Pappus on the Progressions.

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[The present paper is a translation of the second part of the third book of Pappus's *Mathematical Collection*. Pappus's date is uncertain, but 300 A.D. may be taken as an approximation to it.

Throughout the translation I have used the word "progression" as a rendering of the Greek μεσότης, which has no English equivalent. The only other alternative was to employ the term *mediety*, from the Latin *medietas*.

The account of the various progressions given by Nicomachus, in his *Arithmetical Introduction*, differs somewhat from that of Pappus. I hope to have something to say about Nicomachus in a future paper.]

The second problem was this:

Figure 28.

Some other person said that the three progressions could be obtained in a semicircle thus. He described a semicircle ABC whose centre was E; taking any point D in AC, and drawing DB at right angles to EC, he joined EB, and from D drew DF perpendicular to it. The three progressions, he maintained, were exhibited in a simple manner in the semicircle; for EC was the arithmetical mean, DB the geometrical, and BF the harmonical.

That BD is the mean of AD, DC in a geometrical proportion, and EC the mean of AD, DC in an arithmetical progression is evident. For $\quad AD : DB = DB : DC;$

and
$$AD : AD = AD - AE : EC - CD,$$
$$= AD - EC : EC - CD.$$

But how BF is the mean of the harmonical progression, or of what straight lines it is the mean, he has not said; but only that it is the third proportional of EB, BD, not knowing that the harmonical progression is formed from EB, BD, BF when they are in a geometrical proportion*. For it will be shown by us later on that 2EB + 3DB + BF is the greater extreme, 2DB + BF is the mean, and DB + BF is the least extreme of the harmonical progression.

But first, in order that we may more fully discuss the proposed demonstration, one must treat of the three progressions, and, after

* This censure from Pappus seems to be quite undeserved.

this, of the progressions in a semicircle, then of the three others which, according to the ancients, are opposed to these, and lastly, in accordance with the views of more recent geometers, of the four devised by them; and how it is possible to find by means of a geometrical proportion each of the ten progressions.

The three progressions.

A progression differs from a proportion in this, that every proportion is a progression, but not conversely. For there are three progressions—the arithmetical, the geometrical, the harmonical.

A progression is called arithmetical when there are three terms and the mean exceeds one of the terms by the same as it is exceeded by the remaining term, as 6 with reference to 9 and 3; or when the first term is to itself as the first difference to the second.

A progression is called geometrical, that is a proportion strictly, when the mean is to one of the terms as the remaining term is to the mean, as 6 with reference to 12 and 3; and otherwise, when the first term is to the second as the first difference to the second.

A progression is harmonical when the mean exceeds one of the extremes by the same fraction as it is exceeded by the remaining extreme, as 3 with reference to 2 and 6; or when the first term is to the third as the first difference is to the second difference.

These things having been laid down, we shall find the three progressions together in the five minimum straight lines, after premising the following.

First, having given the straight lines AB, BC, let it be proposed to find the mean according to a geometrical proportion.

Figure 29.

Let CD be drawn at right angles, and let AB be bisected at E. With centre E let a circle be described through B cutting CD at D; join BD and cut off BF equal to it. BF is the required mean.

If DA be joined, it contains a right angle with BD, because both BE and EA are equal to DE. Now the angle at C is right; therefore the triangle ABD is equiangular to the triangle BCD, and accordingly the sides about their common angle at B are proportional. Hence AB : DB = BD : BC, and BD or BF is the mean between AB and BC.

Given AB and BF let it be proposed to find the less extreme.

Figure 30.

Let AB be bisected at E; with centre E let a circle be described through B, and let this circle be cut at D by a circle described through F with centre B. Let a perpendicular DC be drawn; then BC is the third proportional to AB, BF.

The proof is similar to that regarding the mean.

Given FB, BC let it be proposed to find the greater extreme.

Figure 31.

Let CH be drawn at right angles, and with centre B let a circle described through F cut CH at H. Join BH, and draw AH at right angles to it. Then AB is the third proportional to CB, BF.

This is obvious from what has been proved before.

Figure 32.

Again, let there be two straight lines AB, BC, and let DAE be at right angles to AB, so that AD is equal to AE. Let BD, ECF be joined, and from F let FG be drawn perpendicular to CB. Then $AB : BG = AB - BC : CB - BG$.

For $\quad AB : BG = DA : FG$

$= AE : FG$, since $AE = AD$;

$= AC : CG$, on account of the triangles ACE, CFG being equiangular.

Now $\quad AC = AB - BC$, and $CG = CB - BG$;

therefore $\quad AB : BG = AB - BC : CB - BG$.

Figure 32.

But if the extremes AB, BG be given and we seek the mean, join BD, and from G draw FG at right angles. From F to E draw FCE, and we shall have CB the mean between AB and BG.

The proof is obvious.

Figure 33.

