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ON A COMMUTATIVITY THEOREM FOR SEMI-SIMPLE RINGS

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In this note a theorem proved by Abu-Khuzam and Yaqub has been improved as follows: let R be a semi-simple ring such that for all x, y in R there exists a positive integer $n = n(x, y)$ for which either $(xy)^n + (yx)^n$ or $(xy)^n - (yx)^n$ is central. Then R is commutative.

1. Introduction

Abu-Khuzam and Yaqub [1] proved that if R is a division ring such that for all x, y in R there exists a positive integer $n = n(x, y)$ for which $(xy)^n - (yx)^n$ is in the centre of R, then R is commutative. In this note we give a much shorter and simpler proof for this theorem. We improve the theorem as follows. "Let R be a semi-simple ring such that for all x, y in R there exists a positive integer $n = n(x, y)$ for which either $(xy)^n + (yx)^n$ or $(xy)^n - (yx)^n$ is central. Then R is commutative." Moreover we give an example which shows that the result does not hold for arbitrary rings.

As usual $Z(R)$ denotes the centre of the ring R and for any $x, y \in R$, $[x, y] = xy - yx$.

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2.

The following lemma is due to Herstein [2].

LEMMA 2.1. *Let R be a ring having no non-zero nil ideals in which for every $x, y \in R$ there exist integers $m = m(x, y) \geq 1$, $n = n(x, y) \geq 1$ such that $[X^m, y^n] = 0$. Then R is commutative.*

Now we begin with the following

LEMMA 2.2. *Let R be a division ring such that for all x, y in R there exists a positive integer $n = n(x, y)$ for which $(xy)^n + (yx)^n \in Z(R)$. Then R is commutative.*

Proof. Let x, y be non-zero elements of R. By hypothesis, there exists a positive integer $n = n(xy^{-1}, y)$ such that

$$\left(\left(xy^{-1}\right)y\right)^n + \left(y\left(xy^{-1}\right)\right)^n \in Z(R) .$$

This implies that $x^n + yx^ny^{-1} \in Z(R)$ and hence

$$\left(x^n+yx^ny^{-1}\right)y = y\left(x^n+yx^ny^{-1}\right) ,$$

which gives

$$x^ny + yx^n = yx^n + y^2x^ny^{-1}$$

and we have

$$\left(x^ny+yx^n\right)y = \left(yx^n+y^2x^ny^{-1}\right)y ,$$

that is, $[x^n, y^2] = 0$. Hence, by Lemma 2.1, R is commutative.

3.

The following lemma is due to Posner [3].

LEMMA 3.1. *Let R be a prime ring of ch $\neq 2$ and d_1, d_2 derivations of R such that the iterate $d_1 \cdot d_2$ is also a derivation. Then one at least of d_1, d_2 is zero.*

Now we give an alternate and simple proof of the following result proved by Abu-Khuzam and Yaqub [1].

LEMMA 3.2. *Let R be a division ring such that for all x, y in R there exists a positive integer $n = n(x, y)$ for which $(xy)^n - (yx)^n \in Z(R)$. Then R is commutative.*

Proof. If $\mathrm{ch}\, R = 2$, then we are through by Lemma 2.2. If $\mathrm{ch}\, R \neq 2$, then proceeding on the same lines as in the case of Lemma 2.2, we get

$$x^n y^2 + y^2 x^n - 2yx^n y = 0 . \tag{1}$$

With $y = x + y$ in (1) we have

$$[x^n, [x, y]] = 0 . \tag{2}$$

Let I_r denote the inner derivation by $I_r : x \to [r, x]$; then (2) becomes

$$I_{x^n} I_x(y) = 0 .$$

Thus, by Lemma 3.1, we have either $x \in Z(R)$ or $x^n \in Z(R)$. If $x^n \in Z(R)$ then $[x^n, y] = 0$ and by Lemma 2.1, we get $x \in Z(R)$. Hence in every case $x \in Z(R)$ and thus R is commutative.

LEMMA 3.3. *Let R be a primitive ring such that for all $x, y \in R$ there exists a positive integer $n = n(x, y)$ for which either $(xy)^n + (yx)^n$ or $(xy)^n - (yx)^n$ is central. Then R is commutative.*

Proof. If R is not a division ring, then, since R is a primitive ring for which either $(xy)^n + (yx)^n$ or $(xy)^n - (yx)^n$ is central, the ring D_2 of all 2×2 matrices over some division ring D will be a homomorphic image of a subring of R and will satisfy either $(xy)^n + (yx)^n \in Z(R)$ or $(xy)^n - (yx)^n \in Z(R)$. In particular if we choose $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, neither $(xy)^n + (yx)^n \in Z(R)$ nor $(xy)^n - (yx)^n \in Z(R)$, which gives a contradiction. Hence R must be a division ring. Consequently, by Lemma 2.2 and Lemma 3.2 respectively, R is commutative.

Further if R is a semi-simple ring such that for all $x, y \in R$

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there exists a positive integer $n = n(x, y)$ for which either $(xy)^n + (yx)^n$ or $(xy)^n - (yx)^n$ is central, then R is a sub-direct sum of primitive rings R_α each of which as a homomorphic image of R satisfies the hypothesis placed on R and hence, by Lemma 3.3, R is commutative. This proves our main theorem.

THEOREM. *Let R be a semi-simple ring such that for all x, y in R there exists a positive integer $n = n(x, y)$ for which either $(xy)^n + (yx)^n$ or $(xy)^n - (yx)^n$ is central. Then R is commutative.*

The ring of 3×3 strictly upper triangular matrices over a ring provides an example to show that the above theorem is not valid for arbitrary rings.

References

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