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Non-commutative thickening of moduli spaces of stable sheaves

Yukinobu Toda

Abstract

We show that the moduli spaces of stable sheaves on projective schemes admit certain non-commutative structures, which we call quasi-NC structures, generalizing Kapranov's NC structures. The completion of our quasi-NC structure at a closed point of the moduli space gives a pro-representable hull of the non-commutative deformation functor of the corresponding sheaf developed by Laudal, Eriksen, Segal and Efimov–Lunts–Orlov. We also show that the framed stable moduli spaces of sheaves have canonical NC structures.

1. Introduction

1.1 Background

Let X be a projective scheme over $\mathbb{C}$ and α a Hilbert polynomial of some coherent sheaf on X. It is well known that (cf. [HL97]) the isomorphism classes of stable sheaves

$$M_\alpha := \{\text{Stable sheaves on } X \text{ with Hilbert polynomial } \alpha\}/(\text{isom})$$

have a structure of a quasi-projective scheme, called the *moduli space of stable sheaves*. Under some primitivity condition of α, which we always assume in this paper, the scheme M_α is projective and represents the functor of flat families of stable sheaves on X with Hilbert polynomial α.

The moduli space M_α plays important roles in several places of algebraic geometry, e.g. the constructions of holomorphic symplectic manifolds [Muk87], Fourier–Mukai transforms [Muk81] and Donaldson–Thomas (DT) invariants [Tho00]. Recently, the moduli scheme M_α turned out to be the truncation of a smooth derived moduli scheme [TV07, BFHR14], which has a shifted symplectic structure if X is a Calabi–Yau manifold [PTVV13]. The shifted symplectic structure in [PTVV13] was used in [BBBJ15] to construct algebraic Chern–Simons functions describing M_α locally as its critical locus, which is crucial in the wall crossing of DT invariants [JS12, KS08]. It is now an important subject to find such hidden structures on the moduli spaces of stable sheaves, and give applications to an enumerative geometry. The goal of this paper is to construct such a hidden structure on M_α in a different direction, that is, a certain *non-commutative structure*, whose existence was known only at the formal level.

1.2 Non-commutative deformations of sheaves

Let us recall the formal deformation theory of sheaves in terms of dg-algebras and A_∞-algebras. For example, we refer to Segal's paper [Seg08] on some details of this subject.

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Let F be a stable sheaf on X giving a closed point of M_α, and $\widehat{\mathcal{O}}_{M_\alpha,[F]}$ the completion of $\mathcal{O}_{M_\alpha}$ at the closed point $[F] \in M_\alpha$. The algebra $\widehat{\mathcal{O}}_{M_\alpha,[F]}$ pro-represents the commutative deformation functor

$$\mathrm{Def}_F : \mathcal{A}rt^{\mathrm{loc}} \to \mathcal{S}et$$

sending R to the isomorphism classes of flat deformations of F to $\mathcal{F} \in \mathrm{Coh}(X \times \operatorname{Spec} R)$. Here $\mathcal{A}rt^{\mathrm{loc}}$ is the category of local Artinian $\mathbb{C}$-algebras. It is well known that (for example, see [FIM12]) the functor Def_F is governed by the dg-algebra $\mathbf{R}\mathrm{Hom}(F,F)$, that is,

$$\mathrm{Def}_F(R) = \mathrm{MC}(\mathbf{R}\mathrm{Hom}(F,F) \otimes \mathbf{m})/(\text{gauge equivalence}),$$

where $\mathbf{m} \subset R$ is the maximal ideal and $\mathrm{MC}(\mathfrak{g}^\bullet)$ for the dg-Lie algebra $\mathfrak{g}^\bullet$ is the solution of the Maurer–Cartan equation

$$\mathrm{MC}(\mathfrak{g}^\bullet) = \{x \in \mathfrak{g}^1 : dx + \tfrac{1}{2}[x,x] = 0\}.$$

On the other hand, by the minimal model theorem of A_∞-algebras, we have a quasi-isomorphism

$$\mathbf{R}\mathrm{Hom}(F,F) \sim (\mathrm{Ext}^*(F,F), \{m_n\}_{n\geqslant 2}),$$

where the right-hand side is a minimal A_∞-algebra. The formal solution of the Mauer–Cartan equation of the right-hand side is then $\operatorname{Spec} R_F$, where R_F is the commutative algebra

$$R_F = \frac{\widehat{\mathrm{Sym}^\bullet}(\mathrm{Ext}^1(F,F)^\vee)}{(\sum_{n\geqslant 2} \overline{m}_n^\vee)}. \tag{1.2.1}$$

Here $\overline{m}_n^\vee$ is the dual of the A_∞-product

$$m_n^\vee : \mathrm{Ext}^2(F,F)^\vee \to \mathrm{Ext}^1(F,F)^{\vee\otimes n} \tag{1.2.2}$$

composed with the symmetrization map

$$\mathrm{Ext}^1(F,F)^{\vee\otimes n} \twoheadrightarrow \mathrm{Sym}^n(\mathrm{Ext}^1(F,F)^\vee). \tag{1.2.3}$$

By a general theory of deformation theory, R_F also pro-represents Def_F and hence we have the isomorphism

$$\widehat{\mathcal{O}}_{M_\alpha,[F]} \cong R_F.$$

However, the dual of the A_∞-products (1.2.2) takes values in the tensor products rather than symmetric products, so the algebra R_F may lose some information of the A_∞-products under the maps (1.2.3). Instead of the commutative algebra R_F, the possibly non-commutative algebra

$$R_F^{\mathrm{nc}} = \frac{\prod_{n\geqslant 0}(\mathrm{Ext}^1(F,F)^\vee)^{\otimes n}}{(\sum_{n\geqslant 2} m_n^\vee)} \tag{1.2.4}$$

is more natural and keeps the information of the A_∞-products which the algebra R_F may lose. In fact, the algebra (1.2.4) appears in the context of *non-commutative deformation theory* of sheaves. The algebra (1.2.4) is a pro-representable hull of the non-commutative deformation functor

$$\mathrm{Def}_F^{\mathrm{nc}} : \mathcal{N}^{\mathrm{loc}} \to \mathcal{S}et, \tag{1.2.5}$$

where $\mathcal{N}^{\mathrm{loc}}$ is the category of finite-dimensional (not necessarily commutative) local $\mathbb{C}$-algebras. The functor $\mathrm{Def}_F^{\mathrm{nc}}$ sends $\Lambda \in \mathcal{N}^{\mathrm{loc}}$ to the isomorphism classes of flat deformations of F to $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_X \otimes_{\mathbb{C}} \Lambda)$. Such a non-commutative deformation theory was studied by Laudal [Lau02] for modules over algebras, and later developed by Eriksen [Eri10], Segal [Seg08] and Efimov *et al.* [ELO09, ELO10, ELO11] in geometric contexts.

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1.3 Global non-commutative moduli spaces of stable sheaves

Note that the algebra R_F^{nc} reconstructs R_F by taking its abelianization. Therefore, at the formal level, we see that the structure sheaf $\mathcal{O}_{M_\alpha}$ admits a possibly non-commutative enhancement R_F^{nc} in the sense that

$$(R_F^{\mathrm{nc}})^{\mathrm{ab}} \cong \widehat{\mathcal{O}}_{M_\alpha,[F]}.$$

The purpose of this paper is to give a globalization of the above isomorphism. Namely, we would like to construct a sheaf of non-commutative algebras on M_α whose formal completion at F gives the algebra (1.2.4). We formulate this problem using the notion of Kapranov's *NC schemes* [Kap98]. Roughly speaking, an NC scheme is a ringed space $(Y, \mathcal{O}_Y)$ whose structure sheaf $\mathcal{O}_Y$ is possibly non-commutative, that is, formal in the non-commutative direction. Its abelianization $(Y, \mathcal{O}_Y^{\mathrm{ab}})$ is a usual scheme, and $(Y, \mathcal{O}_Y)$ is interpreted as a formal non-commutative thickening of $(Y, \mathcal{O}_Y^{\mathrm{ab}})$. We refer to [PT14, Ore14] for the recent developments on Kapranov's NC schemes. We call an NC scheme $(Y, \mathcal{O}_Y)$ an *NC structure* on $(Y, \mathcal{O}_Y^{\mathrm{ab}})$. The following problem is the main interest in this paper.

Problem 1.1. Is there an NC structure $(M_\alpha, \mathcal{O}_{M_\alpha}^{\mathrm{nc}})$ on M_α such that $\widehat{\mathcal{O}}_{M_\alpha,[F]}^{\mathrm{nc}} \cong R_F^{\mathrm{nc}}$ for any $[F] \in M_\alpha$?

The above problem was first addressed by Kapranov [Kap98, Proposition 5.4.3] when any $[F] \in M_\alpha$ is a vector bundle without obstruction space, i.e. $\mathrm{Ext}^2(F, F) = 0$. However, it was pointed out by Polishchuk and Tu [PT14, Remark 4.1.4] that the proof of the above result by Kapranov has a gap, so Problem 1.1 is still open even in the unobstructed case. So far, Problem 1.1 is only known when M_α is the moduli space of line bundles [PT14]. The main result of this paper, stated in the following theorem, is to prove a weaker version of Problem 1.1 in a much more general situation.

Theorem 1.2 (Theorem 4.15). *There exist an affine open covering $\{U_i\}_{i \in \mathbb{I}}$ of M_α, NC structures $U_i^{\mathrm{nc}} = (U_i, \mathcal{O}_{U_i}^{\mathrm{nc}})$ on each U_i and isomorphisms of NC schemes*

$$\phi_{ij} : U_i^{\mathrm{nc}}|_{U_{ij}} \xrightarrow{\cong} U_j^{\mathrm{nc}}|_{U_{ij}}, \quad \phi_{ij}^{\mathrm{ab}} = \mathrm{id} \tag{1.3.1}$$

such that for any $[F] \in U_i$, we have $\widehat{\mathcal{O}}_{U_i,[F]}^{\mathrm{nc}} \cong R_F^{\mathrm{nc}}$.

We call NC structures $\{U_i^{\mathrm{nc}}\}_{i \in \mathbb{I}}$ satisfying the condition (1.3.1) as a *quasi-NC structure* on M_α. It gives a global NC structure on M_α if the isomorphisms (1.3.1) satisfy the cocycle condition. So, the existence of a quasi-NC structure is weaker than the global NC structure, but it is enough for the applications to the enumerative geometry, as we will mention in § 1.5. The obstruction for the cocycle condition of (1.3.1) lies in H^2 sheaf cohomology of M_α and hence Theorem 1.2 implies Problem 1.1 if $\dim M_\alpha \leqslant 1$. The quasi-NC structure (1.3.1) may be treated as a twisted sheaf, and the gluing problem of $\{U_i\}_{i \in \mathbb{I}}$ is something similar to the existence problem of the universal sheaf over $X \times M_\alpha$.

We also note that, on any smooth quasi-projective variety, Kapranov [Kap98, Theorem 1.6.1] proved the existence of a smooth quasi-NC structure on it. However, even if M_α is smooth, the resulting quasi-NC structure in (1.3.1) may not be smooth if $\mathrm{Ext}^2(F, F) \neq 0$ for $[F] \in M_\alpha$. So, even in this case, the result of Theorem 1.2 does not directly follow from [Kap98].

For each i, the NC structure U_i^{nc} satisfies a certain weak universality property with respect to a certain functor, which we call the *NC hull* (see Definition 2.10). In particular, U_i^{nc} is uniquely

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determined, but it is up to non-canonical isomorphisms. So, the choices of ϕ_{ij} in (1.3.1) are not canonical and have some ambiguities. On the other hand, the isomorphisms ϕ_{ij} satisfy the cocycle condition after taking the subquotients of the NC filtration, defined in § 2.1. Hence, we obtain the commutative scheme over M_α (cf. Remark 2.7)

$$\bigcup_{i\in\mathbb{I}} \mathrm{Spec}\,(\mathrm{gr}_F^\bullet(\mathcal{O}_{U_i}^{\mathrm{nc}})) \to M_\alpha, \tag{1.3.2}$$

which is canonically attached to M_α. The scheme (1.3.2) provides a new geometric structure of M_α which captures the non-commutative deformation theory of sheaves.

As we will see in Remark 3.9, the gluing issue in Theorem 1.2 is caused by the possible existence of automorphisms of flat families of stable sheaves over a non-commutative base which do not extend to automorphisms of further deformations. If we instead consider a moduli problem without any automorphism, then this issue is fixed. One of the classical ways to kill automorphisms of sheaves is to add data of *framing*. For an integer p, we consider the moduli space $M^\star_{\alpha,p}$ of pairs (F, s), where F is a semistable sheaf with Hilbert polynomial α and s is an element (called framing) of $H^0(F(p))$, satisfying some stability condition. We have the following result.

Theorem 1.3 (Theorem 4.22). *For $p \gg 0$, the framed moduli scheme $M^\star_{\alpha,p}$ has a canonical NC structure.*

Similarly to Theorem 1.2, the NC structure in Theorem 1.3 locally represents the non-commutative deformation functor of framed sheaves. As a corollary of Theorem 1.3, we have the canonical NC structure on the Hilbert scheme $\mathrm{Hilb}^n(X)$ of n-points on X (cf. § 5.5).

1.4 Idea of the proof

As described in [HL97], the classical way to construct the moduli space M_α is to take the GIT (Geometric Invariant Theory) quotient of the Grothendieck Quot scheme. Rather recently, Álvarez-Cónsul and King [AK07] gave another construction of M_α using the moduli space of representations of a quiver. The argument of [AK07] was improved by Behrend *et al.* [BFHR14] to construct the derived moduli scheme of stable sheaves. Our strategy is to construct a quasi-NC structure on the moduli space of representations of a quiver, and pull it back to M_α by extending the arguments of [BFHR14] to non-commutative thickenings.

The basic idea is as follows: we associate a stable sheaf $[F] \in M_\alpha$ with the vector space

$$\Gamma_{[p,q]}(F) := \bigoplus_{i=p}^{q} \Gamma(X, F(i)). \tag{1.4.1}$$

The vector space $\Gamma_{[p,q]}(F)$ is a representation of a certain quiver $Q_{[p,q]}$ with relation I, defined by the graded algebra structure of the homogeneous coordinate ring of X (cf. Figure 1). For $q \gg p \gg 0$, the result of [BFHR14] shows that the map $F \mapsto \Gamma_{[p,q]}(F)$ gives an open immersion

$$\Gamma_{[p,q]} : M_\alpha \subset M_{Q,I},$$

where $M_{Q,I}$ is the moduli space of representations of $(Q_{[p,q]}, I)$. We then construct a quasi-NC structure on $M_{Q,I}$ in the following way. We first embed $M_{Q,I}$ into the smooth scheme

$$M_{Q,I} \subset M_Q,$$

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where M_Q is the moduli space of representations of Q without relation. Let $U \subset M_Q$ be an affine open subset. By Kapranov [Kap98], the smooth affine scheme U has an NC smooth thickening $(U, \mathcal{O}_U^{\mathrm{nc}})$, which is unique up to non-canonical isomorphisms. We construct the NC structure on $V = U \cap M_{Q,I}$ by taking the quotient algebra $\mathcal{O}_U^{\mathrm{nc}}/\mathcal{J}_{I,U}$ to be its NC structure sheaf, where $\mathcal{J}_{I,U} \subset \mathcal{O}_U^{\mathrm{nc}}$ is the two-sided ideal determined by the relations in I. We prove that the above local NC structures

$$V^{\mathrm{nc}} = (V, \mathcal{O}_U^{\mathrm{nc}}/\mathcal{J}_{I,U})$$

on $M_{Q,I}$ are pulled back to M_α to give a desired quasi-NC structure. The key technical ingredient is to construct the derived left adjoint functor of (1.4.1) for flat families of $(Q_{[p,q]}, I)$-representations over non-commutative bases, which enables us to compare non-commutative deformations of sheaves with those of representations of $(Q_{[p,q]}, I)$.

1.5 Possible applications

As we mentioned, the main motivation of Theorem 1.2 is the application of non-commutative deformation theory for enumerative geometry. Let $X \to Y$ be a flopping contraction from a smooth 3-fold X, whose exceptional locus is a smooth rational curve C. In this situation, the algebras $R_{\mathcal{O}_C}, R_{\mathcal{O}_C}^{\mathrm{nc}}$ are finite dimensional, so in particular $\operatorname{Spec} R_{\mathcal{O}_C}$ is the global moduli space of stable sheaves. The dimension of $R_{\mathcal{O}_C}$ is Katz's genus-zero Gopakumar–Vafa invariants [Kat08] with curve class $[C]$, which is a special case of DT invariants. So, the dimension of $R_{\mathcal{O}_C}^{\mathrm{nc}}$ may be interpreted as a certain enumerative invariant of sheaves which captures non-commutative deformations of $\mathcal{O}_C$. In the paper [Tod15a], the author described the dimension of $R_{\mathcal{O}_C}^{\mathrm{nc}}$ in terms of Katz's genus-zero Gopakumar–Vafa invariants on X with curve class $j[C]$ for $j \geqslant 1$. This phenomenon suggests that, in general, there might be a DT-type theory which captures non-commutative deformations of sheaves on Calabi–Yau 3-folds, and has some relations to the usual DT theory. The result of Theorem 1.2 may lead to a foundation of such a theory.

In the subsequent paper [Tod15b] and a further paper in preparation, we develop two kinds of approaches toward this problem. In [Tod15b], when there are no higher obstruction spaces, we show that the construction of quasi-NC structures in Theorem 1.2 yields interesting virtual structure sheaves $\mathcal{O}_{M_\alpha}^{\mathrm{vir}} \in K_0(M_\alpha)$ which capture non-commutative deformations of sheaves. In a future work, we will construct certain invariants using Hilbert schemes of points on a quasi-NC structure in Theorem 1.2. If X is a Calabi–Yau 3-fold, using a wall-crossing argument, we can relate these invariants with generalized DT invariants counting semistable sheaves on X [JS12, KS08]. This result gives an intrinsic understanding of the above dimension formula of $R_{\mathcal{O}_C}$ given in [Tod15a].

In the paper [DW16], Donovan and Wemyss used the algebra $R_{\mathcal{O}_C}^{\mathrm{nc}}$ to construct a non-commutative twist functor of $D^b \operatorname{Coh}(X)$, which describes Bridgeland–Chen's flop–flop autoequivalence [Bri02, Che02]. On the other hand, one may try to construct a similar autoequivalence associated to a divisorial contraction $X \to Y$ for a Calabi–Yau 3-fold X, contracting a divisor $E \subset X$ to a curve $Z \subset Y$. Since $\dim Z = 1$, Theorem 1.2 should yield an NC structure Z^{nc} on Z induced by non-commutative deformation theory of fibers of $E \to Z$. Then, similarly to the work [DW16], one may be able to construct an autoequivalence of $D^b \operatorname{Coh}(X)$ generalizing an EZ-spherical twist [Hor05], using the NC structure Z^{nc} on Z. When Y is a spectrum of a complete local $\mathbb{C}$-algebra, such an autoequivalence was constructed by Wemyss [Wem14, § 4.4], and our construction of Z^{nc} should give a globalization of his result. We also refer to [DW15, Kaw15, BB15, HT16] for the recent works related to this topic.

1.6 Plan of the paper

In § 2, we review Kapranov's NC schemes, introduce quasi-NC structures and discuss the related notions. In § 3, we construct quasi-NC structures on the moduli spaces of representations of quivers with relations. In § 4, we construct quasi-NC structures on the moduli spaces of stable sheaves using those on representations of quivers with relations. In § 5, we describe some examples.

1.7 Notation and convention

In this paper, all the varieties or schemes are defined over $\mathbb{C}$. The category $\mathcal{C}om$ is the category of commutative $\mathbb{C}$-algebras. An algebra always means an associative, not necessarily commutative, $\mathbb{C}$-algebra. For an algebra Λ, the category Λ mod is the category of finitely generated left Λ-modules. We denote by Λ^{ab} the abelianization of Λ and, for $N \in \Lambda$ mod, we write $N^{\mathrm{ab}} := \Lambda^{\mathrm{ab}} \otimes_{\Lambda} N$. Also, for a left Λ-module homomorphism $\phi : N_1 \to N_2$, we set $\phi^{\mathrm{ab}} = \Lambda^{\mathrm{ab}} \otimes_{\Lambda} \phi : N_1^{\mathrm{ab}} \to N_2^{\mathrm{ab}}$.

2. NC structures and quasi-NC structures

In this section, we recall the definition of NC schemes introduced by Kapranov [Kap98]. We also introduce quasi-NC structures, NC hulls and prove a way to construct them via functors.

2.1 NC nilpotent algebras

Let Λ be an algebra. We regard Λ as a Lie algebra by setting $[a, b] = ab - ba$. The subspace $\Lambda_k^{\mathrm{Lie}} \subset \Lambda$ is defined to be spanned by the elements of the form

$$[x_1, [x_2, \dots, [x_{k-1}, x_k] \cdots]]$$

for $x_i \in \Lambda$, $1 \leqslant i \leqslant k$. The *NC filtration* of Λ is the decreasing filtration

$$\Lambda = F^0\Lambda \supset F^1\Lambda \supset \cdots \supset F^d\Lambda \supset \cdots ,$$

where $F^d\Lambda$ is the two-sided ideal of Λ defined by

$$F^d\Lambda := \sum_{m \geqslant 0} \sum_{i_1 + \cdots + i_m = m+d} \Lambda \cdot \Lambda_{i_1}^{\mathrm{Lie}} \cdot \Lambda \cdot \cdots \cdot \Lambda_{i_m}^{\mathrm{Lie}} \cdot \Lambda.$$

Note that $\Lambda/F^1\Lambda$ is the abelianization Λ^{ab} of Λ. We set $\Lambda^{\leqslant d} := \Lambda/F^{d+1}\Lambda$, and denote $N^{\leqslant d} := \Lambda^{\leqslant d} \otimes_{\Lambda} N$ for $N \in \Lambda$ mod.