Given EB, BC we shall find the greater extreme by drawing from E, DEF at right angles, making $DE = EF$, joining BF, DC, and producing them to G.

For GH the perpendicular drawn from G to BC produced will cut off HB equal to what is sought.

Again, given two straight lines AB, C, of which AB is the greater, we shall find the equidifferent mean thus.

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Figure 34.

Make DB = C, bisect DA at E, and make F = EB. It is obvious that F is the straight line sought.

Similarly if F, C be given, by adding their difference to F we shall have a straight line equal to AB.

Again, if AB, F be given, their difference subtracted from F will give C the third.

Figure 35.

If therefore F be the equidifferent mean of AB, C, the straight lines AB, F, C will form an arithmetical progression. As F is to C so make C to G; then the straight lines F, C, G will form a geometrical progression, that is, a proportion strictly. And if, according to what has been proved before, having two straight lines C, G, the greater of which is C, we make H such that

$$C : H = C - G : G - H,$$

then the straight lines C, G, H will form a harmonical progression. Now AB : C = C : H, AB and C being the extreme terms in the arithmetical progression, and C and H in the harmonical; there will therefore be five minimum straight lines containing the three progressions [and these may be incommensurable with one another].

Now let it be proposed to form the three progressions with the minimum five numbers, and according to what are called multiple, superparticular, and other ratios, unity being supposed indivisible. When the ratio of AB to C is 2, for instance, the minimum numbers which effect what is proposed will be 12, 9, 6, 4, 3; when the ratio is 3, the minimum numbers will be 18, 12, 6, 3, 2. And it is evident how with other ratios also, the minimum numbers for the three progressions must be found. Now if one should wish to express separately each of the progressions, that is clear from what has been previously written; the three terms of the arithmetical progression being in the minimum numbers 3, 2, 1, of the geometrical 4, 2, 1, and the numbers which, according to the given ratio, are fundamental being transformed into equimultiples and superparticulars and the rest. For example, if AB has to C the ratio of 2 to 1, instead of 2 we shall put 4, and instead of 1 we shall put 2. And since the mean between these must exceed and be exceeded by the same amount, the straight line F consists of 3 units. Now the ratio of F to C is that of 3 to 2, and if the ratio of C to G be made equal to it, the problem is not done, because unity remains indivisible. Let everything then be tripled, and 12 is obtained for 4, 9 for 3, 6 for 2. The straight line G then becomes one of 4 units, and H manifestly of 3, and the numbers for the three progressions are 12, 9, 6, 4, 3.

So much, then, concerning the three progressions according to the ancients. Thence it is evident that it is possible to exhibit the three progressions together in a semicircle in the minimum six straight lines.

Figure 36.

Let a semicircle be described having BD perpendicular, and EB a radius, and DF perpendicular to EB. Through B draw HG touching the circle, produce EC to G, make BH equal to BG, and join DKH. Then in the harmonical progression EK is the mean between BE and EF, the greatest term being BE and the least EF.

Since the angles at B and F are right, DF is parallel to HG, and the triangle EBG is equiangular to the triangle EFD, and the triangle BHK to the triangle FKD;

therefore $BE : EF = GB : FD,$

$= HB : FD,$ because $BG = BH$;

$= BK : KF.$

Now $BK = BE - EK, \quad KF = KE - EF$;

therefore $BE : EF = BE - EK : KE - EF.$

The straight lines BE, EK, EF then contain the harmonical progression, the mean being EK, the greatest BE, and the least EF. And AD, EC, CD were shown to contain the arithmetical progression, AD, BD, DC the geometrical. The three progressions therefore have been arranged in a semicircle.

Since Nicomachus,* the Pythagorean, and some others have spoken not only of the first three progressions, which are the most useful in the study of ancient authors, but also of the other three which were in vogue among the ancients, and since, in addition to these six, other four have been invented by more recent writers, we shall endeavour to speak of these somewhat carefully (?), following, however, the older writers who began from the greater term. . . . . . [The Greek text is here corrupt.]

For when the third term is to the first as the excess of the first term is to that of the second, they call the progression contra-harmonical.

* In his *Arithmetical Introduction*. Nicomachus's date is about 100 A.D.

But when the third term is to the second as the excess of the first is to that of the second, the progression is called the fifth and contro-geometrical, for some name it so.

When the second term is to the first as the excess of the first is to that of the second, the progression is called the sixth. It also is called contra-geometrical, because in the sequence of the ratios the order is reversed. Thus, according to them, there are six progressions.

By more recent writers, as we said, other four have been found, in some respect useful, and their discoverers employ their own definitions. For they call the excess of the first term above the second the first difference, that of the second above the third the second difference, and that of the first above the third the third difference, the greatest term being understood and spoken of, as we explained at the outset, as the first, the mean as the second, and the least as the third.

