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MEAN VALUE THEOREM FOR THE m-INTEGRAL OF DINCULEANU

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1. **Introduction.** The classical mean value theorem asserts that if f is a real, bounded, Riemann integrable function defined on a finite real interval $a \leq t \leq b$, then $\int_a^b f(t)\, dt = (b-a)y_0$, where $\inf_{a \leq t \leq b} f(t) \leq y_0 \leq \sup_{a \leq t \leq b} f(t)$. The extensions of Choquet [3], Price [15], and of this paper generalize the fact that y_0 belongs to the closure of the convex hull of $f([a, b])$. The version of Choquet ([3, p. 38]) applies to a *continuous* function on a compact interval with values in a Banach space; that of Price ([15, p. 24]) applies to a bilinear integral of a special type containing the Birkhoff integral [2]. The m-integral of Dinculeanu [6] (specialization of Bartle's $*$-integral [1]) leaves intact the Lebesgue dominated convergence theorem and is strong enough to support an extended development. The paper is organized as follows: the object of §2 is to express the integral of a bounded m-integrable function as a limit of Riemann sums; §3 gives Price's generalization of "convex hull" [15]; the theorem of the paper is established in §4; §5 gives applications to vector differentiation which, for continuously differentiable functions, contain results of Dieudonné [5] and McLeod [13].

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2. **Riemann expression of the m-integral.** Let $m: \tau \to X$ be a measure of bounded variation $|m|$ defined on a σ-algebra τ of subsets of a set T, with values in a Banach space X. The functions in question are $|m|$-measurable applications of T into a Banach space Y. We suppose a continuous bilinear application of $X \times Y$ into a Banach space Z. Then, if $f: T \to Y$ is $|m|$-measurable and essentially bounded, it is m-integrable and $\int f\, dm$ is an element of Z ([1], [6]). This is the context of Dinculeanu [6], slightly specialized in that we assume the space T to be $|m|$-integrable.

LEMMA 2.1. *Given an m-integrable function f and arbitrary* $\epsilon > 0$, *there is a finite partition* $(T_i)_{1 \leq i \leq n}$ of T, $T_i \in \tau$, *such that* $|m|(T_n) < \epsilon$ *and, for* $1 \leq i < n$, *the oscillation of f on* T_i *is* $\leq \epsilon$.

Proof. There is a mean Cauchy sequence (f_n) of τ-step functions converging $|m|$-almost everywhere to f. By the Egorov theorem, there exists $B \in \tau$ such that

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$|m|(T-B)<\epsilon$ and (f_n) converges uniformly to f on B. So there is an index n_0 such that, for the corresponding step function

$$f_{n_0} = \sum_{j=1}^{p} \chi_{B_j} x_j, \qquad |f_{n_0}(t)-f(t)| < \epsilon/2$$

for all $t \in B$. We may suppose the B_j disjoint, then it suffices to let $T_1, \ldots, T_{n-1}$ be the nonempty intersections of the form $B_j \cap B$, and $T_n = T-B$.

LEMMA 2.2. *Given a bounded $|m|$-measurable function f, there exists a sequence of finite partitions of T:*

$(T_i^n)_{1 \le i \le k_n}$ $(T_i^n \in \tau)$, $n=1, 2, \ldots$ *such that, for* $t_i^n \in T_i^n$,

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} m(T_i^n) f(t_i^n) = \int f\, dm,$$

uniformly with respect to the choice of the $t_i^n \in T_i^n$.

PROOF. Put $M = \sup_{t\in T} |f(t)|$, and let $\epsilon > 0$ be arbitrary. Let $(T_i)_{1 \le i \le n}$ be the partition of Lemma 2.1. For arbitrary $t_i \in T_i$,

$$\begin{aligned} \left|\sum_1^n m(T_i) f(t_i) - \int f\, dm\right| &= \left|\sum_1^n \left(\int_{T_i} f(t_i)\, dm - \int_{T_i} f(t)\, dm\right)\right| \\ &\le \sum_1^n \int_{T_i} |f(t_i)-f(t)|\, d|m| \\ &= \sum_1^{n-1} \int_{T_i} |f(t_i)-f(t)|\, d|m| + \int_{T_n} |f(t_n)-f(t)|\, d|m| \\ &\le \epsilon\, |m|\, (T) + 2M\epsilon. \end{aligned}$$

It suffices to take a sequence of partitions of this form with $\epsilon = \epsilon_n$, $\epsilon_n \to 0$.

3. **Generalized convex hull.** Consider the set $C_1 = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \ge 0, \sum_1^n \lambda_i = 1, n = 1, 2, \ldots\}$. If E is a subset of a Banach space X, every $(\lambda_1, \ldots, \lambda_n) \in C_1$ determines the set of vectors $\sum_1^n \lambda_i x_i$ $(x_i \in E)$, called *convex combinations* of elements of E. Now let $\mathscr{L}(X, X)$ be the Banach space of all continuous linear maps of the Banach space X into itself, and consider the set

$$C = \left\{(T_1, \ldots, T_n) \mid T_i \in \mathscr{L}(X, X), \quad \sum_1^n T_i = 1_X, \quad n = 1, 2, \ldots\right\}.$$

Given two elements $\phi_1 = (T_{11}, \ldots, T_{1m})$, $\phi_2 = (T_{21}, \ldots, T_{2n})$ of C.

$$\phi_1\phi_2 = (T_{11}T_{21}, \ldots, T_{11}T_{2n}, \ldots, T_{1m}T_{21}, \ldots, T_{1m}T_{2n})$$

belongs to C. A subset C^* of C *is multiplicatively closed* if $\phi_1, \phi_2 \in C^*$ implies

$\phi_1\phi_2 \in C^*$. Considering $\lambda \geq 0$ as the element $x \to \lambda x$ $(x \in X)$ of $\mathscr{L}(X, X)$, C_1 is a multiplicatively closed subset of C.

