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TOPOLOGICAL ENTROPY AND PERIODIC POINTS OF A FACTOR OF A SUBSHIFT OF FINITE TYPE

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§0. Introduction

Let X be a compact space and f be a continuous map from X into itself. The topological entropy of $f, h(f)$, was defined by Adler, Konheim and McAndrew [1]. After that Bowen [4] defined the topological entropy for uniformly continuous maps of metric spaces, and proved that the two entropies coincide when the spaces are compact. The definition of Bowen is useful in calculating entropy of continuous maps.

By improving on the definition of entropy given in [1] and [4], we have the following results.

THEOREM 1. *Let $f: X \to X$ be a continuous map on a compact space X and let $g: Y \to Y$ be a continuous map on a compact Hausdorff space Y. Suppose there is a continuous map $\pi: X \to Y$ such that $\pi(X) = Y$ and $g \circ \pi = \pi \circ f$. Then*

$$h(f) \leqq h(g) + \sup_{y \in Y} h(f, \pi^{-1}(y)) .$$

This is a generalization of the result of Bowen (Theorem 1.7, [4]). As a corollary of Theorem 1 we have

COROLLARY 2. *Let E, X and G be compact Hausdorff spaces. Suppose that $\pi: E \to X$ is a projection of a fiber bundle with the total space E, the base space X and the structure group G. If $f: E \to E$ is a bundle map and $f': X \to X$ is its base map, then $h(f) = h(f')$ holds.*

Introducing a new method of calculating entropy, we have

THEOREM 3. *Let $\sigma: \Sigma \to \Sigma$ be a topologically mixing subshift of finite type and let $f: X \to X$ be a continuous map on a compact metric space X. Suppose there exists a continuous map $\pi: \Sigma \to X$ such that $\pi(\Sigma) = X$ and*

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$f \circ \pi = \pi \circ \sigma$. *Then*

$$h(f) \leqq \liminf_{n \to \infty} (1/n) \log N_n(f) ,$$

where $N_n(f)$ *is the cardinal number of the set* $\{x \in X;\, f^n(x) = x\}$ *and* $\log N_n(f) = \infty$ *if* $N_n(f)$ *is not finite.*

As a corollary of Theorem 3, we can give a partial answer for a problem stated in Walters (p. 180, [8]). More precisely

COROLLARY 4. *Let* X *be a compact metric space and* $f\colon X \to X$ *be an expansive homeomorphism. If* (X, f) *is a factor of a topologically mixing subshift of finite type, then*

$$h(f) = \lim_{n \to \infty} (1/n) \log N_n(f) .$$

Bowen proved in ((2.8), [2]) that if f is expansive then

$$h(f) \geqq \limsup_{n \to \infty} (1/n) \log N_n(f) .$$

From this result together with Theorem 4, Corollary 4 is obtained.

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§ 1. Definitions and basic properties

Hereafter X is a compact space and f is a continuous map of X into itself. By (X, f) we denote the dynamical system of X and f.

Let α be a finite open covering of X and ${}^{\#}\alpha$ denote the cardinality of α. For K a subset, we put

$$N_K(\alpha) = \min \{{}^{\#}\beta\colon \beta \subset \alpha,\ K \subset \bigcup_{B \in \beta} B\}$$

and for $n > 0$

$$\alpha_f^n = \left\{\bigcap_{i=0}^{n-1} f^{-i}(A_i)\colon A_i \in \alpha,\ 0 \leqq i \leqq n-1\right\} .$$

Further we define

$$h(f, K, \alpha) = \limsup_{n \to \infty} (1/n) \log N_K(\alpha_f^n) ,$$

$$h(f, K) = \sup_{\alpha} h(f, K, \alpha)$$

where the supremum is taken for all the finite open covering of X. If in particular $K = X$, then we write $h(f) = h(f, K)$. This is in the case given in [1].

Remark 1. If K is closed and $f(K) \subset K$, then the existence of

$$\lim_{n\to\infty} (1/n) \log N_K(\alpha_f^n)$$

is easily checked (cf. see [1]).

Remark 2. If (X, d) is the metric space and K is closed, then $h(f, K)$ coincides with entropy given in [4].