When the third difference is to the first as the second term to the third, they call the progression the seventh.

But, while the ratio of the differences remains the same, if it be as the first term to the second, they call the progression the eighth.

If the third difference be to the first as the first term to the third, they call the progression the ninth.

If the third difference be to the second as the second term to the third, they call the progression the tenth.

Having laid down these definitions, we shall explain the origins of the ten progressions, as we said, by means of a geometrical proportion.

The geometrical progression then, since it takes its first origin from equality, will constitute both itself and the other progressions, showing, as saith the divine Plato, that the nature of proportion is the cause of the harmony of all things, and of a rational and ordered creation. For he says that there is one bond of all the sciences. Now the cause of creation and the bond by which all created things are held together is the divine nature of proportion. The constitution of the ten progressions will be shown by means of the geometrical proportion, the following being premised.

Let there be three terms A, B, C proportional, and let $D = A + 2B + C$, $E = B + C$, $F = C$, then the terms D, E, F are proportional.

Since $$A : B = B : C,$$

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by composition $\quad A+B:B=B+C:C$;
therefore the sum of the antecedents is to the sum of the consequents in the same ratio,
that is, $\quad A+2B+C:B+C=B+C:C.$
Now $\quad D=A+2B+C,\ E=B+C,$ and $F=C$;
therefore $\quad D:E=E:F.$

Hence if A, B, C be supposed equal, D, E, F will be in a double proportion; for $A+2B+C=2(B+C)$ and $B+C=2C$. But if A, B, C be supposed to be in a double proportion and A to be the greatest term among them, D, E, F will be in a triple proportion; and if A be the least term, D, E, F will be in sesquialterate proportion. For if $A=2B$, then $A+B=3B$; and if $A=\frac{1}{2}B$, then $A+B=\frac{3}{2}B$.
And so from the succeeding ratios, the numbers which follow, both multiples and superparticulars, will be found.

And again, if A, B, C were units, the geometrical progression formed by D, E, F would be said to be in minimum numbers 4, 2, 1.

[The translators of Pappus, Commandinus, and Hultsch, in the belief that a lacuna exists here in the Greek text, have inserted a proposition showing how the arithmetical progression is constituted by means of a proportion.

Commandinus puts

$$D=2A+2B+C,\ E=A+B+C,\ F=C,$$

and finds the minimum numbers 5, 3, 1. This, however, does not agree with the entries for the arithmetical progression in the table at the end, as given in his edition of Pappus.

Hultsch puts

$$D=2A+3B+C,\ E=A+2B+C,\ F=B+C,$$

and finds the minimum numbers 6, 4, 2. These are the numbers given for the arithmetical progression in the tables of all the MSS. of Pappus which I have examined.]

The harmonical progression is thus constituted by means of a proportion.

Let three terms A, B, C be supposed proportional,
and let $\quad D=2A+3B+C,\ E=2B+C,\ F=B+C$;
then D, E, F form the harmonical progression.

Since A, B, C are proportional,
therefore $\quad 2A+B:B=2B+C:C.$
Taking the sum of the antecedents and the sum of the consequents,

$$2A+3B+C:B+C=2A+B:B,$$

that is $$D:F=2A+B:B.$$

Now $$D-E=2A+B, \text{ and } E-F=B;$$

and when $$D:F=D-E:E-F,$$

the progression is harmonical.

And it is evident, if A, B, C be supposed to be units, that the harmonical progression is constituted in minimum numbers 6, 3, 2.

The contra-harmonical progression is thus constituted from a proportion.

Let the terms A, B, C be supposed proportional,

and let $$D=2A+3B+C, \; E=2A+2B+C, \; F=B+C;$$

then D, E, F form the said progression.

For again, similarly to what has been shown before,

$$D:F=2A+B:B.$$

And $$E-F=2A+B, \; D-E=B;$$

therefore $$F:D=D-E:E-F,$$

which is what characterises the contra-harmonical progression.

And it is evident, if A, B, C be supposed units, that the progres sion is constituted in minimum numbers 6, 5, 2.

The fifth progression is thus constituted from a proportion.

Let the three terms A, B, C be supposed proportional,

and let $$D=A+3B+C, \; E=A+2B+C, \; F=B+C;$$

then D, E, F are in the fifth progression.

Since, on account of the proportion,

$$A+B:B=B+C:C$$

taking the sum of the antecedents and the sum of the consequents,

$$A+2B+C:B+C=A+B:B,$$

that is $$E:F=A+B:B.$$

Now, $$E-F=A+B, \text{ and } D-E=B;$$

therefore $$F:E=D-E:E-F,$$

which is what happens in the fifth progression.

And if A, B, C be supposed units, the progression would be said to be in minimum numbers 5, 4, 2.

The sixth progression is thus constituted from a proportion.