Definition 2.1. (i) An algebra Λ is called NC nilpotent of degree d (respectively NC nilpotent) if $F^{d+1}\Lambda = 0$ (respectively $F^n\Lambda = 0$ for $n \gg 0$).

(ii) The NC completion of an algebra Λ is

$$\Lambda_{[[\mathrm{ab}]]} := \varprojlim \Lambda^{\leqslant d}.$$

(iii) An algebra Λ is called NC complete if the natural map $\Lambda \to \Lambda_{[[\mathrm{ab}]]}$ is an isomorphism.

For an algebra Λ, its subquotient

$$\mathrm{gr}_F^{\bullet}(\Lambda) := \bigoplus_{d \geqslant 0} F^d\Lambda/F^{d+1}\Lambda$$

is a commutative algebra.

Lemma 2.2. *For an algebra Λ and an algebra homomorphism $\phi : \Lambda \to \Lambda$, suppose that the induced homomorphism $\phi^{\mathrm{ab}} : \Lambda^{\mathrm{ab}} \to \Lambda^{\mathrm{ab}}$ is the identity. Then $\mathrm{gr}_F^{\bullet}(\phi) = \mathrm{id}$ and, if Λ is NC complete, ϕ is an isomorphism.*

Proof. The assumption $\phi^{\mathrm{ab}} = \mathrm{id}$ implies that $\phi(x) - x \in F^1\Lambda$ for any $x \in \Lambda$. Therefore, $\phi(x) - x \in F^{d+1}\Lambda$ if $x \in F^d\Lambda$ and hence $\mathrm{gr}_F^{\bullet}(\phi) = \mathrm{id}$. The fact that $\mathrm{gr}_F^{\bullet}(\phi) = \mathrm{id}$ together with the induction on d shows that $\phi^{\leqslant d} : \Lambda^{\leqslant d} \to \Lambda^{\leqslant d}$ is an isomorphism for any $d > 0$. Therefore, ϕ is an isomorphism if Λ is NC complete. □

2.2 NC schemes

Let Λ be an NC complete algebra. The *affine NC scheme*

$$\mathrm{Spf}\,\Lambda = (\mathrm{Spec}\,\Lambda^{\mathrm{ab}}, \mathcal{O}_Y) \tag{2.2.1}$$

is a ringed space defined in the following way. For any multiplicative set $S \subset \Lambda^{\mathrm{ab}}$, its pull-back by the natural surjection $\Lambda^{\leqslant d} \twoheadrightarrow \Lambda^{\mathrm{ab}}$ determines the multiplicative set in $\Lambda^{\leqslant d}$, which satisfies the Ore localization condition (cf. [Kap98, Proposition 2.1.5]). Therefore, similarly to the case of usual affine schemes, the NC nilpotent algebra $\Lambda^{\leqslant d}$ determines the sheaf of algebras $\mathcal{O}_Y^{\leqslant d}$ on $\mathrm{Spec}\,\Lambda^{\mathrm{ab}}$. The sheaf of algebras $\mathcal{O}_Y$ is defined by

$$\mathcal{O}_Y := \varprojlim \mathcal{O}_Y^{\leqslant d}.$$

Definition 2.3 [Kap98, Definition 2.2.5]. A ringed space Y is called an NC scheme if it is locally isomorphic to an affine NC scheme of the form (2.2.1).

The category of NC schemes is the full subcategory of ringed spaces consisting of NC schemes. For an NC scheme Y, the category $\mathrm{Coh}(Y)$ is defined to be the category of coherent left $\mathcal{O}_Y$-modules on Y. Note that the structure sheaf $\mathcal{O}_Y$ has a filtration by sheaves of two-sided ideals $F^n\mathcal{O}_Y \subset \mathcal{O}_Y$, which defines a topology on the set of sections of $\mathcal{O}_Y$. For a sheaf of two-sided ideals $\mathcal{J} \subset \mathcal{O}_Y$, we denote by $\overline{\mathcal{J}} \subset \mathcal{O}_Y$ its topological closure. Then the quotient $\mathcal{O}_Y/\overline{\mathcal{J}}$ is the sheaf of NC complete algebras and hence determines the NC closed subscheme of Y. In particular, the quotient $\mathcal{O}_Y^{\leqslant d} := \mathcal{O}_Y/F^{d+1}\mathcal{O}_Y$ defines the NC subscheme $Y^{\leqslant d} \subset Y$, and $Y^{\mathrm{ab}} := Y^{\leqslant 0}$ is a usual scheme.

Definition 2.4. An NC structure on a commutative scheme M is an NC scheme

$$M^{\mathrm{nc}} = (M, \mathcal{O}_M^{\mathrm{nc}}) \tag{2.2.2}$$

such that $(\mathcal{O}_M^{\mathrm{nc}})^{\mathrm{ab}} = \mathcal{O}_M$.

Here, in the right-hand side of (2.2.2), M is just regarded as a topological space. In general, it is not easy to construct non-trivial NC structures on a given algebraic variety. Instead, we introduce the following weaker notion of NC structures.

Definition 2.5. Let M be a commutative scheme. A quasi-NC structure on M consists of an affine open cover $\{U_i\}_{i\in\mathbb{I}}$ of M with affine $U_{ij} := U_i \cap U_j$, affine NC structures $(U_i, \mathcal{O}_{U_i}^{\mathrm{nc}})$ on U_i for each $i \in \mathbb{I}$ and isomorphisms of affine NC schemes

$$\phi_{ij} : (U_{ij}, \mathcal{O}_{U_j}^{\mathrm{nc}}|_{U_{ij}}) \xrightarrow{\cong} (U_{ij}, \mathcal{O}_{U_i}^{\mathrm{nc}}|_{U_{ij}}) \tag{2.2.3}$$

satisfying $\phi_{ij}^{\mathrm{ab}} = \mathrm{id}$.

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Remark 2.6. A quasi-NC structure gives rise to the NC structure if and only if the isomorphisms ϕ_{ij} in (2.2.3) satisfy the cocycle condition, i.e.

$$\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \mathrm{id}.$$

Remark 2.7. Although (2.2.3) may not satisfy the cocycle condition, Lemma 2.2 implies that $\mathrm{gr}_F^{\bullet}(\phi_{ij})$ always satisfies the cocycle condition. Hence, $\mathrm{gr}_F^{\bullet}(\mathcal{O}_{U_i}^{\mathrm{nc}})$ glue together to give a sheaf of algebras on M. In particular, we have the commutative scheme over M

$$\bigcup_{i \in \mathbb{I}} \mathrm{Spec}\,(\mathrm{gr}_F^{\bullet}(\mathcal{O}_{U_i}^{\mathrm{nc}})) \to M.$$

2.3 Smooth NC schemes

Let $\mathcal{N}_d$ be the category of NC nilpotent algebras of degree d and $\mathcal{N}$ the category of NC nilpotent algebras. We have the following inclusions:

$$\mathcal{C}om := \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_d \subset \cdots \subset \mathcal{N}. \tag{2.3.1}$$

Let $\Lambda' \twoheadrightarrow \Lambda$ be a surjection in $\mathcal{N}$ and take the exact sequence

$$0 \to J \to \Lambda' \to \Lambda \to 0. \tag{2.3.2}$$

The sequence (2.3.2) is called a *central extension* if $J^2 = 0$ and J lies in the center of Λ'. In particular, J is a Λ^{ab}-module.

DEFINITION 2.8. For functors $h_1, h_2 : \mathcal{N} \to \mathcal{S}et$, a natural transform $\phi : h_1 \to h_2$ is called formally smooth if for any central extension (2.3.2) in $\mathcal{N}$, the map

$$h_1(\Lambda') \to h_2(\Lambda') \times_{h_2(\Lambda)} h_1(\Lambda)$$

is surjective.

An NC scheme Y defines a covariant functor

$$h_Y : \mathcal{N} \to \mathcal{S}et$$

sending Λ to $\mathrm{Hom}(\mathrm{Spf}\,\Lambda, Y)$. In the situation of Definition 2.8, a functor $h : \mathcal{N} \to \mathcal{S}et$ is called *formally smooth* if the natural transform $h \to h_{\mathrm{Spec}\,\mathbb{C}}$ is formally smooth, i.e. for any central extension (2.3.2), the map $h(\Lambda') \to h(\Lambda)$ is surjective. If the same condition holds only for central extensions (2.3.2) in $\mathcal{N}_d$, we call the functor $h|_{\mathcal{N}_d}$ *formally d-smooth*.

DEFINITION 2.9. (i) An NC scheme Y is called smooth if h_Y is formally smooth.
(ii) An NC scheme Y is called d-smooth if $F^{d+1}\mathcal{O}_Y = 0$ and $h_Y|_{\mathcal{N}_d}$ is formally d-smooth.

It is easy to see that if an NC scheme Y is smooth, then $Y^{\leqslant d}$ is d-smooth. In particular, Y_{ab} is a smooth scheme in the usual sense. If Y is NC smooth, we say that Y is an *NC smooth thickening* of Y_{ab}. By [Kap98, Theorem 1.6.1], any smooth affine scheme has an NC smooth thickening, which is unique up to non-canonical isomorphisms. In particular, any smooth quasi-projective variety has a smooth quasi-NC structure in the sense of Definition 2.5.

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2.4 (Quasi) NC structures via functors

We introduce the notion of an NC hull to construct quasi-NC structures.

Definition 2.10. Let $h : \mathcal{N} \to \mathcal{S}et$ be a functor. An NC scheme Y together with a natural transform $\phi : h_Y \to h$ is called an NC hull of h if ϕ is formally smooth and an isomorphism on $\mathcal{C}om$. If Y is an affine NC scheme, we call it an affine NC hull.

We have the following lemma.

Lemma 2.11. *An affine NC hull is unique up to non-canonical isomorphisms.*

Proof. For $i = 1, 2$, let $\phi_i : h_{Y_i} \to h$ be affine NC hulls, and write $Y_i = \operatorname{Spf} \Lambda_i$. Note that, by the formal smoothness of NC hulls, the natural map

$$\varprojlim h_{Y_i}(\Lambda^{\leqslant d}) \to \varprojlim h(\Lambda^{\leqslant d}) \tag{2.4.1}$$

is surjective for any NC complete algebra Λ. Since the left-hand side of (2.4.1) coincides with $\operatorname{Hom}(\Lambda_i, \Lambda)$, there exist algebra homomorphisms $u : \Lambda_1 \to \Lambda_2$, $v : \Lambda_2 \to \Lambda_1$ which commute with ϕ_1, ϕ_2. Then the compositions

$$v \circ u : \Lambda_1 \to \Lambda_1, \quad u \circ v : \Lambda_2 \to \Lambda_2 \tag{2.4.2}$$

satisfy $(v \circ u)^{\mathrm{ab}} = \mathrm{id}$, $(u \circ v)^{\mathrm{ab}} = \mathrm{id}$. Therefore, Lemma 2.2 implies that (2.4.2) are algebra isomorphisms and hence u, v are isomorphisms. □

Let M be a commutative scheme and $h : \mathcal{N} \to \mathcal{S}et$ a functor. If $h|_{\mathcal{C}om} = h_M$, we have the natural transform $h \to h_M$ applying h to $\Lambda \to \Lambda^{\mathrm{ab}}$ for $\Lambda \in \mathcal{N}$. For a subscheme $U \subset M$, we define

$$h|_U := h \times_{h_M} h_U.$$

The following corollary obviously follows from Lemma 2.11.

Corollary 2.12. *In the above situation, suppose that M has an affine open cover $\bigcup_{i \in \mathbb{I}} U_i$ with affine U_{ij}, affine NC structures U_i^{nc} on each U_i and NC hulls $h_{U_i^{\mathrm{nc}}} \to h|_{U_i}$. Then $\{U_i^{\mathrm{nc}}\}_{i \in \mathbb{I}}$ gives a quasi-NC structure on M.*

Remark 2.13. In the situation of Corollary 2.12, suppose that each $h_{U_i^{\mathrm{nc}}} \to h|_{U_i}$ is an isomorphism. Then the isomorphism of U_i^{nc} and U_j^{nc} over U_{ij} is canonically determined by h, so the quasi-NC structure $\{U_i\}_{i \in \mathbb{I}}$ gives an NC structure on M.

We next describe some condition for the functor $h_Y \to h$ to give an NC hull. For a functor $h : \mathcal{N} \to \mathcal{S}et$ and a Cartesian diagram of algebras in $\mathcal{N}$

$$\begin{tikzcd} \Lambda_{12} \arrow[r, "q_1"] \arrow[d, "q_2"'] & \Lambda_1 \arrow[d, "p_1"] \\ \Lambda_2 \arrow[r, "p_2"'] & \Lambda, \end{tikzcd}$$

we consider the natural map

$$h(\Lambda_{12}) \to h(\Lambda_1) \times_{h(\Lambda)} h(\Lambda_2). \tag{2.4.3}$$

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Proposition 2.14. *Let $h : \mathcal{N} \to \mathcal{S}et$ be a functor, Y a smooth NC scheme and $\phi : h_Y \to h$ a natural transform. Suppose that ϕ is an isomorphism on $\mathcal{C}om$, and the following conditions hold on the map (2.4.3).*

(1) *The map (2.4.3) is surjective if $\Lambda_1 = \Lambda_2$ and $p_1 = p_2$ is a central extension.*
(2) *The map (2.4.3) is bijective if Λ is commutative and $\Lambda_2 = \Lambda \oplus J$ for a Λ-module J. Here $\Lambda \oplus J$ is the trivial extension, i.e. $(a_1, m_1)(a_2, m_2) = (a_1 a_2, m_1 a_2 + a_1 m_2)$.*

Then ϕ is formally smooth, i.e. $\phi : h_Y \to h$ is an NC hull of h.

Proof. The proof is similar to the classical argument on the existence of pro-representable hulls (cf. [Sch68, Theorem 2.11]). We consider a central extension in $\mathcal{N}$

$$0 \to J \to \Lambda' \xrightarrow{p} \Lambda \to 0$$

and show that the map

$$h_Y(\Lambda') \to h(\Lambda') \times_{h(\Lambda)} h_Y(\Lambda) \tag{2.4.4}$$

is surjective. An element of the right-hand side of (2.4.4) is given by a commutative diagram

$$\begin{tikzcd} h_Y \arrow[r, "\phi"] & h \\ \operatorname{Spf}\Lambda \arrow[u, "\alpha"] \arrow[r] & \operatorname{Spf}\Lambda' \arrow[u, "\beta"'] \arrow[ul, dotted, "\gamma"] \end{tikzcd} \tag{2.4.5}$$

We need to find a dotted arrow γ so that both of the lower and upper triangles in (2.4.5) are commutative. Since h_Y is smooth, there is a dotted arrow γ in (2.4.5) so that the lower triangle is commutative. The set of possible choices of such a γ is a principal homogeneous space over $\operatorname{Der}(\mathcal{O}_{Y^{\mathrm{ab}}}, J)$. Here we regard J as an $\mathcal{O}_{Y^{\mathrm{ab}}}$-module by the algebra homomorphism $\alpha^{\mathrm{ab}} : \mathcal{O}_Y^{\mathrm{ab}} \to \Lambda^{\mathrm{ab}}$ induced by α.

On the other hand, we have the isomorphism

$$\Lambda' \times_{\Lambda} \Lambda' \xrightarrow{\cong} \Lambda' \times_{\Lambda^{\mathrm{ab}}} (\Lambda^{\mathrm{ab}} \oplus J)$$

given by $(x, y) \mapsto (x, x^{\mathrm{ab}} + y - x)$. Therefore, the assumption on the map (2.4.3) implies the surjection

$$h(\Lambda') \times_{h(\Lambda^{\mathrm{ab}})} h(\Lambda^{\mathrm{ab}} \oplus J) \twoheadrightarrow h(\Lambda') \times_{h(\Lambda)} h(\Lambda'). \tag{2.4.6}$$

Let $\eta \in h(\Lambda)$ be the element corresponding to the composition $\phi \circ \alpha$ in the diagram (2.4.5). The surjection (2.4.6) restricts to the surjection

$$h(p)^{-1}(\eta) \times h(q)^{-1}(\eta^{\mathrm{ab}}) \twoheadrightarrow h(p)^{-1}(\eta) \times h(p)^{-1}(\eta). \tag{2.4.7}$$

Here $q : \Lambda^{\mathrm{ab}} \oplus J \to \Lambda^{\mathrm{ab}}$ is the projection. Since $\eta^{\mathrm{ab}} \in h(\Lambda^{\mathrm{ab}})$ is identified with the morphism $\alpha^{\mathrm{ab}} : \mathcal{O}_Y^{\mathrm{ab}} \to \Lambda^{\mathrm{ab}}$, we have

$$h(q)^{-1}(\eta^{\mathrm{ab}}) = \operatorname{Der}(\mathcal{O}_{Y^{\mathrm{ab}}}, J).$$

Therefore, the surjection (2.4.7) shows that $\operatorname{Der}(\mathcal{O}_{Y^{\mathrm{ab}}}, J)$ acts on $h(p)^{-1}(\eta)$ transitively.

Note that if γ is a dotted arrow in (2.4.5) so that the lower triangle is commutative, then we have

$$\phi \circ \gamma \in h(p)^{-1}(\eta), \quad \beta \in h(p)^{-1}(\eta).$$

Therefore, by acting $\operatorname{Der}(\mathcal{O}_{Y^{\mathrm{ab}}}, J)$ on γ, we can also make the upper triangle of (2.4.5) commutative. $\square$

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Finally, we state the construction of NC structures via functors. In the following proposition, we put $\mathcal{N}_\infty = \mathcal{N}$.

Proposition 2.15. *For $d \in [0, \infty]$, let $h : \mathcal{N}_d \to \mathcal{S}et$ be a functor, Y a d-smooth NC scheme and $\phi : h_Y|_{\mathcal{N}_d} \to h$ a natural transform. Suppose that ϕ is an isomorphism on $\mathcal{N}_e$ for some $e < d$, and the following conditions hold on the map (2.4.3).*

(1) *The map (2.4.3) is a bijection if $\Lambda_1 = \Lambda_2$ and $p_1 = p_2$ is a central extension with $\Lambda \in \mathcal{N}_i$, $e \leqslant i < d$.*
(2) *The map (2.4.3) is a bijection if Λ is commutative and $\Lambda_2 = \Lambda \oplus J$ for a Λ-module J.*

Then ϕ is an isomorphism of functors.

Proof. The result for the $e = 0$ case is given in [Kap98, Lemma 2.3.6]. The $e > 0$ case is similarly proved without any major modification. □

2.5 NC hull and pro-representable hull

Let

$$\mathcal{N}^{\mathrm{loc}} \subset \mathcal{N}$$

be the subcategory of local finite-dimensional $\mathbb{C}$-algebras, i.e. an object of $\mathcal{N}^{\mathrm{loc}}$ is a finite-dimensional $\mathbb{C}$-algebra Λ having the unique two-sided maximal ideal $\mathbf{n} \subset \Lambda$. Note that a complete local $\mathbb{C}$-algebra $\widehat{R}$ defines the functor

$$h_{\widehat{R}} : \mathcal{N}^{\mathrm{loc}} \to \mathcal{S}et$$

by sending Λ to the set of local $\mathbb{C}$-algebra homomorphisms $\widehat{R} \to \Lambda$. We recall the notion of a pro-representable hull in [Sch68].

Definition 2.16. Let $h^{\mathrm{loc}} : \mathcal{N}^{\mathrm{loc}} \to \mathcal{S}et$ be a functor. A pro-representable hull of h^{loc} is a complete local $\mathbb{C}$-algebra $\widehat{R}$ together with a formally smooth natural transform $h_{\widehat{R}} \to h^{\mathrm{loc}}$, which is bijective on $\mathbb{C}[t]/t^2$.

The proof similar to Lemma 2.11 shows that a pro-representable hull is unique up to non-canonical isomorphisms (cf. [Sch68, Proposition 2.9]).

Let $h : \mathcal{N} \to \mathcal{S}et$ be a functor, and suppose that it has an NC hull $h_Y \to h$ for an NC scheme Y. Note that $h_Y(\mathbb{C}) \to h(\mathbb{C})$ is bijective, and they are identified with the set of closed points in Y^{ab}. Given a closed point $y \in Y^{\mathrm{ab}}$, we define the functor

$$h_y^{\mathrm{loc}} : \mathcal{N}^{\mathrm{loc}} \to \mathcal{S}et \tag{2.5.1}$$

by sending $(\Lambda, \mathbf{n})$ to the preimage of $y \in h(\mathbb{C})$ under the map

$$h(\Lambda) \to h(\Lambda/\mathbf{n}) = h(\mathbb{C}).$$

Let $m_y^{\mathrm{ab}} \subset \mathcal{O}_{Y^{\mathrm{ab}}}$ be the ideal sheaf which defines y, and $m_y \subset \mathcal{O}_Y$ the two-sided ideal sheaf given by the pull-back of m_y^{ab} by the surjection $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_{Y^{\mathrm{ab}}}$. We denote by $\widehat{\mathcal{O}}_{Y,y}$ the completion of $\mathcal{O}_Y$ by m_y. We also set

$$\mathcal{A}rt^{\mathrm{loc}} := \mathcal{N}^{\mathrm{loc}} \cap \mathcal{C}om,$$

i.e. $\mathcal{A}rt^{\mathrm{loc}}$ is the category of local Artinian $\mathbb{C}$-algebras.

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Lemma 2.17. *In the above situation, the functor* h_y^{loc} *has a pro-representable hull* $h_{\widehat{\mathcal{O}}_{Y,y}} \to h_y^{\mathrm{loc}}$, *which is an isomorphism on* $\mathcal{A}rt^{\mathrm{loc}}$.