In the rest of this section we investigate the properties of the definition of entropy given here.

Let $\alpha = \{A_1, \cdots, A_t\}$ be a finite family of closed sets of X and denote by $M = (M_{ij})$ a $t \times t$-matrix of 0's and 1's. We say that a pair (α, M) is a *CM-pair for f* if for any $x \in X$ there is a sequence

$$x^* = (x_0, x_1, \cdots) \in \prod_{i=0}^{\infty} S(i) \qquad (S(i) = \{1, \cdots, t\},\ i \geqq 0)$$

such that $M_{x_i x_{i+1}} = 1$ $(i \geqq 0)$ and $x \in \bigcap_{i=0}^{\infty} f^{-i}(A_{x_i})$.

Remark 3. Under the notations, let us put

$$\Sigma = \left\{x = (x_0, x_1, \cdots) \in \prod_0^{\infty} S(i) \colon M_{x_i x_{i+1}} = 1\ (i \geqq 0)\right\}.$$

Then Σ is closed. We define a shift σ usually by

$$\sigma(x)_i = x_{i+1} \qquad (i \geqq 0).$$

Obviously $\sigma(\Sigma) = \Sigma$. Such a shift $\sigma \colon \Sigma \to \Sigma$ is called to be a one side subshift of finite type.

Fix $n \geqq 1$. A finite sequence $(x_0, x_1, \cdots, x_{n-1})$ is said to be an *admissible* sequence of length n if $M_{x_i x_{i+1}} = 1$ for $0 \leqq i \leqq n-1$. Let E_n denote a set of admissible sequences of length n. We say that E_n is *separated* if for any distinct points $(x_0, \cdots, x_{n-1}), (y_0, \cdots, y_{n-1}) \in E_n$ there is $0 \leqq j \leqq n-1$ such that $A_{x_j} \cap A_{y_j} = \phi$. For K a subset we say that an admissible sequence $(x_0, \cdots, x_{n-1})$ is *attached* to K if $K \cap \bigcap_{i=0}^{n-1} f^{-i}(A_{x_i}) \neq \phi$.

Denote by S_n $((\alpha, M), K)$ the largest cardinality of any separated set E of admissible sequences of length n attached to K, and put

$$S_f((\alpha, M), K) = \limsup_{n\to\infty} (1/n) \log S_n((\alpha, M), K).$$

PROPERTY 1. Under the notations and the assumptions, $S_f((\alpha, M), K) \leqq h(f, K)$.

Proof. For $x \in X$ we put

$$O(x) = \bigcap\{X\backslash A_i,\ x \notin A_i.\ A_i \in \alpha\}\ .$$

Then $\beta = \{O(x); x \in X\}$ is an open covering of X. For fixed $n > 0$, let E_n be a separated set of admissible sequences of length n attached to K with the maximal cardinality for (α, M). Let γ be a covering of K such that $\gamma \subset \beta_f^n$ and $K \subset \bigcup_{C\in\gamma} C$. Then for each $\tilde{x}^* = (x_0, \cdots, x_{n-1}) \in E_n$ there is a $C \in \gamma$ such that $\tilde{x}^*$ is attached to C. It is easy to see that $\tilde{x}^*$ is only one point of E_n attached to the C. Indeed, write $C = \bigcap_{i=0}^{n-1} f^{-i}(O_i)$ for some $O_i \in \beta$ $(0 \leqq i \leqq n-1)$. If $\tilde{y}^* = (y_0, \cdots, y_{n-1}) \in E_n$ is attached to C, then $A_{x_i} \cap O_i \neq \phi$ and $A_{y_i} \cap O_i \neq \phi$ for all $0 \leqq i \leqq n-1$. So we have $x^* = y^*$.