Let the proportion of the terms A, B, C be the same,

and let $$D=A+3B+2C, \; E=A+2B+C, \; F=A+B-C;$$

then D, E, F form the proposed progression.

Since, on account of the proportion,

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$$A+2B : A+B = B+2C : B+C;$$

taking the sum of the antecedents and the sum of the consequents,

$$A+3B+2C : A+2B+C = B+2C : B+C,$$

that is $\qquad D : E = B+2C : B+C.$

Now $\qquad E-F = B+2C$, and $D-E = B+C$;

therefore $\qquad E : D = D-E : E-F,$

so that D, E, F form the sixth progression.

And if A, B, C be supposed units, it is similarly constituted in minimum numbers 6, 4, 1.

[Here, also, a lacuna has been presumed to exist in the Greek text by Pappus's commentators.

Commandinus puts

$$D = A+2B+2C,\ E = A+B+C,\ F = B+C,$$

and finds the minimum numbers 5, 3, 2. This again does not agree with the entries for the seventh progression in the table, as given in his edition.

Hultsch puts

$$D = A+B+C,\ E = A+B,\ F = C,$$

and finds the minimum numbers 3, 2, 1.

I have given the table at the end, which is much corrupted in the MSS., as it exists in Hultsch's edition, vol. I., pp. 102-103, although I have long entertained some suspicion of its genuineness, as well as of the need for filling up the presumed lacunae.]

The eighth progression is thus constituted from a proportion.

Let the three terms A, B, C be supposed proportional,

and let $\qquad D = 2A+3B+C,\ E = A+2B+C,\ F = 2B+C;$

then D, E, F are according to the eighth progression.

Since, on account of the proportion,

$$2A+B : A+B = 2B+C : B+C;$$

taking the sum of the antecedents and the sum of the consequents,

$$2A+3B+C : A+2B+C = 2A+B : A+B,$$

that is $\qquad D : E = 2A+B : A+B.$

Now $\qquad D-F = 2A+B$, and $D-E = A+B$;

therefore $\qquad D : E = D-F : D-E,$

which constitutes the eighth progression.

And if A, B, C be supposed units, it would be said to be in minimum numbers 6, 4, 3.

The ninth progression is thus constituted from a proportion.

Let A, B, C be supposed proportional,

and let $D = A + 2B + C$, $E = A + B + C$, $F = B + C$;
then D, E, F contain the ninth progression.

Since $A + B : B = B + C : C$;
taking the sum of the antecedents and the sum of the consequents,
$$A + 2B + C : B + C = A + B : B,$$
that is $D : F = A + B : B$.
Now $D - F = A + B$, and $D - E = B$;
therefore $D : F = D - F : D - E$,
which is the characteristic of the ninth progression.

And if A, B, C be similarly supposed units, the minimum numbers 4, 3, 2 contain the progression.

The tenth progression is thus constituted from a proportion.

Again let the three A, B, C be proportional,
and let $D = A + B + C$, $E = B + C$, $F = C$;
then D, E, F are according to the tenth progression.

For $B + C : C = A + B : B$,
that is $E : F = A + B : B$.
Now $D - F = A + B$, and $E - F = B$;
therefore $E : F = D - F : E - F$,
which happens in the tenth progression.

And if A, B, C be supposed units, the minimum numbers 3, 2, 1 form the progression.

For the sake of convenience there are set out the successive numbers by which each term of the proportion is multiplied so as to form each progression, and beside them are placed the minimum numbers containing the progressions. For instance, in the table of the sixth progression the first row 1, 3, 2 means this, that the first term of the proportion taken once, the second thrice, and the third twice, complete the first term of the progression; the second row of the table 1, 2, 1 means that the first term of the proportion taken once, the second twice, and the third once, complete the second term of the progression. The third row of the table in the remaining progressions is composed simply as has been described; exceptionally, however, in this progression the row 1, 1, 1 signifies, as has been said before, that the third term of the progression is obtained from the difference by which the first term of the proportion taken once, and the second taken once, exceed the third term taken once. In the third part of the table the numbers 6, 4, 1 contain the progression itself. Let similar things be understood regarding the remaining tables.

Progressions.	Coefficients of the Terms A, B, C.			Three Minimum Numbers containing the Progressions.		
Arithmetical	2 1	3 2 1	1 1 1	6	4	2
Geometrical	1	2 1	1 1 1	4	2	1
Harmonical	2	3 2 1	1 1 1	6	3	2
Contra-harmonical	2 2	3 2 1	1 1 1	6	5	2
Fifth	1 1	3 2 1	1 1 1	5	4	2
Sixth	1 1 1	3 2 1	2 1 1	6	4	1
Seventh	1	1 1	1 1 1	3	2	1
Eighth	2 1	3 2 2	1 1 1	6	4	3
Ninth	1 1	2 1 1	1 1 1	4	3	2
Tenth	1	1 1	1 1 1	3	2	1