Proof. The NC hull $h_Y \to h$ induces the natural transform $h_{Y,y}^{\mathrm{loc}} \to h_y^{\mathrm{loc}}$, which is formally smooth and an isomorphism on $\mathcal{A}rt^{\mathrm{loc}}$ by the definition of NC hull. For $(\Lambda, \mathbf{n}) \in \mathcal{N}^{\mathrm{loc}}$, giving a morphism $\operatorname{Spf}\Lambda \to Y$ sending $\operatorname{Spec}\Lambda/\mathbf{n}$ to y is equivalent to giving a local $\mathbb{C}$-algebra homomorphism $\widehat{\mathcal{O}}_{Y,y} \to \Lambda$. Hence, $h_{Y,y}^{\mathrm{loc}} = h_{\widehat{\mathcal{O}}_{Y,y}}$, and $h_{\widehat{\mathcal{O}}_{Y,y}} \to h_y^{\mathrm{loc}}$ is a pro-representable hull. $\square$

3. NC thickening of moduli spaces of quiver representations

In this section, we construct quasi-NC structures on the moduli spaces of representations of quivers with relations.

3.1 Representations of quivers

Recall that a *quiver* consists of data

$$Q = (Q_0, Q_1, h, t),$$

where Q_0 and Q_1 are finite sets (called the sets of *vertices* and *arrows*, respectively) and $h, t : Q_1 \to Q_0$ are maps. The maps h, t indicate the vertices at the head and tail of each arrow, respectively.

Definition 3.1. A representation of a quiver Q over an NC scheme $(Y, \mathcal{O}_Y)$ consists of data

$$\mathcal{W} = (\{\mathcal{W}_v\}_{v \in Q_0}, \{\phi_a\}_{a \in Q_1}) \tag{3.1.1}$$

where each $\mathcal{W}_v$ is an object of $\operatorname{Coh}(Y)$, and $\phi_a : \mathcal{W}_{t(a)} \to \mathcal{W}_{h(a)}$ is a morphism of coherent left $\mathcal{O}_Y$-modules.

Given a representation (3.1.1) of Q over Y, we set

$$\mathcal{W}_\bullet := \bigoplus_{v \in Q_0} \mathcal{W}_v.$$

Let $\mathbb{C}[Q]$ be the path algebra of Q. By the definition, we have the natural algebra homomorphism

$$\mathbb{C}[Q] \to \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{W}_\bullet, \mathcal{W}_\bullet) \tag{3.1.2}$$

sending $a \in Q_1$ to ϕ_a. Recall that a quiver with relation is a pair (Q, I), where Q is a quiver and $I \subset \mathbb{C}[Q]$ is a two-sided ideal.

Definition 3.2. Let (Q, I) be a quiver with relation. A representation of (Q, I) over an NC scheme Y is a representation of Q over Y such that the map (3.1.2) is zero on I.

For another representation $\mathcal{W}'$ of (Q, I) over Y, the set of morphisms $\operatorname{Hom}(\mathcal{W}, \mathcal{W}')$ consists of coherent left $\mathcal{O}_Y$-module homomorphisms $\mathcal{W}_v \to \mathcal{W}'_v$ for each $v \in Q_0$ which commute with ϕ_a and ϕ'_a. The category

$$\operatorname{Rep}((Q, I)/Y) \tag{3.1.3}$$

is defined to be the category of representations of (Q, I) over Y. For objects $\mathcal{W}, \mathcal{W}'$ in (3.1.3), we also have the sheaf homomorphisms $\mathcal{H}om(\mathcal{W}, \mathcal{W}')$ on Y by associating an open subset $U \subset Y$ with $\operatorname{Hom}(\mathcal{W}|_U, \mathcal{W}'|_U)$, which is an object of $\operatorname{Coh}(Y)$ if Y is a commutative scheme. If $I = \{0\}$, or $Y = \operatorname{Spf}\Lambda$ for $\Lambda \in \mathcal{N}$, we simply write (3.1.3) as $\operatorname{Rep}(Q/Y)$ or $\operatorname{Rep}((Q, I)/\Lambda)$, respectively. If further $Y = \operatorname{Spec}\mathbb{C}$, we write (3.1.3) as $\operatorname{Rep}(Q, I)$, and $\operatorname{Rep}(Q)$ if $I = \{0\}$.

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Remark 3.3. For $\Lambda \in \mathcal{N}$, the category $\mathrm{Rep}((Q,I)/\Lambda)$ is equivalent to the category of collections of finitely generated left Λ-modules $\{W_v\}_{v\in Q_0}$ together with left Λ-module homomorphisms $\phi_a : W_{t(a)} \to W_{h(a)}$ such that the natural map $I \to \mathrm{Hom}_\Lambda(W_\bullet, W_\bullet)$ is zero. We call such a collection $\{W_v\}_{v\in Q_0}$ a representation of (Q,I) over Λ.

Let Q be a quiver and $\mathcal{W}$ a representation of it over an NC scheme Y. We call it *flat* if each $\mathcal{W}_v$ is a flat left $\mathcal{O}_Y$-module. This is equivalent to that each $\mathcal{W}_v$ is a locally free left $\mathcal{O}_Y$-module. We prepare the following lemma.

LEMMA 3.4. (i) *Let $\mathcal{W}$ be a flat representation of (Q,I) over a commutative scheme T and $\mathcal{J}$ a coherent sheaf on T. Suppose that for any $t \in T$, we have $\mathrm{Hom}(\mathcal{W}|_t, \mathcal{W}|_t) = \mathbb{C}$. Then the morphism*

$$\mathcal{J} \to \mathcal{H}om(\mathcal{W}, \mathcal{J} \otimes_{\mathcal{O}_T} \mathcal{W})$$

given by $u \mapsto u \otimes \mathrm{id}$ is an isomorphism.

(ii) *Let $0 \to J \to \Lambda' \to \Lambda \to 0$ be a central extension in $\mathcal{N}$ and $\mathcal{W}$ a flat representation of (Q,I) over Λ'. Then any automorphism of $\mathcal{W}$ which is the identity over Λ is given by the left multiplication of a central element $1+u$ for $u \in J$.*

Proof. (i) We may assume that $I = \{0\}$. For $t \in T$, we have the isomorphism in $D(\mathrm{Rep}(Q))$:

$$\mathcal{O}_t \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_T} \mathbf{R}\mathcal{H}om(\mathcal{W}, \mathcal{W}) \cong \mathbf{R}\mathrm{Hom}(\mathcal{W}|_t, \mathcal{W}|_t).$$

Hence, we have the spectral sequence

$$E_2^{p,q} = \mathcal{T}or_{-p}^{\mathcal{O}_T}(\mathcal{O}_t, \mathcal{E}xt^q(\mathcal{W}, \mathcal{W})) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{W}|_t, \mathcal{W}|_t).$$

Since $\mathbb{C}[Q]$ is hereditary, we have $\mathrm{Ext}^{\geqslant 2}(\mathcal{W}|_t, \mathcal{W}|_t) = 0$. Using the above spectral sequence and the assumption $\mathrm{Hom}(\mathcal{W}|_t, \mathcal{W}|_t) = \mathbb{C}$, we see that

$$\mathcal{E}xt^{\geqslant 2}(\mathcal{W}, \mathcal{W}) = 0, \quad E_2^{-1,1} = 0, \quad E_2^{0,0} = \mathbb{C}.$$

The vanishing of $E_2^{-1,1}$ shows that $\mathcal{E}xt^1(\mathcal{W}, \mathcal{W})$ is flat over $\mathcal{O}_T$, and $E_2^{0,0} = \mathbb{C}$ shows that the natural map $\mathcal{O}_T \to \mathcal{H}om(\mathcal{W}, \mathcal{W})$ is an isomorphism.[1] In particular, for $\mathcal{J} \in \mathrm{Coh}(T)$, we have the isomorphism

$$\mathcal{H}^0(\mathcal{J} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_T} \mathbf{R}\mathcal{H}om(\mathcal{W}, \mathcal{W})) \cong \mathcal{J}.$$

Then the result follows by taking the zeroth cohomology of the following isomorphism in $D(\mathrm{Coh}(T))$:

$$\mathcal{J} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_T} \mathbf{R}\mathcal{H}om(\mathcal{W}, \mathcal{W}) \cong \mathbf{R}\mathcal{H}om(\mathcal{W}, \mathcal{J} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_T} \mathcal{W}).$$

(ii) Let g be an automorphism of $\mathcal{W}$ which is the identity over Λ. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{W}^{\mathrm{ab}} \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & \mathcal{W} \arrow[r, "p"] \arrow[d, "g"] & \Lambda \otimes_{\Lambda'} \mathcal{W} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & J \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{W}^{\mathrm{ab}} \arrow[r, "\iota"] & \mathcal{W} \arrow[r, "p"] & \Lambda \otimes_{\Lambda'} \mathcal{W} \arrow[r] & 0.
\end{tikzcd}$$

[1] Here we use the assumption that T is commutative, as otherwise there is no natural map $\mathcal{O}_T \to \mathcal{H}om(\mathcal{W}, \mathcal{W})$

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By the above commutative diagram, the isomorphism g is written as $\mathrm{id} + \iota \circ \alpha \circ p$ for some morphism $\alpha : \Lambda \otimes_{\Lambda'} \mathcal{W} \to J \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{W}^{\mathrm{ab}}$. The morphism α descends to the morphism $\mathcal{W}^{\mathrm{ab}} \to J \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{W}^{\mathrm{ab}}$, which is given by $u \otimes \mathrm{id}$ for some $u \in J$ by (i). Therefore, g is the left multiplication of $1 + u$. $\square$

Let $\mathcal{W}$ be a representation of Q over an NC scheme Y. We consider the natural morphism of sheaves of $\mathcal{O}_Y$ bi-modules

$$\mathcal{W}_\bullet \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{W}_\bullet, \mathcal{W}_\bullet) \otimes_{\mathbb{C}} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{W}_\bullet, \mathcal{O}_Y) \to \mathcal{O}_Y$$

given by $e \otimes f \otimes g \mapsto g \circ f(e)$. By composing it with (3.1.2) and the inclusion $I \subset \mathbb{C}[Q]$, we obtain the morphism of sheaves of $\mathcal{O}_Y$ bi-modules

$$\mathcal{W}_\bullet \otimes_{\mathbb{C}} I \otimes_{\mathbb{C}} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{W}_\bullet, \mathcal{O}_Y) \to \mathcal{O}_Y. \tag{3.1.4}$$

Definition 3.5. In the above situation, we define the ideal of relations in I to be the image of (3.1.4), and denote it by $\mathcal{J}_I \subset \mathcal{O}_Y$.

If $\mathcal{W}$ is flat, then $\mathcal{J}_I$ is locally the two-sided ideal generated by the matrix components of the morphism (3.1.2) restricted to I. In particular, the map (3.1.2) is zero on I if and only if $\mathcal{J}_I = 0$. If Z is the closed NC subscheme of Y defined by $\mathcal{O}_Z := \mathcal{O}_Y / \mathcal{J}_I$, the representation of Q

$$\mathcal{W}|_Z := (\{\mathcal{O}_Z \otimes_{\mathcal{O}_Y} \mathcal{W}_v\}_{v \in Q_0}, \{\mathrm{id} \otimes \phi_a\}_{a \in Q_1})$$

over Z is a representation of (Q, I).

3.2 Moduli spaces of representations of quivers

Let (Q, I) be a quiver with relation. We set $\Gamma_Q := \mathbb{Z}^{Q_0}$, and define its inner product by

$$\gamma \cdot \gamma' = \sum_{v \in Q_0} \gamma_v \cdot \gamma'_v.$$

For an object W in $\mathrm{Rep}(Q)$, its dimension vector $\dim W$ is defined by

$$\dim W := (\dim W_v)_{v \in Q_0} \in \Gamma_Q.$$

We recall the notion of King's θ-stability on $\mathrm{Rep}(Q, I)$.

Definition 3.6 [Kin94]. For $\theta \in \Gamma_Q$, a representation W of (Q, I) is called θ-(semi)stable if $\theta \cdot \dim W = 0$ and, for any subobject $0 \neq W' \subsetneq W$ in $\mathrm{Rep}(Q, I)$, we have the inequality

$$\theta \cdot \dim W' > (\geqslant) 0.$$

For $\gamma, \theta \in \Gamma_Q$ with $\theta \cdot \gamma = 0$, the 2-functor

$$\mathfrak{M}_{Q,I,\theta}(\gamma) : \mathcal{S}ch/\mathbb{C} \to \mathcal{G}roupoid \tag{3.2.1}$$

is defined by sending a $\mathbb{C}$-scheme T to the groupoid of flat representations $\mathcal{W}$ of (Q, I) over T such that $\mathcal{W}|_t$ for any $t \in T$ is a θ-semistable representation of (Q, I) with dimension vector γ. The 2-functor (3.2.1) is known to be an algebraic stack of finite type over $\mathbb{C}$. Indeed, let W be the affine space given by

$$W := \prod_{a \in Q_1} \mathrm{Hom}(W_{t(a)}, W_{h(a)}).$$

By the construction of W, there exists a tautological representation of Q over W. Let $M \subset W$ be the subscheme defined by the ideal of relations in I, whose closed points correspond to (Q, I)-representations. The subset of points $M^{ss} \subset M$ corresponding to θ-semistable representations forms an open subset of M. Also, the group $G := \prod_{v \in Q_0} \mathrm{GL}(W_v)$ acts on W by

$$(g_v) \cdot (\phi_a) = (g_{h(a)}^{-1} \circ \phi_a \circ g_{t(a)}).$$

The G-action on W preserves M^{ss}, and the stack (3.2.1) is given by the quotient stack

$$\mathfrak{M}_{Q,I,\theta}(\gamma) = [M^{ss}/G].$$

On the other hand, let

$$\mathcal{M}_{Q,I,\theta}(\gamma) : \mathcal{S}ch/\mathbb{C} \to \mathcal{S}et \tag{3.2.2}$$

be the functor defined by sending a $\mathbb{C}$-scheme T to the set of equivalence classes of θ-stable flat representations of (Q, I) over T. Here $\mathcal{W}$ and $\mathcal{W}'$ are called *equivalent* if there is a line bundle $\mathcal{L}$ on T such that $\mathcal{W}$ and $\mathcal{W}' \otimes \mathcal{L}$ are isomorphic in $\mathrm{Rep}((Q, I)/T)$. The following result was proved by King.

Theorem 3.7 [Kin94]. *If $\gamma \in \Gamma_Q$ is primitive, the functor $\mathcal{M}_{Q,I,\theta}(\gamma)$ is represented by a quasi-projective scheme $M_{Q,I,\theta}(\gamma)$, which is projective if Q does not contain a loop. If $I = \{0\}$, the moduli space $M_{Q,\theta}(\gamma) := M_{Q,\{0\},\theta}(\gamma)$ is non-singular.*

Indeed, let $M^s \subset M^{ss}$ be the open subset consisting of θ-stable representations. For $w \in M^s$, the subgroup of G which fixes w coincides with the diagonal subgroup $\mathbb{C}^* \subset G$. Hence, the group $\overline{G} := G/\mathbb{C}^*$ acts on M^s without fixed points, and $M_{Q,I,\theta}(\gamma)$ is given by

$$M_{Q,I,\theta}(\gamma) = M^s/\overline{G}. \tag{3.2.3}$$

In particular, the open substack of $\mathfrak{M}_{Q,I,\theta}(\gamma)$ consisting of stable representations is a $\mathbb{C}^*$-gerbe over (3.2.3). By the construction, we have the closed embedding

$$M_{Q,I,\theta}(\gamma) \hookrightarrow M_{Q,\theta}(\gamma)$$

such that $M_{Q,I,\theta}(\gamma)$ is defined by the ideal of relations in I on the smooth moduli space $M_{Q,\theta}(\gamma)$. Since $M_{Q,I,\theta}(\gamma)$ represents the functor (3.2.2), there exists a universal (Q, I)-representation, i.e. a family of representations of (Q, I)

$$\mathcal{V} = (\{\mathcal{V}_v\}_{v \in Q_0}, \{\phi_a\}_{a \in Q_1}) \tag{3.2.4}$$

over $M_{Q,I,\theta}(\gamma)$, such that the map $f \mapsto f^*\mathcal{V}$ gives the functorial isomorphism

$$\mathrm{Hom}(T, M_{Q,I,\theta}(\gamma)) \xrightarrow{\cong} \mathcal{M}_{Q,I,\theta}(\gamma)(T)$$

for any $\mathbb{C}$-scheme T.

3.3 Construction of NC hull

Let $\gamma \in \Gamma_Q$ be a primitive element. We consider the smooth moduli space $M_{Q,\theta}(\gamma)$ of representations of Q without relation (cf. Theorem 3.7), and a universal representation $\mathcal{V}$ on it given by (3.2.4) for $I = \{0\}$. We take an affine open subset

$$U \subset M_{Q,\theta}(\gamma)$$

such that each $\mathcal{V}_v|_U$ is isomorphic to $\mathcal{O}_U \otimes_{\mathbb{C}} W_v$, where W_v is a $\mathbb{C}$-vector space with dimension γ_v. Since U is smooth, it admits an NC smooth thickening $U^{\mathrm{nc}} = (U, \mathcal{O}_U^{\mathrm{nc}})$, which is unique up to non-canonical isomorphisms (cf. [Kap98, Theorem 1.6.1]). We set

$$\mathcal{V}_{U,v}^{\mathrm{nc}} := \mathcal{O}_U^{\mathrm{nc}} \otimes_{\mathbb{C}} W_v, \quad v \in Q_0,$$

which we regard as left $\mathcal{O}_U^{\mathrm{nc}}$-modules. Since $\mathcal{O}_U^{\mathrm{nc}} \twoheadrightarrow \mathcal{O}_U$ is surjective, we can lift each universal morphism $\phi_a : \mathcal{V}_{t(a)} \to \mathcal{V}_{h(a)}$ restricted to U to a left $\mathcal{O}_U^{\mathrm{nc}}$-module homomorphism

$$\phi_{U,a}^{\mathrm{nc}} : \mathcal{V}_{U,t(a)}^{\mathrm{nc}} \to \mathcal{V}_{U,h(a)}^{\mathrm{nc}}, \quad a \in Q_1. \tag{3.3.1}$$

Then the data

$$\mathcal{V}_U^{\mathrm{nc}} := (\{\mathcal{V}_{U,v}^{\mathrm{nc}}\}_{v\in Q_0}, \{\phi_{U,a}^{\mathrm{nc}}\}_{a\in Q_1}) \tag{3.3.2}$$

is a representation of Q over U^{nc}.

We define the functor

$$h_{Q,\theta}(\gamma) : \mathcal{N} \to \mathcal{S}et \tag{3.3.3}$$

by sending $\Lambda \in \mathcal{N}$ to the isomorphism classes of triples $(f, \mathcal{W}, \psi)$:

- f is a morphism $\operatorname{Spec}\Lambda^{\mathrm{ab}} \to M_{Q,\theta}(\gamma)$ of schemes;
- $\mathcal{W}$ is a flat representation of Q over Λ;
- ψ is an isomorphism $\psi : \mathcal{W}^{\mathrm{ab}} \xrightarrow{\cong} f^*\mathcal{V}$ as representations of Q over Λ^{ab}.

An isomorphism $(f, \mathcal{W}, \psi) \to (f', \mathcal{W}', \psi')$ exists if $f = f'$, and there is an isomorphism $\mathcal{W} \to \mathcal{W}'$ as representations of Q over Λ commuting ψ, ψ'. Note that we have

$$h_{Q,\theta}(\gamma)|_{\mathcal{C}om} = h_{M_{Q,\theta}(\gamma)}. \tag{3.3.4}$$

Proposition 3.8. *The natural transformation*

$$h_{U^{\mathrm{nc}}} \to h_{Q,\theta}(\gamma)|_U \tag{3.3.5}$$

sending $g : \operatorname{Spf}\Lambda \to U^{\mathrm{nc}}$ *to* $(g^{\mathrm{ab}}, g^*\mathcal{V}_U^{\mathrm{nc}}, \mathrm{id})$ *is an NC hull of* $h_{Q,\theta}(\gamma)|_U$.