Therefore

$$S_n((\alpha, M), K) = {}^{\#}E \leqq N\ (\beta_f^n)$$

and so

$$S_f((\alpha, M), K) \leqq h(f, K, \beta) \leqq h(f, K)\ .$$

Let γ be an open covering of X. For $C \in \gamma$ we define

$$\mathrm{st}(C) = \bigcup\{C' \in \gamma;\ C \cap C' \neq \phi\}\ ,$$
$$\mathrm{st}(\gamma) = \{\mathrm{st}(C);\ C \in \gamma\}\ .$$

PROPERTY 2. Let α be a finite open covering of X and β be a covering of X. If (β, N) is a CM-pair for f such that $\mathrm{st}(\beta)$ refines α, then for $n \geqq 1$

$$N_K(\alpha_f^n) \leqq S_n((\beta, N), K) \quad \text{and} \quad h(f, K, \alpha) \leqq S_f((\beta, N), K)\ .$$

Proof. Let E_n be a set of admissible sequences of length n for (β, N). Suppose that E_n is separated and attached to K and further has the maximal cardinality. Let $x \in K$. Then there is an admissible sequence $x^* = (x_0, \cdots, x_{n-1})$ such that $x \in \bigcap_{i=0}^{n-1} f^{-i}(B_{x_i})$. Obviously $K \cap \bigcap_{i=0}^{n-1} f^{-i}(B_{x_i}) \neq \phi$. Since E_n is maximal, there is $y^* = (y_0, \cdots, y_{n-1}) \in E_n$ such that $B_{x_i} \cap B_{y_i} \neq \phi$ for $0 \leqq i \leqq n-1$, and so $x \in \bigcap_{i=0}^{n-1} \mathrm{st}(B_{y_i})$. This implies that $\{\bigcap_{i=0}^{n-1} \mathrm{st}(B_{y_i});\ y^* \in E_n\}$ is a covering of K. Since each $\bigcap_{i=0}^{n-1} \mathrm{st}\, B_{y_i}$ contained in at least one element of α_f^n, we have

$$N_K(\alpha_f^n) \leqq {}^{\#}E_n = S_n((\beta, N), K) \quad \text{and so} \quad h(f, \alpha, K) \leqq S_f((\beta, N), K)\ .$$

PROPERTY 3. For a finite open covering α there is a finite open covering γ such that $\mathrm{st}(\gamma)$ refines α.

Proof. Recall that compact Haudorff spaces have the uniform structure. We denote by $N(X)$ the family of all neighborhoods of the diagonal

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subset of $X \times X$. Then there exists $L \in N(X)$ such that for all $x \in X$ there is $A \in \alpha$ such that $\{y \in X;\ (x, y) \in L\} \subset A$.

Take an open set $U \in N(X)$ such that $U \circ U \circ U \subset L$ (here $A \circ B = \{(x, z);\ (x, y) \in A,\ (y, z) \in B\}$). Clearly $U[x] = \{y \in X;\ (x, y) \in U\}$ is an open neighborhood of X, and $\mathrm{st}(U[x]) \subset (U \circ U \circ U)[x] \subset L[x]$ holds. Hence a finite subcovering of $\{L[x];\ x \in X\}$ is the desired one.

PROPERTY 4. Let $\mathscr{F}$ be a family of CM-pairs for f. Suppose that for any finite open covering α of X, there is a CM-pair $(\beta, N) \in \mathscr{F}$ such that β refines α. Then for K a subset

$$h(f, K) = \sup_{(\beta, N) \in \mathscr{F}} S_f((\beta, N), K) .$$

Proof. By Property 1 it is enough to see that

$$h(f, K) \leqq \sup_{(\beta, N) \in \mathscr{F}} S_f((\beta, N), K) .$$

By Property 3 there is a finite open covering γ such that $\mathrm{st}(\gamma)$ refines α. So we choose $(\beta, N) \in \mathscr{F}$ such that β refines γ by assumption. Obviously $\mathrm{st}(\beta)$ refines α. By Property 2 we have

$$h(f, K, \alpha) \leqq S_f((\beta, N), K) \leqq \sup_{(\beta, N) \in \mathscr{F}} S_f((\beta, N), K) .$$

Since α is arbitrary, we obtain the conclusion.

