuous. Moreover, $p$ is the smallest value for which this is true: if $X$ has a configuration* $[1; p - 1]$ *then the measure* $(\delta_{e^X}^\natural)^{p-1}$ *is singular.*

PROOF. We use Propositions 6.4–6.6 and [7, Theorem 5.3].                    □

# 7. Conclusion

With this paper and [6, 7], we have now obtained sharp criteria on singular $X$ and $Y$ for the existence of the density of $\delta_{e^X}^\natural \star \delta_{e^Y}^\natural$ for the root systems of types $A_n$, $B_p$, $C_p$, $D_p$ and $E_6$. Thanks to [5, 7] and Theorem 6.7 of the present paper, sharp criteria are now given for the $l$th convolution powers $(\delta_{e^X}^\natural)^l$ to be absolutely continuous for any $X \neq 0$, $X \in \mathfrak{a}$. It is interesting to note that the eligibility criterion depends strongly on the geometry of the root system. Consequently, a characterization of eligibility that would be applicable for all Riemannian symmetric spaces of noncompact type is unlikely to exist.

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