Proof. We write $h = h_{Q,\theta}(\gamma)|_U$ for simplicity. Since $h_{U^{\mathrm{nc}}}|_{\mathcal{C}om} = h_U$, the natural transformation (3.3.5) is an isomorphism on $\mathcal{C}om$ by (3.3.4). Therefore, by Proposition 2.14, it is enough to show the following: for surjections $p_1 : \Lambda_1 \twoheadrightarrow \Lambda$, $p_2 : \Lambda_2 \twoheadrightarrow \Lambda$ in $\mathcal{N}$, $\Lambda_{12} := \Lambda_1 \times_\Lambda \Lambda_2$, the natural map

$$h(\Lambda_{12}) \to h(\Lambda_1) \times_{h(\Lambda)} h(\Lambda_2) \tag{3.3.6}$$

is surjective. The right-hand side of (3.3.6) consists of triples

$$(f_j, \mathcal{W}_j, \psi_j) \in h(\Lambda_j), \quad j = 1, 2 \tag{3.3.7}$$

which are isomorphic over Λ, i.e. $f := f_1|_{\operatorname{Spec}\Lambda^{\mathrm{ab}}} = f_2|_{\operatorname{Spec}\Lambda^{\mathrm{ab}}}$, and there is an isomorphism of representations of Q over Λ

$$\gamma : \Lambda \otimes_{\Lambda_1} \mathcal{W}_1 \xrightarrow{\cong} \Lambda \otimes_{\Lambda_2} \mathcal{W}_2 \tag{3.3.8}$$

which commutes with $\Lambda^{\mathrm{ab}} \otimes_{\Lambda_j^{\mathrm{ab}}} \psi_j$ for $j = 1, 2$. Let us write $\mathcal{W}_j$ as

$$\mathcal{W}_j = (\{W_{j,v}\}_{v\in Q_0}, \{\phi_{j,a}\}_{a\in Q_1})$$

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as representations of Q over Λ_j. Then the isomorphism γ consists of collections of isomorphisms of left Λ-modules

$$\gamma_v : \Lambda \otimes_{\Lambda_1} W_{1,v} \xrightarrow{\cong} \Lambda \otimes_{\Lambda_2} W_{2,v}$$

for each $v \in Q_1$, which commute with $\Lambda \otimes_{\Lambda_j} \phi_{j,a}$. We set

$$W_{12,v} := \{(x, y) \in W_{1,v} \times W_{2,v} : \gamma_v \circ (1 \otimes x) = 1 \otimes y\}.$$

Then $W_{12,v}$ is a projective left Λ_{12}-module by [Mil71, Theorem 2.1]. Therefore, the data

$$\mathcal{W}_{12} := (\{W_{12,v}\}_{v\in Q_0}, \{\phi_{1,a} \times \phi_{2,a}\}_{a\in Q_1}) \tag{3.3.9}$$

determines a flat representation of Q over Λ_{12}. Also, since $\Lambda_{12}^{\mathrm{ab}} = \Lambda_1^{\mathrm{ab}} \times_{\Lambda^{\mathrm{ab}}} \Lambda_2^{\mathrm{ab}}$, the morphisms f_1, f_2 and f induce the morphism of schemes

$$f_{12} := f_1 \times_f f_2 : \operatorname{Spec} \Lambda_{12}^{\mathrm{ab}} \to U.$$

Finally, since γ commutes with $\Lambda^{\mathrm{ab}} \otimes_{\Lambda_j^{\mathrm{ab}}} \psi_j$, we have the isomorphism

$$\psi_{12} := \psi_1 \times \psi_2 : \mathcal{W}_{12}^{\mathrm{ab}} \xrightarrow{\cong} f_{12}^* \mathcal{V}$$

of representations of Q over $\Lambda_{12}^{\mathrm{ab}}$. Therefore, the triple $(f_{12}, \mathcal{W}_{12}, \psi_{12})$ determines an element of the left-hand side of (3.3.6), which is mapped to the triples (3.3.7) by the map (3.3.6). Therefore, the map (3.3.6) is surjective. □

Remark 3.9. Note that the resulting element $(f_{12}, \mathcal{W}_{12}, \psi_{12}) \in h(\Lambda_{12})$ may also depend on a choice of γ. If it really depends on γ, the map (3.3.6) is not bijective, and (3.3.5) is not isomorphic. In order to show the independence of γ, one needs to show that any automorphism of $\Lambda \otimes_{\Lambda_j} \mathcal{W}_j$ extends to that of $\mathcal{W}_j$, which might not be true if Λ_j is non-commutative. A similar issue occurs in the proof of [Kap98, Proposition 5.4.3], which caused its gap as pointed out in [PT14, Remark 4.1.4].

Remark 3.10. *A priori*, the representation $\mathcal{V}_U^{\mathrm{nc}}$ depends on the choices of lifting (3.3.1). However, Proposition 3.8 implies that different choices of lifting (3.3.1) yield isomorphic representations after pulling back by some automorphism of U^{nc}. See Corollary 3.3.14.

We next consider the moduli space $M_{Q,I,\theta}(\gamma)$ in Theorem 3.7 for the non-zero relation $I \subset \mathbb{C}[Q]$. We embed it into the smooth moduli space

$$M_{Q,I,\theta}(\gamma) \subset M_{Q,\theta}(\gamma)$$

and take an affine open subset $U \subset M_{Q,\theta}(\gamma)$ as before. Let U^{nc} be an NC smooth thickening of U and $\mathcal{V}_U^{\mathrm{nc}}$ a lift of a universal representation (3.3.2) to U^{nc}. The representation $\mathcal{V}_U^{\mathrm{nc}}$, together with the relation I, determines the two-sided ideal $\mathcal{J}_{I,U} \subset \mathcal{O}_U^{\mathrm{nc}}$ of relations in I (cf. Definition 3.5). We set

$$V := M_{Q,I,\theta}(\gamma) \cap U, \quad \mathcal{O}_V^{\mathrm{nc}} := \mathcal{O}_U^{\mathrm{nc}} / \overline{\mathcal{J}}_{I,U}, \quad V^{\mathrm{nc}} := (V, \mathcal{O}_V^{\mathrm{nc}}). \tag{3.3.10}$$

Since $(\mathcal{O}_V^{\mathrm{nc}})^{\mathrm{ab}} = \mathcal{O}_V$, the affine NC scheme V^{nc} is a closed NC subscheme of U^{nc}. The restriction $\mathcal{V}_V^{\mathrm{nc}} := \mathcal{V}_U^{\mathrm{nc}}|_{V^{\mathrm{nc}}}$ is a representation of (Q, I) over V^{nc}.

We also define the functor

$$h_{Q,I,\theta}(\gamma) : \mathcal{N} \to \mathcal{S}et \tag{3.3.11}$$

to be the subfunctor of $h_{Q,\theta}(\gamma)$ in (3.3.3), sending $\Lambda \in \mathcal{N}$ to the set of triples $(f, \mathcal{W}, \psi) \in h_{Q,\theta}(\gamma)(\Lambda)$ such that f factors through $M_{Q,\theta,I}(\gamma)$ and $\mathcal{W}$ is a representation of (Q, I) over Λ. Note that we have

$$h_{Q,I,\theta}(\gamma)|_{\mathcal{C}om} = h_{M_{Q,I,\theta}(\gamma)}.$$

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Proposition 3.11. *The natural transformation*

$$h_{V^{\mathrm{nc}}} \to h_{Q,I,\theta}(\gamma)|_V \tag{3.3.12}$$

sending $g : \operatorname{Spf}\Lambda \to V^{\mathrm{nc}}$ *to* $(g^{\mathrm{ab}}, g^*\mathcal{V}_V^{\mathrm{nc}}, \mathrm{id})$ *is an NC hull of* $h_{Q,I,\theta}(\gamma)|_V$.

Proof. By Proposition 3.8, it is enough to note that the following diagram of functors:

$$\begin{tikzcd} h_{U^{\mathrm{nc}}} \arrow[r] & h_{Q,\theta}(\gamma)|_U \\ h_{V^{\mathrm{nc}}} \arrow[r] \arrow[u, hook] & h_{Q,I,\theta}(\gamma)|_V \arrow[u, hook] \end{tikzcd}$$

is Cartesian. Indeed, for $g : \operatorname{Spf}\Lambda \to U^{\mathrm{nc}}$, suppose that $g^*\mathcal{V}_U^{\mathrm{nc}}$ is a representation of (Q, I) over Λ. This is equivalent to that the natural morphism

$$(\Lambda \otimes_{\mathcal{O}_U^{\mathrm{nc}}} \mathcal{V}_U^{\mathrm{nc}}) \otimes_{\mathbb{C}} I \otimes_{\mathbb{C}} \operatorname{Hom}_\Lambda(\Lambda \otimes_{\mathcal{O}_U^{\mathrm{nc}}} \mathcal{V}_U^{\mathrm{nc}}, \Lambda) \to \Lambda$$

is a zero map, which is equivalent to that

$$\Lambda \otimes_{\mathcal{O}_U^{\mathrm{nc}}} \mathcal{J}_{I,U} \otimes_{\mathcal{O}_U^{\mathrm{nc}}} \Lambda \to \Lambda$$

induced by $\mathcal{J}_{I,U} \subset \mathcal{O}_U^{\mathrm{nc}}$ is a zero map. The last condition is also equivalent to that $g^* : \mathcal{O}_U^{\mathrm{nc}} \to \Lambda$ vanishes on $\mathcal{J}_{I,U}$; hence, g factors through $V^{\mathrm{nc}} \hookrightarrow U^{\mathrm{nc}}$. $\square$

Let $\{U_i\}_{i\in\mathbb{I}}$ be a sufficiently small affine open covering of $M_{Q,\theta}(\gamma)$ with affine U_{ij}, and set $V_i = M_{Q,I,\theta}(\gamma) \cap U_i$. By applying the above construction to $U = U_i$, we obtain the NC schemes

$$U_i^{\mathrm{nc}} := (U_i, \mathcal{O}_{U_i}^{\mathrm{nc}}), \quad V_i^{\mathrm{nc}} := (V_i, \mathcal{O}_{V_i}^{\mathrm{nc}}) \tag{3.3.13}$$

and the representations $\mathcal{V}_{U_i}^{\mathrm{nc}}$ of Q over U_i^{nc}, $\mathcal{V}_{V_i}^{\mathrm{nc}} := \mathcal{V}_{U_i}^{\mathrm{nc}}|_{V_i}$ of (Q, I) over V_i^{nc}, respectively. We have the following corollary.

Corollary 3.12. *There exist isomorphisms of NC schemes*

$$\phi_{ij} : V_j^{\mathrm{nc}}|_{V_{ij}} \xrightarrow{\cong} V_i^{\mathrm{nc}}|_{V_{ij}} \tag{3.3.14}$$

giving a quasi-NC structure on $M_{Q,I,\theta}(\gamma)$*, and isomorphisms of representations of* (Q, I)*:*

$$g_{ij} : \phi_{ij}^* \mathcal{V}_{V_i}^{\mathrm{nc}}|_{V_{ij}} \xrightarrow{\cong} \mathcal{V}_{V_j}^{\mathrm{nc}}|_{V_{ij}}. \tag{3.3.15}$$

Proof. The existence of isomorphisms (3.3.14) follows from Proposition 3.11 and Corollary 2.12. Since one can choose ϕ_{ij} commuting with natural transforms $h_{V_i^{\mathrm{nc}}} \to h_{Q,I,\theta}(\gamma)|_{V_i}$ in Proposition 3.11, we also have isomorphisms (3.3.15). $\square$

3.4 Comparison with the formal deformations of quiver representations

Here we use the notation in § 2.5. For an object $W \in \operatorname{Rep}(Q, I)$, the formal non-commutative deformation functor

$$\operatorname{Def}_W^{\mathrm{nc}} : \mathcal{N}^{\mathrm{loc}} \to \mathcal{S}et \tag{3.4.1}$$

is defined by sending $(\Lambda, \mathbf{n})$ to the set of isomorphism classes $(\mathcal{W}, \psi)$, where $\mathcal{W}$ is a flat representation of (Q, I) over Λ, and $\psi : \Lambda/\mathbf{n} \otimes_\Lambda \mathcal{W} \xrightarrow{\cong} W$ is an isomorphism as representations of (Q, I). The non-commutative deformation functor (3.4.1) was studied by Laudal [Lau02], where the existence of a pro-representable hull was proved. We also refer to [Eri10, Seg08, ELO09, ELO10, ELO11] for details on formal non-commutative deformation theory.

Recall that the formal commutative deformation space of W is given by the solution of the Maurer–Cartan equation of the dg-algebra $\mathbf{R}\mathrm{Hom}(W, W)$, up to gauge equivalence. Let

$$(\mathrm{Ext}^*(W, W), \{m_n\}_{n\geqslant 2}) \tag{3.4.2}$$

be the minimal A_∞-algebra which is A_∞ quasi-isomorphic to $\mathbf{R}\mathrm{Hom}(W, W)$. Here we take the Ext-groups in the category $\mathrm{Rep}(Q, I)$. By [Seg08], the pro-representable hull of (3.4.1) is described in terms of the A_∞-algebra (3.4.2). Let

$$m_n : \mathrm{Ext}^1(W, W)^{\otimes n} \to \mathrm{Ext}^2(W, W)$$

be the nth A_∞-product. Below, for a vector space V, we denote by $\widehat{T}^\bullet(V)$ the completed tensor algebra given by

$$\widehat{T}^\bullet(V) = \prod_{n\geqslant 0} V^{\otimes n}.$$

Let

$$J_W \subset \widehat{T}^\bullet(\mathrm{Ext}^1(W, W)^\vee)$$

be the topological closure of the two-sided ideal of the completed tensor algebra of $\mathrm{Ext}^1(W, W)^\vee$ generated by the image of the map

$$\sum_{n\geqslant 2} m_n^\vee : \mathrm{Ext}^2(W, W)^\vee \to \widehat{T}^\bullet(\mathrm{Ext}^1(W, W)^\vee).$$

The pro-representable hull of (3.4.1) is given by the quotient algebra

$$R_W^{\mathrm{nc}} := \widehat{T}^\bullet(\mathrm{Ext}^1(W, W)^\vee)/J_W.$$

Let us take an open neighborhood $[W] \in V \subset M_{Q,I,\theta}(\gamma)$, and a non-commutative thickening $V^{\mathrm{nc}} = (V, \mathcal{O}_V^{\mathrm{nc}})$ as in the previous subsection (3.3.10).

Lemma 3.13. *The completion $\widehat{\mathcal{O}}_{V,[W]}^{\mathrm{nc}}$ is isomorphic to R_W^{nc}.*

Proof. Let

$$h_{Q,I,\theta}^{\mathrm{loc}}(\gamma)_{[W]} : \mathcal{N}^{\mathrm{loc}} \to \mathcal{S}et$$

be the functor constructed from $h_{Q,I,\theta}(\gamma)$ by (2.5.1), which has a pro-representable hull $h_{\widehat{\mathcal{O}}_{V,[W]}^{\mathrm{nc}}} \to h_{Q,I,\theta}^{\mathrm{loc}}(\gamma)_{[W]}$ by Lemma 2.17. We construct the natural transform

$$h_{Q,I,\theta}^{\mathrm{loc}}(\gamma)_{[W]} \to \mathrm{Def}_W^{\mathrm{nc}} \tag{3.4.3}$$

by sending triples $(f, \mathcal{W}, \psi)$ of the $(\Lambda, \mathbf{n})$-point of the left-hand side of (3.4.3) to the pair $(\mathcal{W}, \Lambda/\mathbf{n} \otimes_\Lambda \psi)$ of the right-hand side. By the uniqueness of a pro-representable hull, it is enough to show that (3.4.3) is formally smooth and bijective on $\mathbb{C}[t]/t^2$. Note that since the functor (3.2.2) is represented by $M_{Q,I,\theta}(\gamma)$, the functor

$$\mathrm{Def}_W := \mathrm{Def}_W^{\mathrm{nc}}|_{\mathcal{A}rt^{\mathrm{loc}}}$$

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is pro-represented by $\widehat{\mathcal{O}}_{V,[W]}$. Therefore, by Lemma 2.17, the natural transform (3.4.3) is an isomorphism on $\mathcal{A}rt^{\mathrm{loc}}$ and in particular bijective on $\mathbb{C}[t]/t^2$. It remains to show that (3.4.3) is formally smooth.

Let $(f, \mathcal{W}, \psi)$ be a $(\Lambda, \mathbf{n})$-point of the left-hand side of (3.4.3), and suppose that $\mathcal{W}$ extends to a flat representation $\mathcal{W}'$ of (Q, I) over Λ', where $\Lambda' \twoheadrightarrow \Lambda$ is a surjection in $\mathcal{N}^{\mathrm{loc}}$. Then $\mathcal{W}'^{\mathrm{ab}}$ is a flat extension of $\mathcal{W}^{\mathrm{ab}}$ to Λ'^{ab}. By the pro-representability of Def_W, the morphism $f : \operatorname{Spec} \Lambda^{\mathrm{ab}} \to V$ uniquely extends to a morphism of schemes $f' : \operatorname{Spec} \Lambda'^{\mathrm{ab}} \to V$ such that there exists an isomorphism $\psi' : \mathcal{W}'^{\mathrm{ab}} \xrightarrow{\cong} f'^{*}\mathcal{V}$. By Lemma 3.4(i), any automorphism of $f^{*}\mathcal{V}$ extends to an automorphism of $f'^{*}\mathcal{V}$ and hence one can choose ψ' so that $\Lambda^{\mathrm{ab}} \otimes_{\Lambda'^{\mathrm{ab}}} \psi' = \psi$ holds. Then the triple $(f', \mathcal{W}', \psi')$ is an extension of $(f, \mathcal{W}, \psi)$ to Λ', showing that (3.4.3) is formally smooth. □

By Corollary 3.12 together with the above lemma, we obtain the following result.

Theorem 3.14. *There exists a quasi-NC structure $\{(V_i, \mathcal{O}^{\mathrm{nc}}_{V_i})\}_{i \in \mathbb{I}}$ on $M_{Q,I,\theta}(\gamma)$ such that for any $[W] \in V_i$, there is an isomorphism of algebras $\widehat{\mathcal{O}}^{\mathrm{nc}}_{V_i,[W]} \cong R^{\mathrm{nc}}_W$.*

3.5 Partial NC thickening of moduli spaces of representations of quivers

It is not clear whether the quasi-NC structures in Corollary 3.12 glue together to give an NC structure. Here we discuss the possibility to extend a given $(d-1)$th-order NC thickening to that of the dth-order NC thickening. Let U^{nc}_i, V^{nc}_i be affine NC schemes given in (3.3.13), and take representations $\mathcal{V}^{\mathrm{nc}}_{U_i}$, $\mathcal{V}^{\mathrm{nc}}_{V_i}$ of (Q, I) as before. For $d \in \mathbb{Z}_{\geqslant 0}$, we set

$$U^d_i := (U^{\mathrm{nc}}_i)^{\leqslant d}, \quad V^d_i := (V^{\mathrm{nc}}_i)^{\leqslant d}, \quad \mathcal{V}^d_{U_i} := (\mathcal{V}^{\mathrm{nc}}_{U_i})^{\leqslant d}, \quad \mathcal{V}^d_{V_i} := (\mathcal{V}^{\mathrm{nc}}_{V_i})^{\leqslant d}.$$

The isomorphisms (3.3.14) and (3.3.15) in Corollary 3.12 induce isomorphisms

$$\phi^{\leqslant d}_{ij} : V^d_j|_{V_{ij}} \xrightarrow{\cong} V^d_i|_{V_{ij}}, \quad g^{\leqslant d}_{ij} : \phi^{\leqslant d *}_{ij} \mathcal{V}^d_{V_j}|_{V_{ij}} \xrightarrow{\cong} \mathcal{V}^d_{V_i}|_{V_{ij}},$$

where $\phi^{\leqslant d}_{ij}$ give a quasi-NC structure on $M_{Q,I,\theta}(\gamma)$, and $g^{\leqslant d}_{ij}$ are isomorphisms of representations of (Q, I). We put the following assumption.

Assumption 3.15. (1) By replacing $\phi^{\leqslant d-1}_{ij}$ if necessary, the quasi-NC structure $\{V^{d-1}_i\}_{i \in \mathbb{I}}$ determines an NC structure $M^{d-1}_{Q,I,\theta}(\gamma)$ on $M_{Q,I,\theta}(\gamma)$.

(2) By replacing $g^{\leqslant d-1}_{ij}$ if necessary, the objects $\{\mathcal{V}^{d-1}_{V_i}\}_{i \in \mathbb{I}}$ glue to give a representation $\mathcal{V}^{d-1}$ of (Q, I) over $M^{d-1}_{Q,I,\theta}(\gamma)$.

We define the functor

$$h^d_{Q,I,\theta}(\gamma) : \mathcal{N}_d \to \mathcal{S}et$$

by sending $\Lambda \in \mathcal{N}_d$ to the isomorphism classes of triples $(f, \mathcal{W}, \psi)$:

- f is a morphism of NC schemes $f : \operatorname{Spf} \Lambda^{\leqslant d-1} \to M^{d-1}_{Q,I,\theta}(\gamma)$;
- $\mathcal{W}$ is a flat representation of (Q, I) over Λ;
- $\psi : \mathcal{W}^{\leqslant d-1} \xrightarrow{\cong} f^{*}\mathcal{V}^{d-1}$ is an isomorphism of representations of (Q, I) over $\Lambda^{\leqslant d-1}$.

An isomorphism $(f, \mathcal{W}, \psi) \to (f', \mathcal{W}', \psi')$ exists if $f = f'$, and there is an isomorphism $\mathcal{W} \to \mathcal{W}'$ as representations of (Q, I) over Λ commuting with ψ, ψ'.

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PROPOSITION 3.16. *The natural transform*

$$h_{V_i^d} \to h^d_{Q,I,\theta}(\gamma)|_{V_i} \tag{3.5.1}$$

sending $g : \operatorname{Spf} \Lambda \to V_i^d$ *for* $\Lambda \in \mathcal{N}_d$ *to* $(g^{\leqslant d-1}, g^*\mathcal{V}^d_{V_i}, \mathrm{id})$ *is an isomorphism of functors.*

Proof. Note that (3.5.1) is an isomorphism on $\mathcal{N}_{<d}$. Similarly to the proof of Proposition 3.11, we have the Cartesian square

$$\begin{tikzcd} h_{U_i^d} \arrow[r] & h^d_{Q,\{0\},\theta}(\gamma)|_{U_i} \\ h_{V_i^d} \arrow[r] \arrow[u, hook] & h^d_{Q,I,\theta}(\gamma)|_{V_i}. \arrow[u, hook] \end{tikzcd} \tag{3.5.2}$$

It is enough to show that the top arrow of (3.5.2) is an isomorphism. We write $h^d = h^d_{Q,\{0\},\theta}(\gamma)|_{U_i}$ for simplicity. By Proposition 2.15, it remains to show the following: for surjections $p_1 : \Lambda_1 \twoheadrightarrow \Lambda$, $p_2 : \Lambda_2 \twoheadrightarrow \Lambda$ in $\mathcal{N}_d$ with $\Lambda \in \mathcal{N}_{d-1}$, the natural map

$$h^d(\Lambda_{12}) \to h^d(\Lambda_1) \times_{h^d(\Lambda)} h^d(\Lambda_2) \tag{3.5.3}$$

is bijective. We follow the same notation and argument as in the proof of Proposition 3.8, replacing h with h^d and $*^{\mathrm{ab}}$ with $*^{\leqslant d-1}$. The difference from the proof of Proposition 3.8 is that, since we have

$$\Lambda \otimes_{\Lambda_j} \mathcal{W}_j = (\Lambda \otimes_{\Lambda_j} \mathcal{W}_j)^{\leqslant d-1}$$

and the isomorphism γ in (3.3.8) should commute with $\Lambda \otimes_{\Lambda_j} \psi_j$, γ is uniquely determined by the right-hand side of (3.5.3). Therefore, sending the triples (3.3.7) to $(f_{12}, \mathcal{W}_{12}, \psi_{12})$ is a well-defined map from the right-hand side to the left-hand side of (3.5.3), giving the inverse of (3.5.3). □

By the above proposition together with Remark 2.13, we have the following corollary.