§ 2. Proof of Theorem 1

Let α be a finite open covering of X and put

$$a = \sup_{y \in Y} \inf_{n > 0} (1/n) \log N_{\pi^{-1}(y)}(\alpha_f^n) .$$

Take and fix $\varepsilon > 0$. For any $y \in Y$ there is $m_y > 0$ such that

$$(1/m_y) \log N_{\pi^{-1}(y)}(\alpha_f^{m_y}) \leqq a + \varepsilon$$

and so

$$N_{\pi^{-1}(y)}(\alpha_f^{m_y}) \leqq e^{m_y(a+\varepsilon)} .$$

From now on we fix $y \in Y$ and choose $\alpha_y \subset \alpha_f^{m_y}$ such that

$$\pi^{-1}(y) \subset \bigcup_{A \in \alpha_y} A , \qquad {}^{\#}\alpha_y = N_{\pi^{-1}(y)}(\alpha_f^{m_y}) .$$

Put $O_y = \bigcup_{A \in \alpha_y} A$ and denote by $C(y)$ the family of the closed neighbourhoods of y. Then we have

$$(X \setminus O_y) \cap (\bigcap_{k \in C(y)} \pi^{-1}(K)) = \phi .$$

Since X is compact, there is $K_y \in C(y)$ such that $\pi^{-1}(K_y) \subset O_y$. If U_y is the interior of K_y, then $\{U_y; y \in Y\}$ is a covering of Y. Hence there is a finite subcovering $\beta = \{U_{y_1}, \cdots, U_{y_\ell}\}$ of Y. For simplicity we write $m_k = m_{y_k}$ and $U_k = U_{y_k}$ for $k = 1, \cdots, \ell$. Remark that

$$N_{\pi^{-1}(U_k)}(\alpha_f^{m_k}) = N_{\pi^{-1}(y_k)}(\alpha_f^{m_k}) \leqq e^{m_k(a+\varepsilon)} .$$

Take and fix $n > 0$. Then $B \in \beta_g^n$ is expressed as $B = \bigcap_{i=0}^{n-1} g^{-i}(B(i))$ for some $B(i) \in \beta$ $(0 \leqq i \leqq n-1)$. We fix this B and define recursively a finite sequence $\{i_s\}$ such that

$$i_0 = 0 , \quad i_{s+1} = i_s + m_k \qquad \text{when } B(i_s) = U_k .$$

Let q be the least integer such that $i_{q+1} \geqq n$ and put $n_s = m_k$ if $B(i_s) = U_k$ for $0 \leqq s \leqq q$. Since for any $x \in \pi^{-1}(B)$ and s with $0 \leqq s \leqq q$, $f^{i_s}(x) \in \pi^{-1}(B_{i_s})$, we have

$$N_{\pi^{-1}(B)}(\alpha_f^n) \leqq \prod_{s=0}^{q} N_{\pi^{-1}(B(i_s))}(\alpha_f^{n_s})$$

and hence

$$\begin{aligned} \log N_{\pi^{-1}(B)}(\alpha_f^n) &\leqq \sum_{s=0}^{q} \log N_{\pi^{-1}(B(i_s))}(\alpha_f^{n_s}) \\ &\leqq \sum_{s=0}^{q} \log e^{n_s(a+\varepsilon)} \\ &\leqq \sum_{s=0}^{q} n_s(a+\varepsilon) \\ &\leqq (n+H)(a+\varepsilon) \end{aligned}$$

where $H = \max\{n_1, \cdots, n_\ell\}$. Therefore

$$N_{\pi^{-1}(B)}(\alpha_f^n) \leqq e^{(n+H)(a+\varepsilon)} .$$

Since B is arbitrary in β_g^n, we have

$$N(\alpha_f^n) \leqq N(\beta_f^n) e^{(n+H)(a+\varepsilon)}$$

and so $h(f, \alpha) \leqq h(g, \varepsilon) + a + \varepsilon \leqq h(g) + a + \varepsilon$. Since α and ε are arbitrary, the conclusion is obtained.

Remark. We give an example such that the equality of Theorem 1 does not holds. Let X and f be as in Theorem 1 and Z be a compact space. Suppose that f has a fixed point $(f(x_0) = x_0)$, $h(f) > 0$ and further

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(Z, φ) is a dynamical system such that $h(\varphi) > 0$.

