A CHARACTERIZATION OF THE NORMAL AND WEIBULL DISTRIBUTIONS

V. Seshadri*

(received April 7, 1968)

1. Introduction. Let X and Y be two independent normal variates each distributed with zero mean and a common variance. Then the quotient X/Y has the Cauchy distribution symmetrical about the origin. Of particular interest in recent years has been the converse problem and examples of non-normal distributions with a Cauchy distribution for the quotient have been illustrated by Mauldon [9], Laha [2; 3; 4] and Steck [10].

Characterization problems for the normal distribution based on the independence of suitable statistics and the sample mean have also been considered by several authors [1; 5; 6; 7; 8]. In Section 2, we obtain a characterization of the normal distribution by considering the independence of the sum of squares of X and Y and their quotient X/Y.

If X and Y are independently distributed as normal variates with zero mean and a common variance, we find that not only does the quotient X/Y follow the Cauchy law but is independent of the random variable $X^2 + Y^2$. This property of independence provides the characterization of the normal law. A similar property of independence between $X^m + Y^m$ and X/Y for the Weibull distribution is studied in Section 3.

2. A characterization of the normal law. We need the following two theorems for proving the main result about the normal law.

* This work was done while the author was a Fellow of the Summer Research Institute of the Canadian Mathematical Congress in 1967.

(a) THEOREM (Lukacs) [7]. Let X_1 and X_2 be two non-degenerate and positive random variables such that X_1 and X_2 are independent. The random variables $U = X_1 + X_2$ and $V = X_1/X_2$ are independently distributed if and only if both X_1 and X_2 have the gamma distribution with the same scale parameter.

(b) THEOREM (Laha) [3]. Let X and Y be two independently and identically distributed random variables having a common distribution function F(x). Let the quotient $w=x/y$ follow the Cauchy law distributed symmetrically about the origin $w=0$. Then F(x) has the following properties:

(i) it is symmetric about $X = 0$;
(ii) it is absolutely continuous and has a continuous density function $f(x) = F'(x) > 0$.

THEOREM. Let X and Y be two independently and identically distributed random variables with a common distribution function F(x). Let the quotient $W = X/Y$ follow the Cauchy law distributed symmetrically about $W = 0$, and be independent of $U = X^2 + Y^2$. Then the random variables X and Y follow the normal law.

Proof. Applying Lukacs' Theorem to the random variables X^2 and Y^2, we find that X^2/Y^2 is independent of $(X^2 + Y^2)$ and hence both X^2 and Y^2 have the gamma distribution with the same scale parameter α. If $X^2 \sim G(\lambda_1, \alpha)$ and $Y^2 \sim G(\lambda_2, \alpha)$ from the fact that $W = X/Y$ is Cauchy it is clear that $V = 1/(1 + W^2)$ has the Beta distribution (0,1) with parameters (1/2, 1/2). But $V = Y^2/(X^2 + Y^2)$ has the Beta distribution with parameters (λ_2, λ_1) and hence $\lambda_1 = \lambda_2 = 1/2$. The density function of X^2 is therefore

$$\text{(A)} \qquad g(x^2) = (\sqrt{\alpha/\pi})x^{-1} \exp(-\alpha x^2).$$

From Laha's Theorem, (i) F(x) is symmetric about $x = 0$, and (ii) F(x) is absolutely continuous with a continuous probability density function $f(x) = F'(x) > 0$. Therefore $f(x) = f(-x)$. From (A), we have (letting G(.) denote the distribution function of x^2) that

$G(\lambda) = P[x^2 < \lambda] = P[-\sqrt{\lambda} < x < \sqrt{\lambda}] = F(\sqrt{\lambda}) - F(-\sqrt{\lambda}) = 2F(\sqrt{\lambda}) - 1$. Hence $g(\lambda) = f(\sqrt{\lambda})/\sqrt{\lambda}$ $(\lambda > 0)$ so that $f(\sqrt{\lambda}) = \sqrt{\lambda}\, g(\lambda)$. Thus $f(x) = \sqrt{\alpha/\pi}\, \exp(-\alpha x^2)$ which is the normal density function.

3. A characterization of the Weibull distribution. If X and Y are independently distributed as gamma variates with parameters (λ_1, α) and (λ_2, α), we observe that $X + Y$ is independent of the scale invariant function X/Y. On the other hand if X and Y are independent normal variables then $X^2 + Y^2$ and X/Y are independent. We find that for the Weibull distribution given by

$$p(x) = \theta\lambda x^{\lambda-1} \exp(-\theta x^{\lambda}), \quad \lambda \geq 1, \quad \theta > 0, \quad x > 0$$

it turns out that $X^{\lambda} + Y^{\lambda}$ is independent of the quotient X/Y. This motivates the following characterization of the Weibull law.

THEOREM 2. Let X and Y be two positive and independently distributed random variables such that the quotient $V = X/Y$ has the p.d.f. given by $f(v) = \lambda v^{\lambda-1}/(1 + v^{\lambda})^2$, where $v > 0$ and $\lambda \geq 1$. The random variables X and Y have the Weibull distribution with the same scale parameter if $X^{\lambda} + Y^{\lambda}$ is independent of X/Y.

Proof. We apply Lukacs' theorem to the positive and non-degenerate random variables X^{λ} and Y^{λ} and note that both X^{λ} and Y^{λ} must have the gamma distribution with the same scale. Let the parameters be (λ_1, θ) and (λ_2, θ) respectively. Since the distribution of V is known we can obtain the distribution of $W = 1/(1 + V^{\lambda})$. It is $g(w) = 1$ $(0 < w < 1)$. W is a Beta variable $(0, 1)$ with parameters $(1, 1)$. Since X^{λ} and Y^{λ} are gamma variables $Y^{\lambda}/(X^{\lambda} + Y^{\lambda}) = W$ has the Beta distribution $(0, 1)$ with parameters (λ_2, λ_1) and so $\lambda_1 = \lambda_2 = 1$. Therefore the distribution of X^{λ} is $p(x^{\lambda}) = \theta \exp(-\theta x^{\lambda})$ and the distribution of X is found to be $p_1(x) = \theta\lambda x^{\lambda-1} \exp(-\theta x^{\lambda})$. The same distribution can be derived for Y and the proof is complete.

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McGill University

https://doi.org/10.4153/CMB-1969-031-2 Published online by Cambridge University Press