COROLLARY 3.17. *Under Assumption 3.15, affine NC structures* $\{V_i^d\}_{i\in\mathbb{I}}$ *glue together to give an NC structure* $M^d_{Q,I,\theta}(\gamma)$ *on* $M_{Q,I,\theta}(\gamma)$.

By the above corollary, if the local (Q, I) representations $\mathcal{V}^d_{V_i}$ glue together to give a global (Q, I) representation, then we can extend the dth-order NC structure $M^d_{Q,I,\theta}(\gamma)$ to a $(d+1)$th-order NC thickening. The obstruction of gluing $\mathcal{V}^d_{V_i}$ is given as follows. We set

$$\mathcal{J}^d := \operatorname{Ker}(\mathcal{O}_{M^d_{Q,I,\theta}(\gamma)} \twoheadrightarrow \mathcal{O}_{M^{d-1}_{Q,I,\theta}(\gamma)}).$$

Note that $\mathcal{J}^d$ is a coherent sheaf on $M_{Q,I,\theta}(\gamma)$.

LEMMA 3.18. *In the situation of Corollary 3.17, there is a class*

$$\mathrm{ob} \in H^2(M_{Q,I,\theta}(\gamma), \mathcal{J}^d) \tag{3.5.4}$$

such that $\mathrm{ob} = 0$ *if and only if* $\{\mathcal{V}^d_{V_i}\}_{i\in\mathbb{I}}$ *glue to give a representation* $\mathcal{V}^d$ *of* (Q, I) *over* $M^d_{Q,I,\theta}(\gamma)$.

Proof. By Lemma 3.4(ii), the automorphism $g^{\leqslant d}_{ij} \circ g^{\leqslant d}_{jk} \circ g^{\leqslant d}_{ki}$ of $\mathcal{V}^d_{V_i}|_{V_{ijk}}$ is given by the multiplication of $1 + u_{ijk}$ for some element $u_{ijk} \in \mathcal{J}_d|_{V_{ijk}}$. Then $\{u_{ijk}\}_{ijk}$ is a Cech 2-cocycle of (3.5.4), giving the desired obstruction class. □

Since Assumption 3.15 is always satisfied for $d = 1$, the affine NC structures $\{V_i^1\}_{i \in \mathbb{I}}$ glue together to give an NC structure $M^1_{Q,I,\theta}(\gamma)$ on $M_{Q,I,\theta}(\gamma)$. Indeed, one of such a thickening is explicitly constructed in the following way. We first construct a 1-smooth NC structure $M^1_{Q,\theta}(\gamma)$ on the smooth moduli space $M_{Q,\theta}(\gamma)$ by

$$\mathcal{O}_{M^1_{Q,\theta}(\gamma)} = \mathcal{O}_{M_{Q,\theta}(\gamma)} \oplus \Omega^2_{M_{Q,\theta}(\gamma)}.$$

The multiplication is given

$$(x, f) \cdot (y, g) = (xy, xg + fy + dx \wedge dy).$$

The universal representations $\mathcal{V}^1_{U_i}$ may not glue together, but the ideals of relations $\mathcal{J}^1_{I,U_i} \subset \mathcal{O}^1_{U_i}$ coincide on U_{ij} and hence determine the two-sided ideal $\mathcal{J}^1_I \subset \mathcal{O}_{M^1_{Q,\theta}(\gamma)}$. Then the NC scheme

$$M^1_{Q,I,\theta}(\gamma) = (M_{Q,I,\theta}(\gamma), \mathcal{O}_{M^1_{Q,\theta}}(\gamma)/\mathcal{J}^1_I)$$

is isomorphic to V_i^1 on V_i.

3.6 NC structures on framed quiver representations

As we observed in Remark 3.9, the issue for having the global NC structure is caused by some automorphisms of sheaves over non-commutative bases. One of the classical ways to kill automorphisms of sheaves is to add additional data called *framing*. Here we show that this classical idea also works for the construction problem of global NC structures. Let (Q, I) be a quiver with relation. We fix a vertex $\star \in Q_0$.

Definition 3.19. A framed representation of (Q, I) over an NC scheme Y is a pair $(\mathcal{W}, \tau)$, where $\mathcal{W}$ is a representation of (Q, I) over Y and $\tau : \mathcal{O}_Y \to \mathcal{W}_\star$ is a morphism of coherent left $\mathcal{O}_Y$-modules.

A framed representation $(\mathcal{W}, \tau)$ over Y is called *flat* if $\mathcal{W}$ is flat. Let $(\mathcal{W}', \tau')$ be another framed representation of (Q, I) over Y. An isomorphism from $(\mathcal{W}, \tau)$ to $(\mathcal{W}', \tau')$ is an isomorphism $g : \mathcal{W} \xrightarrow{\cong} \mathcal{W}'$ in $\mathrm{Rep}((Q, I)/Y)$ such that $g_\star \circ \tau = \tau'$.

Remark 3.20. For $\Lambda \in \mathcal{N}$, giving a framed representation of (Q, I) over $\mathrm{Spf}\,\Lambda$ is equivalent to giving a representation W of (Q, I) over Λ together with a left Λ-module homomorphism $\tau : \Lambda \to W_\star$. The data (W, τ) is called a framed representation of (Q, I) over Λ. If $\Lambda = \mathbb{C}$, we just call it a framed representation of (Q, I).

Definition 3.21. For $\theta \in \Gamma_Q$, a framed representation (W, τ) of (Q, I) is called θ-stable if $\theta \cdot \dim W = 0$, W is θ-semistable and, for any subobject $0 \neq W' \subset W$ in $\mathrm{Rep}(Q)$ with $\mathrm{Im}\,\tau \in W'_\star$, we have $\theta \cdot \dim W' > 0$.

For $\theta, \gamma \in \Gamma_Q$ with $\theta \cdot \gamma = 0$, the functor

$$\mathcal{M}^\star_{Q,I,\theta}(\gamma) : \mathcal{S}ch/\mathbb{C} \to \mathcal{S}et$$

is defined by sending a $\mathbb{C}$-scheme T to the set of isomorphism classes of flat framed representations $(\mathcal{W}, \tau)$ over T such that for any $t \in T$, the restriction $(\mathcal{W}|_t, \tau|_t)$ is a θ-stable framed representation of (Q, I) with $\dim \mathcal{W}|_t = \gamma$.

Proposition 3.22. *The functor* $\mathcal{M}^{\star}_{Q,I,\theta}(\gamma)$ *is represented by a quasi-projective scheme* $M^{\star}_{Q,I,\theta}(\gamma)$*, which is non-singular if* $I=\{0\}$.

Proof. Let $Q^{\diamond}$ be a quiver defined by adding one vertex $\diamond$ and one arrow $\diamond\to\star$ to the quiver Q. Note that the relation I naturally determines the relation in $Q^{\diamond}$. By the construction, giving a framed representation of (Q,I) is equivalent to giving a representation of $(Q^{\diamond},I)$ whose dimension vector at the vertex $\diamond$ is one.

For a rational number $0<\varepsilon\ll 1$, let $\theta^{\diamond}_{\varepsilon}$ be an element of $\Gamma_{Q^{\diamond}}\otimes\mathbb{Q}$ given by

$$\theta^{\diamond}_{\varepsilon,v}=\theta_v+\varepsilon \text{ for } v\in Q_0,\quad \theta^{\diamond}_{\varepsilon,\diamond}=-\varepsilon\sum_{v\in Q_0}\gamma_v.$$

Then $\theta^{\diamond}_{\varepsilon}\cdot\gamma^{\diamond}=0$, where $\gamma^{\diamond}=(1,\gamma)\in\Gamma_{Q^{\diamond}}$ and 1 is the dimension vector at the vertex $\diamond$. Let $(\mathcal{W},\tau)$ be a flat framed representation of (Q,I) over a $\mathbb{C}$-scheme T such that $\dim\mathcal{W}|_t=\gamma$ for any $t\in T$. Then we have the associated flat representation $\mathcal{W}^{\diamond}$ of $(Q^{\diamond},I)$ over T by putting $\mathcal{W}^{\diamond}_{\diamond}=\mathcal{O}_T$ and the morphism corresponding to $\diamond\to\star$ is τ. It is easy to show that $(\mathcal{W},\tau)$ is a family of framed θ-stable representations of (Q,I) if and only if $\mathcal{W}^{\diamond}$ is a family of $\theta^{\diamond}$-stable representations of $(Q^{\diamond},I)$. Therefore, $(\mathcal{W},\tau)\mapsto\mathcal{W}^{\diamond}$ gives the natural transform

$$\mathcal{M}^{\star}_{Q,I,\theta}(\gamma)\to\mathcal{M}_{Q^{\diamond},I,\theta^{\diamond}_{\varepsilon}}(\gamma^{\diamond}). \tag{3.6.1}$$

For a T-valued point $\mathcal{W}^{\diamond}$ of the right-hand side of (3.6.1), it is equivalent to a unique element $\mathcal{W}'^{\diamond}$ such that $\mathcal{W}'^{\diamond}_{\diamond}=\mathcal{O}_T$, up to isomorphisms. Therefore, the natural transform (3.6.1) is indeed an isomorphism of functors, and the result follows from Theorem 3.7. $\square$

Let $(\mathcal{V},\iota)$ be the universal framed representation of (Q,I) over $M^{\star}_{Q,I,\theta}(\gamma)$. We define the functor

$$h^{\star}_{Q,I,\theta}(\gamma):\mathcal{N}\to\mathcal{S}et \tag{3.6.2}$$

by sending $\Lambda\in\mathcal{N}$ to the isomorphism classes of triples $(f,(\mathcal{W},\tau),\psi)$:

- f is a morphism of schemes $\operatorname{Spec}\Lambda^{\mathrm{ab}}\to M^{\star}_{Q,I,\theta}(\gamma)$;
- $(\mathcal{W},\tau)$ is a flat framed representation of (Q,I) over Λ;
- ψ is an isomorphism $\psi:(\mathcal{W}^{\mathrm{ab}},\tau^{\mathrm{ab}})\xrightarrow{\cong}f^{*}(\mathcal{V},\iota)$ as framed representations of (Q,I) over Λ^{ab}.

We use the following lemma.

Lemma 3.23. *For* $(f,(\mathcal{W},\tau),\psi)\in h^{\star}_{Q,I,\theta}(\gamma)(\Lambda)$*, we have* $\operatorname{Aut}(\mathcal{W},\tau)=\mathrm{id}$.

Proof. We prove the lemma by induction on the degree of the NC nilpotence of Λ. First suppose that Λ is commutative. We use the notation in the proof of Proposition 3.22. Let $\mathcal{W}^{\diamond}$ be the $\operatorname{Spec}\Lambda$-valued point of $\mathcal{M}_{Q^{\diamond},I,\theta^{\diamond}_{\varepsilon}}(\gamma^{\diamond})$ corresponding to $(\mathcal{W},\tau)$ under the natural transform (3.6.1). Then, by Lemma 3.4(i), any automorphism of $\mathcal{W}^{\diamond}$ is given by a multiplication of an element in Λ^{*}. Therefore, it commutes with τ if and only if it is an identity, proving the lemma when Λ is commutative.

Next suppose that $\Lambda\in\mathcal{N}_d$ and the lemma holds for $\Lambda^{\leqslant d-1}$. Let g be an automorphism of $(\mathcal{W},\tau)$. Then g induces an automorphism of $\mathcal{W}^{\diamond}$ which induces an identity on $(\mathcal{W}^{\diamond})^{\leqslant d-1}$. Hence, by Lemma 3.4(ii), g is a left multiplication by some central element in Λ. Since g commutes with τ, it follows that g must be an identity. $\square$

Using the above lemma, we show the following result on the existence of global NC structures on the moduli spaces of stable framed representations.

Theorem 3.24. *The framed moduli space $M^{\star}_{Q,I,\theta}(\gamma)$ has an NC structure which represents the functor $h^{\star}_{Q,I,\theta}(\gamma)$.*

Proof. In order to simplify the notation, we write $M = M^{\star}_{Q,I,\theta}(\gamma)$ and $N = M^{\star}_{Q,\{0\},\theta}(\gamma)$. Note that N is non-singular. Let $U \subset N$ be a sufficiently small affine open subset, and write $h^{\star} = h^{\star}_{Q,\{0\},\theta}(\gamma)|_U$. Let U^{nc} be an NC smooth thickening of U. Similarly to Proposition 3.8, we can construct a natural transform $h_{U^{\mathrm{nc}}} \to h^{\star}$, which is an isomorphism on $\mathcal{C}om$. We show that $h_{U^{\mathrm{nc}}} \to h^{\star}$ is indeed an isomorphism on $\mathcal{N}$. By Proposition 2.15, it is enough to show the following: for surjections $p_j : \Lambda_j \twoheadrightarrow \Lambda$ in $\mathcal{N}$ with $j = 1, 2$, $\Lambda_{12} := \Lambda_1 \times_{\Lambda} \Lambda_2$, the natural map

$$h^{\star}(\Lambda_{12}) \to h^{\star}(\Lambda_1) \times_{h^{\star}(\Lambda)} h^{\star}(\Lambda_2) \tag{3.6.3}$$

is a bijection. Let us take an element of the right-hand side of (3.6.3), i.e. elements of $h^{\star}(\Lambda_1)$ and $h^{\star}(\Lambda_2)$ which are isomorphic over Λ. Following the proof of Proposition 3.8, one can lift it to an element of the left-hand side of (3.6.3). By Lemma 3.23, the framed isomorphism over Λ is uniquely determined and hence the similar argument of the proof of Proposition 3.16 shows that the above lift is uniquely determined. Thus, we obtain a map from the right-hand side to the left-hand side of (3.6.3), which obviously gives the inverse of (3.6.3). Therefore, $h_{U^{\mathrm{nc}}} \to h^{*}$ is an isomorphism.

By Remark 2.13, the affine NC structures U^{nc} on each affine open subset $U \subset N$ glue together to give the NC structure $(N, \mathcal{O}_N^{\mathrm{nc}})$ on N. Again by Lemma 3.23, the local framed universal representations on U^{nc} also glue to give the global universal framed representation $(\mathcal{V}^{\mathrm{nc}}, \tau^{\mathrm{nc}})$ on N^{nc}, which induces the functorial isomorphism $h_{N^{\mathrm{nc}}} \to h_{Q,\{0\},\theta}(\gamma)$ by the above argument.

Now we consider the subscheme $M \subset N$. Let $\mathcal{J}_I \subset \mathcal{O}_{N^{\mathrm{nc}}}$ be the ideal of relation in I, and set $\mathcal{O}_M^{\mathrm{nc}} = \mathcal{O}_N^{\mathrm{nc}}/\overline{\mathcal{J}}_I$. Then, as in the proof of Proposition 3.11, the NC scheme $(M, \mathcal{O}_M^{\mathrm{nc}})$ is the desired NC structure by the Cartesian square

$$\begin{tikzcd}
h_{N^{\mathrm{nc}}} \arrow[r] & h_{Q,\{0\},\theta}(\gamma) \\
h_{M^{\mathrm{nc}}} \arrow[r] \arrow[u, hook] & h_{Q,I,\theta}(\gamma). \arrow[u, hook]
\end{tikzcd}$$ □

Remark 3.25. By the proof of Theorem 3.24, the framed smooth moduli space $M^{\star}_{Q,\{0\},\theta}(\gamma)$ has a canonical NC smooth thickening. Here the NC smoothness follows from $\mathrm{Ext}^2(F, F) = 0$ for any $Q^{\diamond}$-representation F, where $Q^{\diamond}$ is the quiver in the proof of Proposition 3.22, as we set the relation I to be zero. These framed moduli spaces give many examples of varieties which admit NC smooth thickening.

3.7 An example

We describe an example of an NC thickening of the moduli space of representations of a quiver. We consider the quiver Q described as

$$\begin{tikzcd}
\diamond \arrow[r] & \star \arrow[loop above, "y"] \arrow[loop below, "x"]
\end{tikzcd}$$

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with relation I given by

$$xy = yx. \tag{3.7.1}$$

We also set the vectors γ and θ in Γ_Q to be

$$\gamma = (\gamma_\diamond, \gamma_\star) = (1, 2), \quad \theta = (\theta_\diamond, \theta_\star) = (-2, 1).$$

A representation W of (Q, I) with dimension vector γ is given by the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] & \mathbb{C}^2 \arrow[loop above, "B"] \arrow[loop below, "A"] \end{tikzcd} \tag{3.7.2}$$

where f, A, B are linear maps satisfying $AB = BA$. It is easy to see that a (Q, I)-representation (3.7.2) is θ-stable if and only if $f(1)$ generates $\mathbb{C}^2$ as a $\mathbb{C}[x, y]$-module. Hence, we have the natural identification

$$M_{Q,I,\theta}(\gamma) = \mathrm{Hilb}^2(\mathbb{C}^2),$$

where the right-hand side is the Hilbert scheme of two points on $\mathbb{C}^2$. On the other hand, the smooth moduli space $M_{Q,\theta}(\gamma)$ is given by

$$M_{Q,\theta}(\gamma) = (\mathbb{C}^2 \times M_2(\mathbb{C}) \times M_2(\mathbb{C}))^s/\mathrm{GL}_2(\mathbb{C}).$$

Here $(-)^s$ means the θ-stable part, and the $\mathrm{GL}_2(\mathbb{C})$ action is given by

$$g(v, A, B) = (g^{-1}v, g^{-1}Ag, g^{-1}Bg).$$

By the θ-stability, we have $v \neq 0$ and hence we have

$$M_{Q,\theta}(\gamma) = (\{(1, 0)^t\} \times M_2(\mathbb{C}) \times M_2(\mathbb{C}))^s/G, \tag{3.7.3}$$

where G is the stabilizer of $(1, 0)^t$:

$$G = \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix}, \quad u, v \in \mathbb{C}.$$

We omit $\{(1, 0)^t\}$ in the notation of the right-hand side of (3.7.3). Then we have

$$(M_2(\mathbb{C}) \times M_2(\mathbb{C}))^s = \left\{ A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} : a_3 \neq 0 \text{ or } b_3 \neq 0 \right\}.$$

The open subsets $a_3 \neq 0$, $b_3 \neq 0$ are G-invariants and hence we obtain the open covering

$$M_{Q,\theta}(\gamma) = U_A \cup U_B,$$

where U_A, U_B are quotients of $a_3 \neq 0$, $b_3 \neq 0$, respectively. We also obtain the open cover

$$M_{Q,I,\theta}(\gamma) = V_A \cup V_B, \quad V_* = U_* \cap M_{Q,I,\theta}(\gamma).$$

For example, U_A is given by

$$U_A = \left\{ A = \begin{pmatrix} 0 & a_2 \\ 1 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} : a_i, b_i \in \mathbb{C} \right\},$$

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so $U_A \cong \mathbb{C}^6$. A smooth NC thickening of U_A is given by

$$U_A^{\mathrm{nc}} = \mathrm{Spf}\ \mathbb{C}\langle a_2, a_4, b_1, b_2, b_3, b_4\rangle_{[[\mathrm{ab}]]}.$$

Then the ideal of relation (3.7.1) in $\mathcal{O}_{U_A}^{\mathrm{nc}}$ is determined by the relation

$$\begin{pmatrix} 0 & a_2 \\ 1 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} 0 & a_2 \\ 1 & a_4 \end{pmatrix},$$

where we regard a_i and b_i as non-commutative variables in $\mathcal{O}_{U_A}^{\mathrm{nc}}$. By expanding the above matrix multiplications, we obtain

$$\begin{pmatrix} a_2 b_3 - b_2 & a_2 b_4 - b_1 a_2 - b_2 a_4 \\ b_1 + a_4 b_3 - b_4 & b_2 + a_4 b_4 - b_3 a_2 - b_4 a_4 \end{pmatrix} = 0. \tag{3.7.4}$$

By definition, the two-sided ideal $\mathcal{J}_A \subset \mathcal{O}_{U_A}^{\mathrm{nc}}$ of relations in I is generated by the matrix components of the left-hand side of (3.7.4)

$$\mathcal{J}_A = (a_2 b_3 - b_2, a_2 b_4 - b_1 a_2 - b_2 a_4, b_1 + a_4 b_3 - b_4, b_2 + a_4 b_4 - b_3 a_2 - b_4 a_4).$$

The quotient $\mathcal{O}_{V_A}^{\mathrm{nc}} = \mathcal{O}_{U_A}^{\mathrm{nc}}/\overline{\mathcal{J}}_A$ gives an NC thickening of V_A:

$$V_A^{\mathrm{nc}} = \mathrm{Spf}\left(\frac{\mathbb{C}\langle a_2, a_4, b_1, b_3\rangle}{([a_2, b_1] + a_2[a_4, b_3], [a_2, b_3] + [a_4, b_1] + a_4[a_4, b_3])}\right)_{[[\mathrm{ab}]]}.$$

Note that $V_A \cong \mathbb{C}^4$, with coordinates (a_2, a_4, b_1, b_3). In the same way, we obtain an NC thickening of $V_B \cong \mathbb{C}^4$:

$$V_B^{\mathrm{nc}} = \mathrm{Spf}\left(\frac{\mathbb{C}\langle b_2', b_4', a_1', a_3'\rangle}{([b_2', a_1'] + b_2'[b_4', a_3'], [b_2', a_3'] + [b_4', a_1'] + b_4'[b_4', a_3'])}\right)_{[[\mathrm{ab}]]}.$$

Note that

$$V_A \cap V_B = \{b_3 \neq 0\} = \{a_3' \neq 0\}.$$

The gluing isomorphism

$$V_A^{\mathrm{nc}}|_{V_A \cap V_B} \xrightarrow{\cong} V_B^{\mathrm{nc}}|_{V_A \cap V_B}$$

is calculated as

$$\begin{aligned}
a_3' &\mapsto b_3^{-1}, \\
a_1' &\mapsto -b_1 b_3^{-1}, \\
b_4' &\mapsto b_1 + b_3^{-1} b_1 b_3 + b_3^{-1} a_4 b_3^2, \\
b_2' &\mapsto a_2 b_3^2 - b_1 b_3^{-1} b_1 b_3 - b_1 b_3^{-1} a_4 b_3^2.
\end{aligned}$$

4. Quasi-NC structures on moduli spaces of stable sheaves

In this section, using the results in the previous section, we show the existence of quasi-NC structures on the moduli spaces of stable sheaves on projective schemes satisfying the desired property in Theorem 1.2.