Now we put $Y = X \cup Z$ (disjoint union) and define a continuous map $g: Y \to Y$ by

$$g(y) = \begin{cases} f(y) & \text{when } y \in X \\ \varphi(y) & \text{when } y \in Z . \end{cases}$$

Set $\pi: Y \to X$ by

$$\pi(y) = \begin{cases} y & \text{when } y \in X \\ x_0 & \text{when } y \in Z . \end{cases}$$

Then $h(g) = \max\{h(f), h(\phi)\}$ (see R. L. Adler, A. G. Konheim and M. H. McAndrew [1]). And $\sup_{x \in X} h(g, \pi^{-1}(x)) = h(\varphi) > 0$. So that $h(g) < h(f) + h(\varphi)$.

Proof of Corollary 2. Since f' is a factor of f, we have $h(f') \leqq h(f)$. Hence it is enough to show that $h(f, \pi^{-1}(x)) = 0$ for all $x \in X$.

Let β be a finite open covering of X such that for all $B \in \beta$ there is a coordinate neighborhood U_B with $\mathrm{cl}(B) \subset U_B$ (here $\mathrm{cl}(B)$ denotes the closure of B in X). For any $B \in \beta$ let ξ_B denote the coordinate map

$$\xi_B: U_B \times F \longrightarrow \pi^{-1}(U_B)$$

where F is the fiber space. Then for fixed $x \in \mathrm{cl}(B)$ the map

$$\xi_{B,x}: F \longrightarrow E$$

is defined by $\xi_{B,x}(y) = \xi_B(x, y)$ for $y \in F$.

Now take and fix a finite open covering α of E. Firstly we show that there is a finite open covering γ_B of F such that γ_B refines $\xi_{B,x}^{-1}(\alpha)$ for all $x \in B$. For any $(x, y) \in \mathrm{cl}(B) \times F$ we can find an open neighborhood $U_y(x) \times U_x(y)$ of (x, y) that is contained in some element of $\xi_B^{-1}(\alpha)$. If $y \in F$ is fixed, then $\mathrm{cl}(B) \subset \bigcup_{i=1}^{t} U_y(x_i)$ for some finite set $\{x_1, \cdots, x_t\}$. So we put $W_y = \bigcap_{i=1}^{t} U_{x_i}(y)$. Since $\{W_y; y \in F\}$ is a covering of F, we have a finite subcovering $\gamma_B = \{W_B\}$ which is the desired one.

We put $\gamma = \{\bigcap_{B \in \beta} W_B;\, W_B \in \gamma_B\}$ and fix $x \in X$. Then for $j \geqq 0$ there is $B(j) \in \beta$ such that $f^j(x) \in B(j)$, and so put

$$g_j = (\xi_{B(j), f^j(x)} | \pi^{-1}(f^j x))^{-1} \circ f^j \circ \xi_{B(j), x} .$$

Then $g_j: F \to F$ coincides with action of some element of G on F. Since G is compact by assumption, the action of G is equicontinuous. Therefore there exists an open covering γ' of F such that $g_j^{-1}(\gamma)$ is refined by γ' for $j \geqq 0$.

Therefore we have $N_{\pi^{-1}(x)}(\alpha_f^j) \leqq N_F(\gamma')$ for all $j \geqq 0$, which implies $h(f, \pi^{-1}(x), \alpha) = 0$. Since α is arbitrary, the conclusion is obtained.

§ 3. Proof of Theorem 3

Since $\sigma: \Sigma \to \Sigma$ is the subshift of finite type, there exist a $p \times p$-matrix M of 0's and 1's and a finite set $S = \{1, 2, \cdots, p\}$ such that

$$\Sigma = \left\{x = (x_i) \in \prod_{j=-\infty}^{\infty} S(j);\ M_{x_i x_{i+1}} = 1 \ (i \in \boldsymbol{Z})\right\}.$$

Here each $S(j)$ denotes the copy of S, i.e. $S(j) = S$ for $j \in \boldsymbol{Z}$. Note that the shift σ is defined by $\sigma(x)_i = x_{i+1}$ for $i \in \boldsymbol{Z}$.

It is well known (cf. (1, 3), [6]) that σ is topologically mixing if and only if $M^n > 0$ (i.e. $M_{ij}^n > 0$ for all i, j) for sufficiently large n.