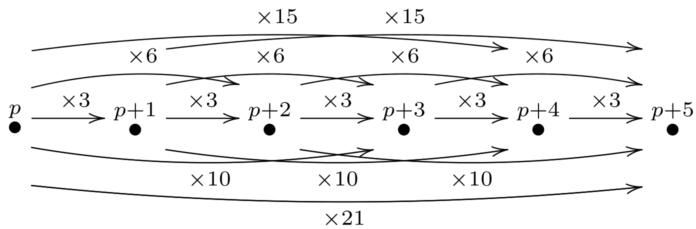

FIGURE 1. Quiver $Q_{[p,p+5]}$ for $X = \mathbb{P}^2$.

4.1 Graded algebras

Let $(X, \mathcal{O}_X(1))$ be a polarized projective scheme over $\mathbb{C}$. Then $X = \mathrm{Proj}(A)$ for the graded $\mathbb{C}$-algebra A given by

$$A = \bigoplus_{i \geqslant 0} H^0(X, \mathcal{O}_X(i)).$$

Below, we set $\mathfrak{m} := A_{>0}$, the maximal ideal of A.

For $\Lambda \in \mathcal{N}$, we set $A_\Lambda := A \otimes_{\mathbb{C}} \Lambda$. Let $A_\Lambda \ \mathrm{mod}_{\mathrm{gr}}$ be the category of finitely generated graded left A_Λ-modules. For $M \in A_\Lambda \ \mathrm{mod}_{\mathrm{gr}}$, we denote by M_i the degree-i part of M, and $M(j)$ the graded left A_Λ-module such that $M(j)_i = M_{j+i}$. For an interval $\mathbb{I} \subset \mathbb{Z}$, we define

$$A_\Lambda \ \mathrm{mod}_{\mathbb{I}} \subset A_\Lambda \ \mathrm{mod}_{\mathrm{gr}}$$

to be the subcategory of graded left A_Λ-modules M such that $M_i = 0$ for $i \notin I$. For $q > p > 0$, we describe the category $A_\Lambda \ \mathrm{mod}_{[p,q]}$ in terms of a quiver with relation.

We define the quiver $Q_{[p,q]}$ whose set of vertices is

$$\{p, p+1, \ldots, q\}.$$

The number of arrows of $Q_{[p,q]}$ from i to j is given by $\dim_{\mathbb{C}} \mathfrak{m}_{j-i}$ (cf. Figure 1). Below, we fix bases of $\mathfrak{m}_k$ for each $k \in \mathbb{Z}_{\geqslant 1}$, and identify the set of arrows from i to j with the set of basis elements of $\mathfrak{m}_{j-i}$. The multiplication

$$\vartheta : \mathfrak{m}_{j-i} \otimes_{\mathbb{C}} \mathfrak{m}_{k-j} \to \mathfrak{m}_{k-i} \tag{4.1.1}$$

in A defines the relation in $Q_{[p,q]}$, by defining the two-sided ideal $I \subset \mathbb{C}[Q_{[p,q]}]$ to be generated by

$$\vartheta(\alpha \otimes \beta) - \alpha \cdot \beta, \quad \alpha \in \mathfrak{m}_{i-j}, \quad \beta \in \mathfrak{m}_{k-j}.$$

Here we have regarded α and β as formal linear combinations of paths from i to j and from j to k, respectively, and $\alpha \cdot \beta$ is the multiplication in $\mathbb{C}[Q_{[p,q]}]$. From the construction of $(Q_{[p,q]}, I)$, sending $(\{W_i\}_{i=p}^q, \{\phi_a\})$ to $\bigoplus_{i=p}^q W_i$ gives the equivalence

$$\mathrm{Rep}((Q_{[p,q]}, I)/\Lambda) \xrightarrow{\sim} A_\Lambda \ \mathrm{mod}_{[p,q]}. \tag{4.1.2}$$

4.2 Moduli stacks of semistable sheaves

For $F \in \mathrm{Coh}(X)$, let $\alpha(F, t)$ be its Hilbert polynomial

$$\alpha(F, t) := \chi(F \otimes \mathcal{O}_X(t))$$

and $\overline{\alpha}(F, t) = \alpha(F, t)/c$ its reduced Hilbert polynomial, where c is the leading coefficient of $\alpha(F, t)$. We recall the notion of (semi)stable sheaves (cf. [HL97]).

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Definition 4.1. A coherent sheaf F on X is called (semi)stable if it is a pure sheaf and, for any subsheaf $0 \subsetneq F' \subsetneq F$, we have

$$\overline{\alpha}(F', k) < (\leqslant) \overline{\alpha}(F, k), \quad k \gg 0. \tag{4.2.1}$$

Let us take a polynomial $\alpha \in \mathbb{Q}[t]$, which is a Hilbert polynomial of some coherent sheaf on X. The moduli stack

$$\mathfrak{M}_\alpha : \mathcal{S}ch/\mathbb{C} \to \mathcal{G}roupoid \tag{4.2.2}$$

is defined by sending a $\mathbb{C}$-scheme T to the groupoid of T-flat sheaves $\mathcal{F} \in \mathrm{Coh}(X \times T)$ such that for any $t \in T$, the sheaf $\mathcal{F}_t := \mathcal{F}|_{X \times \{t\}}$ is θ-semistable with Hilbert polynomial α. The stack (4.2.2) is known to be an algebraic stack of finite type over $\mathbb{C}$.

Following [BFHR14], we relate the moduli stack $\mathfrak{M}_\alpha$ with the moduli stack of semistable representations of $(Q_{[p,q]}, I)$ for $q \gg p \gg 0$. Let T be a $\mathbb{C}$-scheme of finite type and $\mathcal{F} \in \mathrm{Coh}(X \times T)$ an object of $\mathfrak{M}_\alpha(T)$. Below, we write $p_X : X \times T \to X$, $p_T : X \times T \to T$ for the projections, and we set $\mathcal{F}(i) := \mathcal{F} \otimes p_X^* \mathcal{O}_X(i)$ for $i \in \mathbb{Z}$. For $q \gg p \gg 0$ and $i \in [p, q]$, we have $\mathbf{R} p_{T*} \mathcal{F}(i) = p_{T*} \mathcal{F}(i)$, and the sheaf

$$\Gamma_{[p,q]}(\mathcal{F}) := \bigoplus_{i=p}^{q} p_{T*} \mathcal{F}(i) \tag{4.2.3}$$

is a locally free sheaf on T. In fact, the sheaf (4.2.3) is a sheaf of graded $A \otimes_{\mathbb{C}} \mathcal{O}_T$ algebras whose graded pieces are concentrated on $[p, q]$. Hence, by the equivalence (4.1.2), $\Gamma_{[p,q]}(\mathcal{F})$ is a flat representation of $(Q_{[p,q]}, I)$ over T with dimension vector

$$\alpha_{[p,q]} := (\alpha(p), \alpha(p+1), \ldots, \alpha(q)).$$

Now we consider the stability condition on $(Q_{[p,q]}, I)$. We set $\theta \in \Gamma_{Q_{[p,q]}}$ to be

$$\theta_p = -\alpha(q), \quad \theta_q = \alpha(p), \quad \theta_i = 0 \quad \text{for } i \neq p, q.$$

By [BFHR14, Theorem 3.7], the representation (4.2.3) of $(Q_{[p,q]}, I)$ is a flat family of θ-semistable representations over T.

Theorem 4.2 [BFHR14, Corollary 3.4]. *There exist $q \gg p \gg 0$ such that the functor $\mathcal{F} \mapsto \Gamma_{[p,q]}(\mathcal{F})$ defines a morphism of algebraic stacks*

$$\mathfrak{M}_\alpha \to \mathfrak{M}_{Q_{[p,q]}, I, \theta}(\alpha_{[p,q]}), \tag{4.2.4}$$

which is an open immersion.

Let $\mathfrak{M}_{[p,q]}$ be the image of the morphism (4.2.4). By Theorem 4.2, we have the isomorphism of stacks

$$\Gamma_{[p,q]} : \mathfrak{M}_\alpha \xrightarrow{\cong} \mathfrak{M}_{[p,q]}.$$

4.3 Non-commutative thickening of moduli stacks

For $\Lambda \in \mathcal{N}$, we denote by X_Λ the NC scheme defined by $X \times \mathrm{Spf}\, \Lambda$. We define the 2-functor

$$\mathfrak{M}_\alpha^{\mathrm{nc}} : \mathcal{N} \to \mathcal{G}roupoid \tag{4.3.1}$$

by sending $\Lambda \in \mathcal{N}$ to the groupoid of $\mathcal{F} \in \mathrm{Coh}(X_\Lambda)$ which is flat over Λ such that $\mathcal{F}^{\mathrm{ab}} \in \mathfrak{M}_\alpha(\operatorname{Spec}\Lambda^{\mathrm{ab}})$. Similarly, we define the 2-functor

$$\mathfrak{M}^{\mathrm{nc}}_{[p,q]} : \mathcal{N} \to \mathcal{G}roupoid \tag{4.3.2}$$

by sending $\Lambda \in \mathcal{N}$ to the groupoid of $\mathcal{W} \in \mathrm{Rep}((Q_{[p,q]}, I)/\Lambda)$ flat over Λ such that $\mathcal{W}^{\mathrm{ab}} \in \mathfrak{M}_{[p,q]}(\operatorname{Spec}\Lambda^{\mathrm{ab}})$. Note that on the subcategory $\mathcal{C}om \subset \mathcal{N}$, (4.3.1) and (4.3.2) coincide with $\mathfrak{M}_\alpha$ and $\mathfrak{M}_{[p,q]}$, respectively. By Theorem 4.2, the functor $\Gamma_{[p,q]}$ gives the isomorphism for $q \gg p \gg 0$:

$$\Gamma_{[p,q]} : \mathfrak{M}^{\mathrm{nc}}_\alpha|_{\mathcal{C}om} \xrightarrow{\cong} \mathfrak{M}^{\mathrm{nc}}_{[p,q]}|_{\mathcal{C}om}. \tag{4.3.3}$$

Below, we fix such $q \gg p \gg 0$.

Lemma 4.3. *For $\Lambda \in \mathcal{N}$ and $\mathcal{F} \in \mathfrak{M}^{\mathrm{nc}}_\alpha(\Lambda)$, we have $\mathbf{R}\Gamma(\mathcal{F}(i)) = \Gamma(\mathcal{F}(i))$ for $i \in [p,q]$, and the left Λ-module*

$$\Gamma_{[p,q]}(\mathcal{F}) = \bigoplus_{i \in [p,q]} \Gamma(\mathcal{F}(i)) \tag{4.3.4}$$

is flat over Λ. Moreover, the natural morphism

$$u : \Gamma_{[p,q]}(\mathcal{F})^{\mathrm{ab}} \to \Gamma_{[p,q]}(\mathcal{F}^{\mathrm{ab}}) \tag{4.3.5}$$

is an isomorphism of representations of $(Q_{[p,q]}, I)$ over Λ^{ab}.

Proof. Since $\mathcal{F}^{\mathrm{ab}} \in \mathfrak{M}_\alpha(\operatorname{Spec}\Lambda^{\mathrm{ab}})$, we have $\mathbf{R}\Gamma(\mathcal{F}^{\mathrm{ab}}(i)) = \Gamma(\mathcal{F}^{\mathrm{ab}}(i))$ for $i \in [p,q]$ by our choice of $q \gg p \gg 0$. Note that $\mathcal{F}$ admits a filtration whose subquotient is given by $\mathrm{gr}^\bullet_F(\Lambda) \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{F}^{\mathrm{ab}}$. By the projection formula, for $i \in [p,q]$, we have

$$\mathbf{R}\Gamma(\mathrm{gr}^\bullet_F(\Lambda) \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{F}^{\mathrm{ab}}(i)) \cong \mathrm{gr}^\bullet_F(\Lambda) \otimes_{\Lambda^{\mathrm{ab}}} \Gamma(\mathcal{F}^{\mathrm{ab}}(i)).$$

Therefore, we have $\mathbf{R}\Gamma(\mathcal{F}(i)) = \Gamma(\mathcal{F}(i))$ for $i \in [p,q]$. Also, by the derived base change, we have

$$\Lambda^{\mathrm{ab}} \overset{\mathbf{L}}{\otimes}_\Lambda \mathbf{R}\Gamma(\mathcal{F}(i)) \cong \mathbf{R}\Gamma(\mathcal{F}^{\mathrm{ab}}(i)).$$

By combining the above isomorphism with $\mathbf{R}\Gamma(\mathcal{F}^{\mathrm{ab}}(i)) = \Gamma(\mathcal{F}^{\mathrm{ab}}(i))$ and $\mathbf{R}\Gamma(\mathcal{F}(i)) = \Gamma(\mathcal{F}(i))$, we have the isomorphism

$$\Lambda^{\mathrm{ab}} \overset{\mathbf{L}}{\otimes}_\Lambda \Gamma_{[p,q]}(\mathcal{F}) \cong \Gamma_{[p,q]}(\mathcal{F}^{\mathrm{ab}}).$$

Since $\Gamma_{[p,q]}(\mathcal{F}^{\mathrm{ab}})$ is flat over Λ^{ab}, the above isomorphism implies that the left Λ-module (4.3.4) is flat and the morphism (4.3.5) is an isomorphism. $\square$

By the above lemma, the isomorphism (4.3.3) extends to the 1-morphism

$$\Gamma_{[p,q]} : \mathfrak{M}^{\mathrm{nc}}_\alpha \to \mathfrak{M}^{\mathrm{nc}}_{[p,q]} \tag{4.3.6}$$

by sending $\mathcal{F}$ to $\Gamma_{[p,q]}(\mathcal{F})$.

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4.4 The left adjoint functor

The purpose here is to construct the left adjoint of the functor (4.3.4). We first interpret $\Gamma_{[p,q]}$ in a derived categorical way. Let

$$A_\Lambda \operatorname{mod}_{\mathrm{tor}} \subset A_\Lambda \operatorname{mod}_{\mathrm{gr}}$$

be the subcategory of finitely generated graded left A_Λ-modules M with $M_i = 0$ for $i \gg 0$. By Serre's theorem, we have the equivalence

$$\operatorname{Coh}(X_\Lambda) \xrightarrow{\sim} A_\Lambda \operatorname{mod}_{\mathrm{gr}}/A_\Lambda \operatorname{mod}_{\mathrm{tor}} \tag{4.4.1}$$

given by $\mathcal{F} \mapsto \bigoplus_{i\geqslant 0} \Gamma(\mathcal{F}(i))$. Below, we identify both sides of (4.4.1) via the above equivalence.

On the other hand, we have the functor

$$\mathbf{R}\Gamma_{\geqslant p} : D^b(\operatorname{Coh}(X_\Lambda)) \to D^b(A_\Lambda \operatorname{mod}_{\geqslant p}) \tag{4.4.2}$$

defined by

$$\mathbf{R}\Gamma_{\geqslant p}(E) := \bigoplus_{i\geqslant p} \mathbf{R}\Gamma(E(i)). \tag{4.4.3}$$

The functor (4.4.2) has a left adjoint, given by the quotient functor

$$\pi : D^b(A_\Lambda \operatorname{mod}_{\geqslant p}) \to D^b(A_\Lambda \operatorname{mod}_{\mathrm{gr}}/A_\Lambda \operatorname{mod}_{\mathrm{tor}}). \tag{4.4.4}$$

Since $\pi \circ \mathbf{R}\Gamma_{\geqslant p} = \mathrm{id}$, the functor (4.4.3) is fully faithful.

Let $\mathbb{C}(-i)$ be the one-dimensional graded A-module located in degree i. We define the subcategory

$$\mathcal{S}_{[p,q]} \subset D^b(A_\Lambda \operatorname{mod}_{\geqslant p}) \tag{4.4.5}$$

to be the smallest triangulated subcategory which contains objects of the form $\mathbb{C}(-i) \otimes_{\mathbb{C}} M$ for $i \in [p,q]$ and $M \in \Lambda \operatorname{mod}$. Similarly, let

$$\mathcal{P}_{[p,q]} \subset D^b(A_\Lambda \operatorname{mod}_{\geqslant p})$$

be the smallest triangulated subcategory which contains objects of the form $A(-i) \otimes_{\mathbb{C}} M$ for $M \in \Lambda \operatorname{mod}$ and $i \in [p,q]$.

Lemma 4.4. *We have the semiorthogonal decompositions*

$$D^b(A_\Lambda \operatorname{mod}_{\geqslant p}) = \langle \mathcal{S}_{[p,q]}, D^b(A_\Lambda \operatorname{mod}_{>q}) \rangle, \tag{4.4.6}$$
$$D^b(A_\Lambda \operatorname{mod}_{\geqslant p}) = \langle D^b(A_\Lambda \operatorname{mod}_{>q}), \mathcal{P}_{[p,q]} \rangle. \tag{4.4.7}$$

Proof. If $\Lambda = \mathbb{C}$, the result is proved in [Orl09, Lemma 2.3]. Indeed, the same argument works for any $\Lambda \in \mathcal{N}$ to prove (4.4.6) and (4.4.7). □

Note that the standard t-structure on $D^b(A_\Lambda \operatorname{mod}_{\geqslant p})$ restricts to the t-structure on $\mathcal{S}_{[p,q]}$ whose heart is $A_\Lambda \operatorname{mod}_{[p,q]}$. Let $\mathbf{R}\Gamma_{[p,q]}$ be the composition

$$\mathbf{R}\Gamma_{[p,q]} : D^b(\operatorname{Coh}(X_\Lambda)) \xrightarrow{\mathbf{R}\Gamma_{\geqslant p}} D^b(A_\Lambda \operatorname{mod}_{\geqslant p}) \overset{pr_S}{\twoheadrightarrow} \mathcal{S}_{[p,q]}. \tag{4.4.8}$$

Here pr_S is the projection with respect to the decomposition (4.4.6). By taking the zeroth cohomology, we obtain the functor

$$\Gamma_{[p,q]} := \mathcal{H}^0 \mathbf{R}\Gamma_{[p,q]} : \operatorname{Coh}(X_\Lambda) \to A_\Lambda \operatorname{mod}_{[p,q]}, \tag{4.4.9}$$

which coincides with (4.3.4).

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We describe the left adjoint of (4.4.8). Let $\mathbf{L}\wp$ be the composition

$$\begin{aligned}\mathbf{L}\wp : \mathcal{S}_{[p,q]} \stackrel{i}{\hookrightarrow} D^b(A_\Lambda \text{ mod}_{\geqslant p}) &\stackrel{pr_P}{\twoheadrightarrow} \mathcal{P}_{[p,q]} \\ &\stackrel{j}{\hookrightarrow} D^b(A_\Lambda \text{ mod}_{\geqslant p}) \stackrel{\pi}{\twoheadrightarrow} D^b(\mathrm{Coh}(X_\Lambda)).\end{aligned}$$

Here i, j are the natural embeddings, and pr_P is the projection with respect to the decomposition (4.4.7).

LEMMA 4.5. *$\mathbf{L}\wp$ is the left adjoint of $\mathbf{R}\Gamma_{[p,q]}$.*

Proof. Since the functor π in (4.4.4) is the left adjoint of $\mathbf{R}\Gamma_{\geqslant p}$ in (4.4.2), it is enough to show that the composition

$$\mathcal{S}_{[p,q]} \stackrel{i}{\hookrightarrow} D^b(A_\Lambda \text{ mod}_{\geqslant p}) \stackrel{pr_P}{\twoheadrightarrow} \mathcal{P}_{[p,q]} \stackrel{j}{\hookrightarrow} D^b(A_\Lambda \text{ mod}_{\geqslant p})$$

is the left adjoint of $pr_S : D^b(A_\Lambda \text{ mod}_{\geqslant p}) \twoheadrightarrow \mathcal{S}_{[p,q]}$. We take $E \in \mathcal{S}_{[p,q]}$ and $F \in D^b(A_\Lambda \text{ mod}_{\geqslant p})$. By the decomposition (4.4.7), we have the distinguished triangle

$$j \circ pr_P \circ i(E) \to i(E) \to E'$$

for some $E' \in D^b(A_\Lambda \text{ mod}_{>q})$. Applying $\mathrm{Hom}(-, i \circ pr_S(F))$ to the above triangle, the decomposition (4.4.6) shows that

$$\mathrm{Hom}(E, pr_S(F)) \xrightarrow{\cong} \mathrm{Hom}(j \circ pr_P \circ i(E), i \circ pr_S(F)). \tag{4.4.10}$$

By the decomposition (4.4.6), we also have the distinguished triangle

$$F' \to F \to i \circ pr_S(F)$$

for some $F' \in D^b(A_\Lambda \text{ mod}_{>q})$. Applying $\mathrm{Hom}(j \circ pr_P \circ i(E), -)$ to the above triangle, we have the isomorphism

$$\mathrm{Hom}(j \circ pr_P \circ i(E), F) \xrightarrow{\cong} \mathrm{Hom}(j \circ pr_P \circ i(E), i \circ pr_S(F)). \tag{4.4.11}$$

The isomorphisms (4.4.10), (4.4.11) show that $j \circ pr_p \circ i$ is the left adjoint of pr_S. □

LEMMA 4.6. *The functor $\mathbf{L}\wp$ is right t-exact, i.e.*

$$\mathbf{L}\wp(A_\Lambda \text{ mod}_{[p,q]}) \subset D^{\leqslant 0}(\mathrm{Coh}(X_\Lambda)).$$

Proof. By the construction, the functor $\mathbf{R}\Gamma_{[p,q]}$ is left t-exact, i.e. it takes $\mathrm{Coh}(X_\Lambda)$ to $\mathcal{S}^{\geqslant 0}_{[p,q]}$. Hence, $\mathbf{L}\wp$ is right t-exact by Lemma 4.5. □

We define the following functor:

$$\wp := \mathcal{H}^0\mathbf{L}\wp : A_\Lambda \text{ mod}_{[p,q]} \to \mathrm{Coh}(X_\Lambda). \tag{4.4.12}$$

By Lemmas 4.5 and 4.6, the above functor $\wp$ is right exact, and gives the left adjoint functor of $\Gamma_{[p,q]}$ in (4.4.9).

PROPOSITION 4.7. *The functor $\wp$ induces the 1-morphism*

$$\wp : \mathfrak{M}^{\mathrm{nc}}_{[p,q]}|_{\mathcal{C}om} \to \mathfrak{M}^{\mathrm{nc}}_{\alpha}|_{\mathcal{C}om} \tag{4.4.13}$$

giving the inverse of (4.3.3).

Proof. Suppose that $\Lambda \in \mathcal{N}$ is commutative. Since the functor $\wp$ in (4.4.12) is the left adjoint functor of $\Gamma_{[p,q]}$ in (4.4.9), it coincides with the left adjoint of $\Gamma_{[p,q]}$ constructed in [BFHR14, Proposition 3.1]. By [BFHR14, Proposition 3.2], the adjunction $\wp \circ \Gamma_{[p,q]} \to \mathrm{id}$ is an isomorphism on $\mathfrak{M}_\alpha$ and hence $\wp$ gives the inverse of (4.3.3). □

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4.5 The inverse transform

The purpose here is to show that $\wp$ in (4.4.13) extends to give the inverse of (4.3.6). We prepare some lemmas.

Lemma 4.8. *For $\Lambda \in \mathcal{N}$, let M be a Λ bi-module. Then the functors $\mathbf{R}\Gamma_{[p,q]}$ and $\mathbf{L}\wp$ commute with $M \overset{\mathbf{L}}{\otimes}_{\Lambda} -$.*

Proof. The commutativity of $\mathbf{R}\Gamma_{[p,q]}$ and $M \overset{\mathbf{L}}{\otimes}_{\Lambda} -$ follows from the derived base change $M \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathbf{R}\Gamma(\mathcal{F}) \cong \mathbf{R}\Gamma(M \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathcal{F})$ for $\mathcal{F} \in D^b(\mathrm{Coh}(X_{\Lambda}))$. The commutativity of $\mathbf{L}\wp$ and $M \overset{\mathbf{L}}{\otimes}_{\Lambda} -$ follows from the construction of $\mathbf{L}\wp$ and the fact that $M \overset{\mathbf{L}}{\otimes}_{\Lambda} -$ preserves the decomposition (4.4.7). $\square$

Lemma 4.9. *In the situation of Lemma 4.8, we have the following.*

(i) *For $\mathcal{F} \in \mathfrak{M}_{\alpha}^{\mathrm{nc}}(\Lambda)$, we have $\Gamma_{[p,q]}(M \otimes_{\Lambda} \mathcal{F}) \cong M \otimes_{\Lambda} \Gamma_{[p,q]}(\mathcal{F})$.*

(ii) *The functor $\wp$ in (4.4.12) commutes with $M \otimes_{\Lambda} -$.*

Proof. (i) follows from Lemmas 4.3 and 4.8, and (ii) follows from Lemmas 4.6 and 4.8. $\square$

Lemma 4.10. *By replacing $q \gg p \gg 0$ if necessary, for any $\Lambda \in \mathcal{N}$ and $\mathcal{F} \in \mathfrak{M}_{\alpha}^{\mathrm{nc}}(\Lambda)$, the adjunction morphism*

$$\wp \circ \Gamma_{[p,q]}(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism. Moreover, we have $\mathcal{H}^{-1}(\mathbf{L}\wp \circ \Gamma_{[p,q]}(\mathcal{F})) = 0$.

Proof. We take the cone of the adjunction morphism in $D^b(\mathrm{Coh}(X_{\Lambda}))$

$$\mathcal{G} \to \mathbf{L}\wp \circ \mathbf{R}\Gamma_{[p,q]}(\mathcal{F}) \to \mathcal{F}. \tag{4.5.1}$$

For a closed point $z = (x, y) \in X \times \operatorname{Spec} \Lambda^{\mathrm{ab}}$, we regard its structure sheaf $\mathcal{O}_z$ as an object of $\mathrm{Coh}(X_{\Lambda})$. Note that we have

$$\mathbf{R}\Gamma_{[p,q]}(\mathcal{F}) = \Gamma_{[p,q]}(\mathcal{F}), \quad \mathbf{R}\Gamma_{[p,q]}(\mathcal{O}_z) = \Gamma_{[p,q]}(\mathcal{O}_z)$$

by the proof of Lemma 4.3. Hence, applying $\mathbf{R}\mathrm{Hom}_{X_{\Lambda}}(-, \mathcal{O}_z)$ to the triangle (4.5.1), we obtain the distinguished triangle

$$\mathbf{R}\mathrm{Hom}_{X_{\Lambda}}(\mathcal{F}, \mathcal{O}_z) \to \mathbf{R}\mathrm{Hom}_{A_{\Lambda}\mathrm{gr}}(\Gamma_{[p,q]}(\mathcal{F}), \Gamma_{[p,q]}(\mathcal{O}_z)) \to \mathbf{R}\mathrm{Hom}_{X_{\Lambda}}(\mathcal{G}, \mathcal{O}_z). \tag{4.5.2}$$

Since $\mathcal{F}$ is flat over Λ, we have

$$\mathbf{R}\mathrm{Hom}_{X_{\Lambda}}(\mathcal{F}, \mathcal{O}_z) = \mathbf{R}\mathrm{Hom}_{X}(\mathcal{F}_y^{\mathrm{ab}}, \mathcal{O}_x).$$

Here we have set $\mathcal{F}_y^{\mathrm{ab}} := \mathcal{F}^{\mathrm{ab}}|_{X \times \{y\}}$. Also, using Lemma 4.8, we have

$$\mathbf{R}\mathrm{Hom}_{A_{\Lambda}\,\mathrm{gr}}(\Gamma_{[p,q]}(\mathcal{F}), \Gamma_{[p,q]}(\mathcal{O}_z)) = \mathbf{R}\mathrm{Hom}_{A\,\mathrm{gr}}(\Gamma_{[p,q]}(\mathcal{F}_y^{\mathrm{ab}}), \Gamma_{[p,q]}(\mathcal{O}_x)).$$

As $\mathcal{F}_y^{\mathrm{ab}}$ corresponds to a closed point of M_{α}, for $q \gg p \gg 0$ which are independent of $\mathcal{F}$ and (x, y), we have the isomorphisms

$$\mathrm{Ext}^i_X(\mathcal{F}_y^{\mathrm{ab}}, \mathcal{O}_x) \xrightarrow{\cong} \mathrm{Ext}^i_{A\,\mathrm{gr}}(\Gamma_{[p,q]}(\mathcal{F}_y^{\mathrm{ab}}), \Gamma_{[p,q]}(\mathcal{O}_x))$$

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for $i \leqslant 2$ by [FK01, Proposition 4.3.4]. Applying the above isomorphisms to the triangle (4.5.2), we obtain

$$\mathrm{Hom}_{X_\Lambda}(\mathcal{G}, \mathcal{O}_z[i]) = 0, \quad i \leqslant 1, z \in X \times \mathrm{Spec}\,\Lambda^{\mathrm{ab}},$$

which implies that $\mathcal{H}^i(\mathcal{G}) = 0$ for $i \geqslant -1$. By taking the long exact sequence of cohomologies associated to (4.5.1), we obtain the desired result. □

Now we show the following proposition.

PROPOSITION 4.11. *The functor $\wp$ induces the 1-morphism*

$$\wp : \mathfrak{M}^{\mathrm{nc}}_{[p,q]} \to \mathfrak{M}^{\mathrm{nc}}_{\alpha} \tag{4.5.3}$$

giving the inverse of (4.3.6). In particular, (4.3.6) is an isomorphism.

Proof. It is enough to show that, for $\Lambda \in \mathcal{N}$ and $\mathcal{W} \in \mathfrak{M}^{\mathrm{nc}}_{[p,q]}(\Lambda)$, the object $\wp(\mathcal{W}) \in \mathrm{Coh}(X_\Lambda)$ is flat over Λ and the adjunction morphism

$$\mathcal{W} \to \Gamma_{[p,q]} \circ \wp(\mathcal{W}) \tag{4.5.4}$$

is an isomorphism. Indeed, by Proposition 4.7 and Lemma 4.9(ii), we have

$$\wp(\mathcal{W})^{\mathrm{ab}} \cong \wp(\mathcal{W}^{\mathrm{ab}}) \in \mathfrak{M}_\alpha(\mathrm{Spec}\,\Lambda^{\mathrm{ab}}).$$

Therefore, if $\wp(\mathcal{W})$ is flat over Λ, then the object $\wp(\mathcal{W})$ determines an object of $\mathfrak{M}^{\mathrm{nc}}_\alpha(\Lambda)$, and the 1-morphism (4.5.3) is well defined. Moreover, if the morphism (4.5.4) is an isomorphism, then, combined with Lemma 4.10, the 1-morphism (4.5.3) gives the inverse of (4.3.6). Below, we prove the flatness of $\wp(\mathcal{W})$ and the isomorphism (4.5.4) by induction on the NC nilpotence of Λ. The first step of the induction is the case of $\Lambda \in \mathcal{C}om$, which follows from Proposition 4.7.

Suppose that $\Lambda \in \mathcal{N}_d$. By the assumption of the induction, we may assume that $\wp(\mathcal{W}^{\leqslant d-1})$ is flat over $\Lambda^{\leqslant d-1}$ and the morphism (4.5.4) is an isomorphism for $\mathcal{W}^{\leqslant d-1}$. By Lemma 4.8, we have the isomorphism

$$\Lambda^{\leqslant d-1} \overset{\mathbf{L}}{\otimes}_\Lambda \mathbf{L}\wp(\mathcal{W}) \cong \mathbf{L}\wp(\mathcal{W}^{\leqslant d-1}),$$

which yields the spectral sequence

$$E_2^{p,q} = \mathcal{T}or^\Lambda_{-p}(\Lambda^{\leqslant d-1}, \mathcal{H}^q(\mathbf{L}\wp(\mathcal{W}))) \Rightarrow \mathcal{H}^{p+q}(\mathbf{L}\wp(\mathcal{W}^{\leqslant d-1})). \tag{4.5.5}$$

On the other hand, we have the isomorphism

$$\mathbf{L}\wp(\mathcal{W}^{\leqslant d-1}) \cong \mathbf{L}\wp \circ \Gamma_{[p,q]} \circ \wp(\mathcal{W}^{\leqslant d-1})$$

by the assumption of the induction. Applying Lemma 4.10 to the above isomorphism, we obtain the vanishing

$$\mathcal{H}^{-1}(\mathbf{L}\wp(\mathcal{W}^{\leqslant d-1})) = 0. \tag{4.5.6}$$

The spectral sequence (4.5.5) together with the vanishing (4.5.6) shows that

$$\mathcal{T}or^\Lambda_1(\Lambda^{\leqslant d-1}, \wp(\mathcal{W})) = 0. \tag{4.5.7}$$

By Lemma 4.9(ii), we have $\wp(\mathcal{W})^{\leqslant d-1} \cong \wp(\mathcal{W}^{\leqslant d-1})$, which is flat over $\Lambda^{\leqslant d-1}$ by the induction assumption. Therefore, the vanishing (4.5.7) shows that $\wp(\mathcal{W})$ is flat over Λ.

It remains to show the isomorphism (4.5.4). Let J be the kernel of $\Lambda \twoheadrightarrow \Lambda^{\leqslant d-1}$, which is a Λ^{ab}-module. We have the exact sequence

$$0 \to J \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{W}^{\mathrm{ab}} \to \mathcal{W} \to \mathcal{W}^{\leqslant d-1} \to 0.$$

We apply $\wp$ to the above sequence. Since $\wp$ is right exact, using the vanishing (4.5.6) and Lemma 4.9(ii), we obtain the exact sequence

$$0 \to J \otimes_{\Lambda^{\mathrm{ab}}} \wp(\mathcal{W}^{\mathrm{ab}}) \to \wp(\mathcal{W}) \to \wp(\mathcal{W}^{\leqslant d-1}) \to 0.$$

Then we apply $\Gamma_{[p,q]}$ to the above sequence. By Lemmas 4.3 and 4.9(i), we also have the exact sequence

$$0 \to J \otimes_{\Lambda^{\mathrm{ab}}} \Gamma_{[p,q]} \circ \wp(\mathcal{W}^{\mathrm{ab}}) \to \Gamma_{[p,q]} \circ \wp(\mathcal{W}) \to \Gamma_{[p,q]} \circ \wp(\mathcal{W}^{\leqslant d-1}) \to 0.$$

We have the commutative diagram of exact sequences

$$\begin{tikzcd}
J \otimes_{\Lambda^{\mathrm{ab}}} \mathcal{W}^{\mathrm{ab}} \arrow[r] \arrow[d, "\cong"] & \mathcal{W} \arrow[r] \arrow[d] & \mathcal{W}^{\leqslant d-1} \arrow[d, "\cong"] \\
J \otimes_{\Lambda^{\mathrm{ab}}} \Gamma_{[p,q]} \circ \wp(\mathcal{W}^{\mathrm{ab}}) \arrow[r] & \Gamma_{[p,q]} \circ \wp(\mathcal{W}) \arrow[r] & \Gamma_{[p,q]} \circ \wp(\mathcal{W}^{\leqslant d-1}).
\end{tikzcd}$$

Here the right and left vertical arrows are isomorphisms by the assumption of the induction. By the five lemma, the morphism (4.5.4) is an isomorphism. □

4.6 Quasi-NC structures on the moduli space of stable sheaves

Let

$$\mathcal{M}_\alpha : \mathcal{S}ch/\mathbb{C} \to \mathcal{S}et \tag{4.6.1}$$

be the functor defined by sending a $\mathbb{C}$-scheme T to the equivalence classes of objects $\mathcal{F} \in \mathfrak{M}_\alpha(T)$, where $\mathcal{F}$ and $\mathcal{F}'$ are called *equivalent* if there is a line bundle $\mathcal{L}$ on T such that $\mathcal{F} \cong \mathcal{F}' \otimes p_T^*\mathcal{L}$. The moduli functor (4.6.1) is not always representable by a scheme, but, if we assume that

$$\text{g.c.d.}\{\alpha(m) : m \in \mathbb{Z}\} = 1, \tag{4.6.2}$$

then (4.6.1) is represented by a projective scheme M_α (cf. [Muk87]). Below, we call α satisfying the condition (4.6.2) *primitive*. In this case, the stack $\mathfrak{M}_\alpha$ consists of stable sheaves, and is a trivial $\mathbb{C}^*$-gerbe over M_α.

Suppose that α is primitive, and take $q \gg p \gg 0$ as in Theorem 4.2. Let

$$\mathcal{E} \in \mathrm{Coh}(X \times M_\alpha)$$

be a universal sheaf. Applying $\Gamma_{[p,q]}$ to $\mathcal{E}$, we obtain a family of θ-stable representations of $(Q_{[p,q]}, I)$ over M_α. Note that if α is primitive, then $\alpha_{[p,q]}$ is a primitive dimension vector for $q \gg p \gg 0$. Let $M_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$ be the moduli space of representations of $(Q_{[p,q]}, I)$ with dimension vector $\alpha_{[p,q]}$, given in Theorem 3.7. By Theorem 4.2, the functor $\Gamma_{[p,q]}$ induces the morphism

$$\Upsilon : M_\alpha \to M_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]}), \tag{4.6.3}$$

which is an open immersion. We denote by

$$M_{[p,q]} \subset M_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$$

the image of the morphism (4.6.3). The scheme $M_{[p,q]}$ is an open subscheme of $M_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$, such that we have the isomorphism

$$\Upsilon : M_\alpha \xrightarrow{\cong} M_{[p,q]}.$$

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Remark 4.12. Since $Q_{p,q}$ does not contain a loop, the moduli scheme $M_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$ is projective. Hence, $M_{[p,q]}$ consists of a union of connected components of $M_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$ (cf. [BFHR14, Corollary 3.8]).

We define the functor

$$h_\alpha : \mathcal{N} \to \mathcal{S}et$$

by sending $\Lambda \in \mathcal{N}$ to the set of isomorphism classes of triples $(f, \mathcal{F}, \psi)$:

- f is a morphism of schemes $f : \operatorname{Spec} \Lambda^{\mathrm{ab}} \to M_\alpha$;
- $\mathcal{F}$ is an object of $\operatorname{Coh}(X_\Lambda)$, which is flat over Λ;
- ψ is an isomorphism $\psi : \mathcal{F}^{\mathrm{ab}} \xrightarrow{\cong} f^*\mathcal{E}$.

An isomorphism $(f, \mathcal{F}, \psi) \to (f', \mathcal{F}', \psi')$ exists if $f = f'$, and there is an isomorphism $\mathcal{F} \to \mathcal{F}'$ in $\operatorname{Coh}(X_\Lambda)$ commuting ψ, ψ'. We also define the following functor:

$$h_{[p,q]} := h_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})|_{M_{[p,q]}} : \mathcal{N} \to \mathcal{S}et.$$

Here $h_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$ is introduced in (3.3.11).

Proposition 4.13. *The functor $\Gamma_{[p,q]}$ in (4.4.9) induces the isomorphism of functors*

$$\Gamma_{[p,q]} : h_\alpha \xrightarrow{\cong} h_{[p,q]}.$$

Proof. The result obviously follows from the isomorphism $\Upsilon : M_\alpha \xrightarrow{\cong} M_{[p,q]}$ and Proposition 4.11. □

We have the following corollary.

Corollary 4.14. *The moduli scheme M_α admits an affine open cover $\{U_i\}_{i\in\mathbb{I}}$ with affine U_{ij}, affine NC structures $\{U_i^{\mathrm{nc}} = (U_i, \mathcal{O}_{U_i}^{\mathrm{nc}})\}_{i\in\mathbb{I}}$ and NC hulls $h_{U_i^{\mathrm{nc}}} \to h_\alpha|_{U_i}$. In particular, there exist isomorphisms*

$$\phi_{ij} : U_j^{\mathrm{nc}}|_{U_{ij}} \xrightarrow{\cong} U_i^{\mathrm{nc}}|_{U_{ij}} \tag{4.6.4}$$

of NC schemes giving a quasi-NC structure on M_α.

Proof. The result follows from Proposition 3.11, Corollary 3.12 and Proposition 4.13. □

Let $\mathcal{E}_i^{\mathrm{nc}}$ be the object of $\operatorname{Coh}(X \times U_i^{\mathrm{nc}})$ corresponding to $\mathrm{id} \in h_{U_i^{\mathrm{nc}}}(U_i^{\mathrm{nc}})$ under the natural transformation $h_{U_i^{\mathrm{nc}}} \to h_\alpha|_{U_i}$. Similarly to Corollary 3.12, we also have the isomorphisms

$$g_{ij} : \phi_{ij}^* \mathcal{E}_i^{\mathrm{nc}}|_{U_{ij}} \xrightarrow{\cong} \mathcal{E}_j^{\mathrm{nc}}|_{U_{ij}}. \tag{4.6.5}$$

4.7 Comparison with the formal deformations of sheaves

Similarly to § 3.4, we relate the quasi-NC structure in Corollary 4.14 with formal non-commutative deformation algebras of sheaves. For $F \in \operatorname{Coh}(X)$, the formal non-commutative deformation functor

$$\mathrm{Def}_F^{\mathrm{nc}} : \mathcal{N}^{\mathrm{loc}} \to \mathcal{S}et \tag{4.7.1}$$

is defined by sending $(\Lambda, \mathbf{n})$ to the set of isomorphism classes $(\mathcal{F}, \psi)$, where $\mathcal{F} \in \mathrm{Coh}(X_\Lambda)$ is flat over Λ and $\psi : \Lambda/\mathbf{n} \otimes_\Lambda \mathcal{F} \xrightarrow{\cong} F$ is an isomorphism in $\mathrm{Coh}(X)$. It is well known that the formal commutative deformation space of F is given by the solution of the Mauer–Cartan equation of the differential graded algebra $\mathbf{R}\mathrm{Hom}(F, F)$, up to gauge equivalence. Let

$$(\mathrm{Ext}^*(F, F), \{m_n\}_{n\geqslant 2}) \tag{4.7.2}$$

be the minimal A_∞-algebra which is quasi-isomorphic to $\mathbf{R}\mathrm{Hom}(F, F)$. An argument similar to [Seg08] shows that the pro-representable hull of (4.7.1) is described in terms of the A_∞-structure of (4.7.2). Let

$$m_n : \mathrm{Ext}^1(F, F)^{\otimes n} \to \mathrm{Ext}^2(F, F)$$

be the nth A_∞-product, and

$$J_F \subset \widehat{T}^\bullet(\mathrm{Ext}^1(F, F)^\vee)$$

the topological closure of the two-sided ideal generated by the image of the map

$$\sum_{n\geqslant 2} m_n^\vee : \mathrm{Ext}^2(F, F)^\vee \to \widehat{T}^\bullet(\mathrm{Ext}^1(F, F)^\vee).$$

The pro-representable hull of (4.7.1) is given by the quotient algebra

$$R_F^{\mathrm{nc}} := \widehat{T}^\bullet(\mathrm{Ext}^1(F, F)^\vee)/J_F.$$

The following is the main result in this section.