Now we give the proof of Theorem 3. Let $\ell \geqq 0$. For any

$$C = (a_{-\ell}, \cdots, a_\ell) \in \prod_{i=-\ell}^{\ell} S(i)$$

we write

$$[C]_{-\ell} = \{x \in \Sigma;\ x_i = a_i,\ |i| \leqq \ell\},$$

$$[M]_\ell = \left\{(x_{-\ell}, \cdots, x_\ell) \in \prod_{i=-\ell}^{\ell} S(i);\ M_{x_i x_{i+1}} = 1,\ -\ell \leqq i < \ell\right\}.$$

Since $[M]_\ell$ is finite, we put $[M]_\ell = \{C^1, \cdots, C^t\}$. If $B_j = \pi([C^j]_{-\ell})$ for $1 \leqq j \leqq t$, then $\beta = \{B_1, \cdots, B_t\}$ is the family of closed subsets of X and β covers X.

By using this β, we construct a CM-pair for f. Define a $t \times t$-matrix $B = (B_{ij})$ as follows.

In case $\ell = 0$, we put

$$B_{ij} = \begin{cases} 1 & \text{when } M_{a_0^i a_0^j} = 1 \text{ where } C^i = (a_0^i) \text{ and } C^j = (a_0^j), \\ 0 & \text{otherwise} \end{cases}$$

and in case $\ell > 0$

$$B_{ij} = \begin{cases} 1 & \text{when } a_{n+1}^i = a_n^j \ (-\ell \leqq n < \ell) \text{ where} \\ & \qquad C^i = (a_{-\ell}^i, \cdots, a_\ell^i) \text{ and } C^j = (a_{-\ell}^j, \cdots, a_\ell^j) \\ 0 & \text{otherwise}. \end{cases}$$

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It is easy to see that (β, B) is a CM-pair for f.

We denote by (Σ_B, σ_B) the subshift of finite type defined by the matrix B and a finite set $\{1, 2, \cdots, t\}$. Since $\sigma: \Sigma \to \Sigma$ is topologically mixing, there is $m > 0$ such that $M^m > 0$. Then we have $B^L > 0$ where $L = 2\ell + m$. Hence $\sigma_B: \Sigma_B \to \Sigma_B$ is topologically mixing.

Let $x \in \Sigma$. For any $i \in Z$ there is a unique $C^{j_i} \in [M]_\ell$ such that

$$C^{j_i} = (x_{i-\ell}, \cdots, x_{i+\ell}) .$$

So we put $\rho(x)_i = j_i$ $(i \in Z)$ and $\rho(x) = (\rho(x)_i)$. It is easily checked then that $\rho: \Sigma \to \Sigma_B$ is a homeomorphism and the diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \rho\downarrow & & \downarrow\rho \\ \Sigma_B & \xrightarrow[\sigma_B]{} & \Sigma_B \end{array}$$

commutes. Define $\pi' = \pi \circ \rho^{-1}$. Then $\pi': \Sigma_B \to X$ is surjective and for $1 \leqq j \leqq t$

$$\begin{aligned} \pi'([j]_0) &= \pi(\{x \in \Sigma;\ \rho(x)_0 = j\}) \\ &= \pi(\{x \in \Sigma;\ C^j = (x_{-\ell}, \cdots, x_\ell)\}) \\ &= \pi([C^j]_{-\ell}) \\ &= B_j . \end{aligned}$$

For fixed $n \geqq 1$, let E_n be a separated set of admissible sequences attached to X for (β, B). We may assume that E_n is chosen such that ${}^\#E_n$ is maximal. Since $\sigma_B: \Sigma_B \to \Sigma_B$ is topologically mixing, for $x^* = (x_0, \cdots, x_{n-1}) \in E_n$ we can find an admissible sequence $\tilde{x}^* = (x_0, \cdots, x_{L+n-1})$ such that $x_0 = x_{L+n-1}$. Put $\tilde{E}_n = \{\tilde{x}^*;\ x^* \in E_n\}$. Since E_n is separated, so is $\tilde{E}_n$. For any $\tilde{y}^* \in \tilde{E}_n$ define a periodic point $y = (y)_{i \in Z}$ by