Theorem 4.15. *There exists a quasi-NC structure $\{U_i^{\mathrm{nc}} = (U_i, \mathcal{O}_{U_i}^{\mathrm{nc}})\}_{i\in\mathbb{I}}$ on M_α such that for any $[F] \in U_i$, there is an isomorphism of algebras $\widehat{\mathcal{O}}_{U_i,[F]}^{\mathrm{nc}} \cong R_F^{\mathrm{nc}}$.*

Proof. We take the quasi-NC structure on M_α as in Corollary 4.14. Similarly to the proof of Lemma 3.13, the natural transform

$$h_{\alpha\,[F]}^{\mathrm{loc}} \to \mathrm{Def}_F^{\mathrm{nc}}$$

sending triples $(f, \mathcal{F}, \psi)$ to $(\mathcal{F}, \Lambda/\mathbf{n} \otimes_\Lambda \psi)$ is formally smooth and an isomorphism on $\mathbb{C}[t]/t^2$. Hence, $\widehat{\mathcal{O}}_{U_i,[F]}^{\mathrm{nc}} \cong R_F^{\mathrm{nc}}$ follows from Lemma 2.17 and the uniqueness of the pro-representable hull. $\square$

4.8 Partial NC thickening of moduli spaces of sheaves

Let $\{U_i^{\mathrm{nc}}\}_{i\in\mathbb{I}}$ be a quasi-NC structure on M_α given in Corollary 4.14, and $\mathcal{E}_i^{\mathrm{nc}}$ the object in $\mathrm{Coh}(X \times U_i^{\mathrm{nc}})$ given in (4.6.5). For $d \in \mathbb{Z}_{\geqslant 0}$, we set

$$U_i^d = (U_i^{\mathrm{nc}})^{\leqslant d}, \quad \mathcal{E}_i^d = (\mathcal{E}_i^{\mathrm{nc}})^{\leqslant d}.$$

The isomorphisms (4.6.4) and (4.6.5) induce isomorphisms

$$\phi_{ij}^{\leqslant d} : U_j^d|_{U_{ij}} \xrightarrow{\cong} U_i^d|_{U_{ij}}, \quad g_{ij}^{\leqslant d} : \phi_{ij}^{\leqslant d*}\mathcal{E}_i^d|_{U_{ij}} \xrightarrow{\cong} \mathcal{E}_j^d|_{U_{ij}}.$$

Similarly to § 3.5, we assume the following.

Assumption 4.16. (1) By replacing $\phi_{ij}^{\leqslant d-1}$ if necessary, the quasi-NC structure $\{U_i^{d-1}\}_{i\in\mathbb{I}}$ determines an NC structure M_α^{d-1} on M_α.

(2) By replacing $g_{ij}^{\leqslant d-1}$ if necessary, the sheaves $\mathcal{E}_i^{d-1}$ glue to give an object

$$\mathcal{E}^{d-1} \in \mathrm{Coh}(X \times M_\alpha^{d-1}).$$

We have the following result similar to Corollary 3.17.

Proposition 4.17. *Under Assumption 4.16, affine NC structures $\{U_i^d\}_{i\in\mathbb{I}}$ glue to give an NC structure M_α^d on M_α.*

Proof. The result follows from Corollary 3.17 and Proposition 4.11. □

Hence, we always have a 1-thickening M_α^1 of M_α. By Proposition 4.17, if the obstruction extending $\mathcal{E}^{d-1}$ to a dth-order $\mathcal{E}^d$ vanishes, then M_α admits a $(d+1)$th global NC thickening. We set

$$\mathcal{I}^d := \mathrm{Ker}(\mathcal{O}_{M_\alpha}^d \twoheadrightarrow \mathcal{O}_{M_\alpha}^{d-1}),$$

which is a coherent sheaf on M_α. Similarly to Lemma 3.18, the obstruction extending $\mathcal{E}^{d-1}$ to $\mathcal{E}^d$ lies in $H^2(M_\alpha, \mathcal{I}^d)$. In particular, if $\dim M_\alpha \leqslant 1$, then $\{U_i^{\mathrm{nc}}\}_{i\in\mathbb{I}}$ glue to give a global NC structure on M_α.

4.9 NC structures on framed moduli spaces of sheaves

We fix $q \gg p \gg 0$ as in the previous subsections. A pair (F, s) for $F \in \mathrm{Coh}(X)$ and $s \in \Gamma(F(p))$ is called a *framed sheaf*.

Definition 4.18. A framed sheaf (F, s) is called framed stable if F is a semistable sheaf and, for any proper subsheaf $0 \subsetneq F' \subsetneq F$ which contains s, the inequality (4.2.1) is strict.

The functor

$$\mathcal{M}_{\alpha,p}^{\star} : \mathcal{S}ch/\mathbb{C} \to \mathcal{S}et$$

is defined by sending a $\mathbb{C}$-scheme T to the set of isomorphism classes of pairs $(\mathcal{F}, s)$, where $\mathcal{F} \in \mathfrak{M}_\alpha(T)$ and $s \in \Gamma(\mathcal{F}(p))$ are such that the pair $(\mathcal{F}_t, s_t)$ is framed stable for any $t \in T$. An isomorphism from $(\mathcal{F}, s)$ to $(\mathcal{F}', s')$ is an isomorphism of sheaves $g : \mathcal{F} \to \mathcal{F}'$ sending s to s'. It is well known that $\mathcal{M}_{\alpha,p}^{\star}$ is represented by a projective scheme $M_{\alpha,p}^{\star}$ (cf. [JS12, § 12]).

Let $\star$ be the vertex $\{p\}$ in the quiver $Q_{[p,q]}$. For a framed sheaf (F, s), the definition of $\Gamma_{[p,q]}$ naturally gives the framed representation $(\Gamma_{[p,q]}(F), s)$ of $(Q_{[p,q]}, I)$.

Lemma 4.19. *There exist $q \gg p \gg 0$ such that for any semistable sheaf F with Hilbert polynomial α, a framed sheaf (F, s) is framed stable if and only if $(\Gamma_{[p,q]}(F), s)$ is framed θ-stable.*

Proof. Let (F, s) be a framed sheaf such that F is semistable with Hilbert polynomial α. By Theorem 4.2, the representation $\Gamma_{[p,q]}(F)$ of $(Q_{[p,q]}, I)$ is θ-semistable. Moreover, any subsheaf $0 \neq F' \subsetneq F$ with the same reduced Hilbert polynomials gives rise to the subrepresentation $0 \neq W \subsetneq \Gamma_{[p,q]}(F)$ with $\theta \cdot \dim W = 0$, which contains s if F' does. Therefore, if $(\Gamma_{[p,q]}(F), s)$ is framed θ-stable, then (F, s) must be framed stable.

Conversely, suppose that $(\Gamma_{[p,q]}(F), s)$ is not framed θ-stable. Then there is a subrepresentation $0 \neq W \subsetneq \Gamma_{[p,q]}(F)$ of $(Q_{[p,q]}, I)$ which contains s such that $\theta \cdot \dim W = 0$. Let $Q^{\dagger}_{[p,q]}$ be the quiver with vertex $\{p,q\}$ and $\dim_{\mathbb{C}} \mathbf{m}_{q-p}$-arrows from p to q. By [AK07, Theorem 5.10(a)], the representation of $Q^{\dagger}_{[p,q]}$

$$\Gamma^{\dagger}_{[p,q]}(F) = (H^0(F(p)), H^0(F(q)))$$

is $\theta^{\dagger} = (\theta_p, \theta_q)$-semistable. Moreover, W gives rise to the $Q^{\dagger}_{[p,q]}$ subrepresentation $W^{\dagger} = (W_p, W_q)$ of $\Gamma^{\dagger}_{[p,q]}(F)$ with $\theta^{\dagger} \cdot \dim W^{\dagger} = 0$. By [AK07, Theorem 5.10(c)], $W^{\dagger}$ is given by $\Gamma^{\dagger}_{[p,q]}(F')$ for some $0 \neq F' \subsetneq F$ having the same reduced Hilbert polynomials. As $s \in W_p = H^0(F'(p))$, the pair (F, s) is not framed stable. $\square$

By Lemma 4.19, we have the morphism

$$M^{\star}_{\alpha,p} \to M^{\star}_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]}) \tag{4.9.1}$$

sending (F, s) to $(\Gamma_{[p,q]}, s)$. Let $M^{\star}_{[p,q]}$ be the open subscheme of $M^{\star}_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$ given by the Cartesian square

$$\begin{tikzcd} M^{\star}_{[p,q]} \arrow[r, hook] \arrow[d] & M^{\star}_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]}) \arrow[d] \\ \mathfrak{M}_{[p,q]} \arrow[r, hook] & \mathfrak{M}_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]}). \end{tikzcd}$$

Here the bottom morphism is the open immersion, and the right-hand morphism is forgetting the framing. The morphism (4.9.1) factors through the morphism

$$\Upsilon^{\star} : M^{\star}_{\alpha,p} \to M^{\star}_{[p,q]}. \tag{4.9.2}$$

Lemma 4.20. *The morphism $\Upsilon^{\star}$ is an isomorphism. In particular, the morphism (4.9.1) is an open immersion whose image $M^{\star}_{[p,q]}$ consists of a union of components of $M^{\star}_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$.*

Proof. We have the commutative diagram

$$\begin{tikzcd} M^{\star}_{\alpha,p} \arrow[r, "\Upsilon^{\star}"] \arrow[d] & M^{\star}_{[p,q]} \arrow[d] \\ \mathfrak{M}_{\alpha} \arrow[r, "\Gamma_{[p,q]}"] & \mathfrak{M}_{[p,q]}. \end{tikzcd}$$

Here the vertical morphisms are forgetting the framings. By Lemma 4.19, the above diagram is Cartesian. Since the bottom morphism is an isomorphism by Theorem 4.2, the morphism $\Upsilon^{\star}$ is also an isomorphism. $\square$

Let $(\mathcal{E}, \iota)$ be the universal family of framed stable sheaves on $M^{\star}_{\alpha,p}$, i.e.

$$\mathcal{E} \in \mathrm{Coh}(X \times M^{\star}_{\alpha,p}), \quad \iota \in \Gamma(\mathcal{E}(p)).$$

We define the functor

$$h^{\star}_{\alpha,p} : \mathcal{N} \to \mathcal{S}et$$

by sending $\Lambda \in \mathcal{N}$ to the set of isomorphism classes of triples $(f, (\mathcal{F}, s), \psi)$:

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- f is a morphism of schemes $f : \operatorname{Spec} \Lambda^{\mathrm{ab}} \to M^{\star}_{\alpha,p}$;
- $\mathcal{F} \in \mathfrak{M}^{\mathrm{nc}}_{\alpha}(\Lambda)$ and $s \in \Gamma(\mathcal{F}(p))$;
- ψ is an isomorphism $\psi : (\mathcal{F}^{\mathrm{ab}}, s^{\mathrm{ab}}) \xrightarrow{\cong} f^*(\mathcal{E}, \iota)$ of framed sheaves.

We also define

$$h^{\star}_{[p,q]} := h^{\star}_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})|_{M^{\star}_{[p,q]}} : \mathcal{N} \to \mathcal{S}et.$$

Here $h^{\star}_{Q_{[p,q]},I,\theta}(\alpha_{[p,q]})$ is introduced in (3.6.2). Since the functor $\Gamma_{[p,q]}$ takes $\mathfrak{M}^{\mathrm{nc}}_{\alpha}$ to $\mathfrak{M}^{\mathrm{nc}}_{[p,q]}$, we have the natural transform

$$\Gamma_{[p,q]} : h^{\star}_{\alpha,p} \to h^{\star}_{[p,q]}, \tag{4.9.3}$$

defined in an obvious way.

Proposition 4.21. *The natural transform (4.9.3) is an isomorphism of functors.*

Proof. Similarly to the proof of Lemma 4.20, we have the Cartesian diagram

$$\begin{tikzcd} h^{\star}_{\alpha,p} \arrow[r, "\Gamma_{[p,q]}"] \arrow[d] & h^{\star}_{[p,q]} \arrow[d] \\ \mathfrak{M}^{\mathrm{nc}}_{\alpha} \arrow[r, "\Gamma_{[p,q]}"] & \mathfrak{M}^{\mathrm{nc}}_{[p,q]}. \end{tikzcd}$$

Here the left-hand vertical arrow is sending $(f, (\mathcal{F}, s), \psi)$ to $\mathcal{F}$, and the right-hand vertical arrow is similar. Since the bottom arrow is an isomorphism by Proposition 4.11, the top arrow is also an isomorphism. $\square$

Finally, we obtain the following result.

Theorem 4.22. *The framed moduli scheme $M^{\star}_{\alpha,p}$ has a canonical NC structure which represents the functor $h^{\star}_{\alpha,p}$.*

Proof. The result is an immediate consequence of Theorem 3.24 and Proposition 4.21. $\square$

5. Examples

In this section, we discuss some examples of non-commutative thickening of moduli spaces of sheaves.

5.1 Non-commutative moduli spaces of points

Let X be a smooth projective variety over $\mathbb{C}$. If we take $\alpha \in \mathbb{Q}[t]$ to be the constant function $\alpha = 1$, then we have the isomorphism

$$X \xrightarrow{\cong} M_{\alpha}$$

sending $x \in X$ to the skyscraper sheaf $\mathcal{O}_x$. In this case, the dg-algebra $\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x)$ is quasi-isomorphic to the exterior algebra

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x) = \bigoplus_{i \geqslant 0} \bigwedge^{i} T_x X[i].$$

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Therefore, we have (cf. [Seg08])

$$R^{\mathrm{nc}}_{\mathcal{O}_x} = \widehat{T}^{\bullet}((T_xX)^{\vee})/\langle u \otimes v - v \otimes u \rangle$$

for $u, v \in T_xX$. Hence, $R^{\mathrm{nc}}_{\mathcal{O}_x}$ is isomorphic to $\widehat{\mathcal{O}}_{X,x}$, which is a commutative algebra. Let $\{U_i^{\mathrm{nc}}\}_{i \in \mathbb{I}}$ be a quasi-NC structure given in Theorem 4.15. Then $U_i^{\mathrm{nc}} = U_i$ in this case, and they are of course glued to give X, i.e. the global non-commutative moduli space of points in X is X itself.

5.2 Non-commutative moduli spaces of contractible curves

Let X be a quasi-projective 3-fold and

$$f : X \to Y$$

a flopping contraction which contracts a single smooth rational curve $C \subset X$ to a point $p \in Y$. Let $\alpha \in \mathbb{Q}[t]$ be the Hilbert polynomial of $\mathcal{O}_C$, and M_α the commutative moduli space of stable sheaves with Hilbert polynomial α. It is well known that M_α is topologically one point, consisting of $\mathcal{O}_C$ (cf. [Kat08]). Therefore, giving a quasi-NC structure on M_α is equivalent to giving an NC structure, which is equivalent to giving an NC complete algebra whose abelianization is $\mathcal{O}_{M_\alpha}$.

It is well known that the normal bundle $N_{C/X}$ is given by $\mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ such that

$$(a, b) \in \{(-1, -1), (0, -2), (1, -3)\}.$$

The non-commutative moduli space of the object $\mathcal{O}_C$ was studied by Donovan and Wemyss [DW16]. By [DW16], the algebra $R^{\mathrm{nc}}_{\mathcal{O}_C}$ is commutative if and only if C is not a $(1, -3)$-curve, and in this case $R^{\mathrm{nc}}_{\mathcal{O}_C}$ is isomorphic to $\mathbb{C}[t]/t^k$ for some $k \in \mathbb{Z}_{\geqslant 1}$. An example of a $(1, -3)$-curve is given by the exceptional locus of a crepant small resolution of the affine singularity $Y = \operatorname{Spec} R_k$, where R_k is defined by

$$R_k = \mathbb{C}[u, v, x, y]/(u^2 + v^2 y = x(x^2 + y^{2k+1})).$$

In this case, the algebra $R^{\mathrm{nc}}_{\mathcal{O}_C}$ is given by [DW16, Example 3.14]

$$R^{\mathrm{nc}}_{\mathcal{O}_C} = \mathbb{C}\langle x, y \rangle/(xy = -yx, x^2 = y^{2k+1}).$$

In this case, the global NC structure on M_α is given by $\operatorname{Spf} R^{\mathrm{nc}}_{\mathcal{O}_C}$.

5.3 Non-commutative moduli spaces of line bundles

Let X be a smooth projective variety and $\alpha \in \mathbb{Q}[t]$ the Hilbert polynomial of $\mathcal{O}_X$. Then we have

$$M_\alpha = \mathrm{Pic}^0(X),$$

where $\mathrm{Pic}^0(X)$ is the moduli space of line bundles on X with $c_1 = 0$. The moduli space $\mathrm{Pic}^0(X)$ is an abelian variety with dimension $\dim H^1(\mathcal{O}_X)$. Let $\{U_i^{\mathrm{nc}}\}_{i \in \mathbb{I}}$ be a quasi-NC structure in Corollary 4.14. In this case, the moduli space $\mathrm{Pic}^0(X)$ is also interpreted as the moduli space of pairs $(\mathcal{L}, s)$, where $\mathcal{L} \in \mathrm{Pic}^0(X)$ and s is an isomorphism

$$s : \mathbb{C} \xrightarrow{\cong} \mathcal{L}|_p.$$

Note that the different choices of s yield isomorphic pairs $(\mathcal{L}, s)$. The data s behaves like a choice of a framing in § 4.9. Although we omit a detail here, the proof similar to Proposition 4.17 shows that $\{U_i^{\mathrm{nc}}\}_{i \in \mathbb{I}}$ glue to give the NC structure on $\mathrm{Pic}^0(X)$.

In fact, this idea was used by Polischchuk and Tu [PT14] to give a global NC smooth thickening of $\mathrm{Pic}^0(X)$ when $H^2(\mathcal{O}_X) = 0$. In general, using the notion of algebraic NC connections, the global NC structure on $\mathrm{Pic}^0(X)$ was also constructed by Polishchuk and Tu [PT14, § 7.1], satisfying the property of Theorem 4.15.

5.4 Non-commutative moduli spaces of stable sheaves on K3 surfaces

Let X be a smooth projective K3 surface over $\mathbb{C}$ and suppose that $\alpha \in \mathbb{Q}[t]$ is primitive. By the result of Mukai [Muk87], any connected component of the moduli space M_α is a holomorphic symplectic manifold. For a stable sheaf $[F] \in M_\alpha$, the dg-algebra $\mathbf{R}\mathrm{Hom}(F, F)$ is known to be formal (cf. [Zha12, Proposition 1.3]), i.e. there is a quasi-isomorphism of dg-algebras

$$\mathbf{R}\mathrm{Hom}(F, F) \cong (\mathrm{Ext}^*(F, F), m_2).$$

In particular, the higher A_∞-products m_n of the minimal model (4.7.2) vanish for $n \geqslant 3$. Also, the multiplication

$$m_2 : \mathrm{Ext}^1(F, F) \times \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F) \cong \mathbb{C}$$

gives a holomorphic symplectic form on M_α. Therefore, by choosing a suitable basis of $\mathrm{Ext}^1(F, F)$, the algebra R_F^{nc} is given by

$$R_F^{\mathrm{nc}} = \frac{\mathbb{C}\langle\langle x_1, x_2, \dots, x_{2m-1}, x_{2m}\rangle\rangle}{\langle [x_1, x_2] + \cdots + [x_{2m-1}, x_{2m}]\rangle}. \tag{5.4.1}$$

Let $\{U_i^{\mathrm{nc}}\}_{i\in\mathbb{I}}$ be a quasi-NC structure on M_α given in Corollary 4.14. We do not know whether $\{U_i\}_{i\in\mathbb{I}}$ glue to give a global NC structure on M_α. However, by Proposition 4.17, we at least know that the 1-thickenings $\{U_i^1\}_{i\in\mathbb{I}}$ glue to give an NC structure M_α^1 on M_α. By (5.4.1), one of the gluings is given by the sheaf of algebras

$$\mathcal{O}_{M_\alpha^1} = \mathcal{O}_{M_\alpha} \oplus (\Omega^2_{M_\alpha}/\mathcal{O}_{M_\alpha}).$$

Here $\mathcal{O}_{M_\alpha^1} \subset \Omega^2_{M_\alpha}$ is given by the holomorphic symplectic form on M_α, and the algebra structure on $\mathcal{O}^1_{M_\alpha}$ is given by

$$(x, f) \cdot (y, g) = (xy, xg + fy + dx \wedge dy).$$

5.5 Non-commutative thickening of Hilbert schemes of points

Let X be a projective scheme over $\mathbb{C}$ and take $\alpha \in \mathbb{Q}[t]$ to be the constant function $\alpha = n$ for $n \in \mathbb{Z}_{\geqslant 1}$. Let $M^\star_{\alpha,p}$ be the moduli space of framed stable sheaves given in § 4.9. Then $M^\star_{\alpha,p}$ is independent of p, and we have the isomorphism

$$\mathrm{Hilb}^n(X) \xrightarrow{\cong} M^\star_{\alpha,p}$$

sending $Z \subset X$ to $(\mathcal{O}_Z, s)$, where $s \in H^0(\mathcal{O}_Z)$ is the canonical surjection $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$. Here $\mathrm{Hilb}^n(X)$ is the Hilbert scheme of n-points, parameterizing zero-dimensional subschemes $Z \subset X$ with length n. By Theorem 4.22, the Hilbert scheme of points $\mathrm{Hilb}^n(X)$ has a canonical NC structure. For example, one can check that the NC structure on $\mathrm{Hilb}^2(\mathbb{C}^2)$ induced by the open immersion $\mathrm{Hilb}^2(\mathbb{C}^2) \subset \mathrm{Hilb}^2(\mathbb{P}^2)$ coincides with the one given in § 3.7.

Remark 5.1. If X is non-singular with $\dim X \geqslant 2$ and $H^1(\mathcal{O}_X) = 0$, then $\mathrm{Hilb}^n(X)$ is also regarded as a moduli space of unframed sheaves by associating a zero-dimensional subscheme $Z \subset X$ with the ideal sheaf $I_Z \subset \mathcal{O}_X$. However, a quasi-NC structure on $\mathrm{Hilb}^n(X)$ given as the unframed moduli of sheaves is in general different from the above NC structure.

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