$$y_i = y_r \qquad (i = (L + n - 1)p + r,\ 0 \leqq r < L + n - 1,\ p \in Z) .$$

Obviously y has the period $L + n - 1$ and $y \in \Sigma_B$. For $\tilde{y}^*, \tilde{z}^* \in \tilde{E}_n$, as above there exist periodic points $y, z \in \Sigma_B$ corresponding to $\tilde{y}^*, \tilde{z}^*$ repsectively. If $\tilde{y}^* \neq \tilde{z}^*$, then we have $\pi'(y) \neq \pi'(z)$ since $\{\tilde{y}^*, \tilde{z}^*\}$ is separated. Hence we have

$${}^\#E_n = {}^\#\tilde{E}_n \leqq N_{K+n-1}(f) ,$$

i.e. $S_n((\beta, B), X) \leqq N_{L+n-1}(f)$ since ${}^\#E_n$ is maximal.

Now let α be a finite open covering of X. By the construction of β we have that $\operatorname{diam}(\beta) \to 0$ when $\ell \to \infty$ (here $\operatorname{diam}(\beta) = \max\{\operatorname{diam}(B_i); 1 \leqq i \leqq t\}$). Hence there is β such that $\operatorname{st}(\beta)$ refines α. From Property 2 and the above inequality we have

$$\begin{aligned} h(f, \alpha) &= \lim_{n\to\infty} (1/n) \log N(\alpha_f^n) \\ &\leqq \liminf_{n\to\infty} (1/n) \log S_n((\beta, N), X) \\ &\leqq \liminf_{n\to\infty} (1/n) \log N_{L+n-1}(f) \\ &\leqq \liminf_{n\to\infty} (1/n) \log N_n(f)\,, \end{aligned}$$

and therefore the desired inequality is obtained.

Remark. Under the notations and the assumptions of Theorem 3, we can construct an example such that

$$h(f) \lneqq \liminf_{n} (1/n) \log N_n(f)\,.$$

Let Σ be a shift space defined by two symbols $\{0, 1\}$ and as before define a shift $\sigma: \Sigma \to \Sigma$. For $k \in \boldsymbol{Z}$ $(k > 0)$ we consider a point $p^k = (p_i^k) \in \Sigma$ defined by

$$p_i^k = \begin{cases} 1 & \text{when } [i/k] \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

where [ ] denotes the Gauss' symbol. Obviously each p^k is a periodic point of period $2k$. For this point p^k we denote by $O_k(p)$ the orbit of σ.

Consider two the matrices $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and define shift spaces of finite type which are denoted by Σ_A and Σ_B. Let p^ω be a point in $O_\omega = \Sigma_A \cup \Sigma_B$. We define a new space X by

$$X = (\Sigma - (O_1(p) \cup O_2(p) \cup \cdots \cup O_\omega)) \cup \{p^1, p^2, \cdots, p^\omega\}$$

and construct a map $\pi: \Sigma \to X$ by

$$\pi(x) = \begin{cases} p^k & \text{when } x \in O_k(p) \quad (k = 1, 2, \cdots, \omega) \\ x & \text{otherwise}\,. \end{cases}$$

Obviously π is surjective. So we introduce the strongest topology in X for which π is continuous. It is easily checked then that X is *Hausdorff* and compact. Hence the product topological space $\Sigma \times X$ is Hausdorff and compact.

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It is easy to see that a continuous map $f: X \to X$ is induced from σ by π. Then f has infinitely many fixed points. In fact, $f(p^k) = p^k$ for $k = 1, 2, \cdots, \omega$. Put $\rho(x, y) = (x, \pi(y))$ for $(x, y) \in \Sigma \times \Sigma$. Then the diagram

$$\begin{array}{ccc} \Sigma \times \Sigma & \xrightarrow{\sigma \times \sigma} & \Sigma \times \Sigma \\ \rho \downarrow & & \downarrow \rho \\ \Sigma \times X & \xrightarrow[\sigma \times f]{} & \Sigma \times X \end{array}$$

commutes. $(\Sigma \times \Sigma, \sigma \times \sigma)$ is a subshift of finite type and topologically mixing, and $h(\sigma \times f) \leqq h(\sigma \times \sigma) = 2 \log 2$ holds. However $N_n(\sigma \times f)$ is infinite for $n > 0$.

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