Henceforth (until final specialization), C^* denotes an arbitrary multiplicatively closed subset of C. If E is a subset of the Banach space X, every $(T_1, \ldots, T_n) \in C^*$ determines the set of vectors $\sum_1^n T_i x_i$ $(x_i \in E)$, called *C^*-convex combinations* of elements of E. A subset of X is *C^*-convex* if it contains all C^*-convex combinations of its elements; and the smallest C^*-convex set containing a given subset E of X is the *C^*-convex hull* of E, denoted $C_0^*[E]$. The symbol $C_0[E]$ will denote the classical convex hull of E; in the case $C^* = C_1$ we have $C_0^*[E] = C_0[E]$.

LEMMA 3.1. *The operator $C_0^*: 2^X \to 2^X$ has the following properties:*

(i) $E \leq C_0^*[E]$

(ii) $E \leq F$ *implies* $C_0^*[E] \leq C_0^*[F]$

(iii) $C_0^*[C_0^*[E]] = C_0^*[E]$

(iv) $\sum_{i=1}^n T_i C_0^*[E] = C_0^*[E] \quad ((T_1, \ldots, T_n) \in C^*)$

Proof. The only part which is not immediate is the inclusion $\sum_{i=1}^n T_i C_0^*[E] \leq C_0^*[E]$. But an element of the left member is of the form $\sum_{i=1}^n T_i x_i$, $(T_1, \ldots, T_n) \in C^*$, $x_i \in C_0^*[E]$, and, since $C_0^*[E]$ is C^*-convex it contains this C^*-convex combination of its elements.

LEMMA 3.2. *For a subset E of the Banach space X,*

$$C^*[E] = \left\{ \sum_{i=1}^n T_i x_i \mid (T_1, \ldots, T_n) \in C^*, \quad x_i \in E, \quad n = 1, 2, \ldots \right\}$$

is C^*-convex.

Proof. Let $x = \sum_{i=1}^n T_i x_i$, $\phi = (T_1, \ldots, T_n) \in C^*$, $x_i \in C^*[E]$, so that

$$x_i = \sum_{j_i} T_{j_i}^{(i)} x_{j_i}^{(i)}, \qquad \phi_i = (T_1^{(i)}, \ldots, T_{k_i}^{(i)}) \in C^*, \quad x_{j_i}^{(i)} \in E,$$

and

$$x = \sum_{i=1}^n \sum_{j_i} T_i T_{j_i}^{(i)} x_{j_i}^{(i)}.$$

For $i = 1$,

$$\sum_{j_1} T_1 T_{j_1}^{(1)} x_{j_1}^{(1)} = \sum_{j_1} \sum_{j_2} T_1 T_{j_1}^{(1)} T_{j_2}^{(2)} x_{j_1}^{(1)} = \sum_{j_1} \cdots \sum_{j_n} T_1 T_{j_1}^{(1)} \ldots T_{j_n}^{(n)} x_{j_1}^{(1)}.$$

For $i > 1$,

$$\begin{aligned} \sum_{j_i} T_i T_{j_i}^{(i)} x_{j_i}^{(i)} &= \sum_{j_{i-1}} \sum_{j_i} T_i T_{j_{i-1}}^{(i-1)} T_{j_i}^{(i)} x_{j_i}^{(i)} \\ &= \sum_{j_1} \cdots \sum_{j_i} T_i T_{j_1}^{(1)} \ldots T_{j_i}^{(i)} x_{j_i}^{(i)} \\ &= \sum_{j_1} \cdots \sum_{j_n} T_i T_{j_1}^{(1)} \ldots T_{j_n}^{(n)} x_{j_i}^{(i)}. \end{aligned}$$

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We then have

$$x = \sum_{i=1}^{n} \sum_{j_1} \ldots \sum_{j_n} T_i T_{j_1}^{(1)} \ldots T_{j_n}^{(n)} x_{j_i}^{(i)}.$$

Since $x_{j_i}^{(i)} \in E$, and the operators of this last sum are the components of $\phi\phi_1 \ldots \phi_n$, $x \in C^*[E]$.

This lemma identifies C_0^* with the Price operator C^* ([15, p. 8]).

In order to apply this generalized convexity to integration, we will define a special multiplicatively closed subset C^* of C in terms of a given measure $m: \tau \to \mathscr{L}(X, X)$, of bounded variation, satisfying the

Axiom of Price ([15, p. 20]). For every $A \in \tau$, $m(A)=0$ or $m(A)$ is bijective.

We note that every scalar measure on a σ-algebra is of bounded variation ([6, pp. 47, 50]) and obviously satisfies the axiom of Price.

Let Ω denote the set of all finite disjoint sequences $\Delta=(A_i)$, $A_i \in \tau$, $\sum m(A_i) \neq 0$. Let C' denote the subset of C consisting of the finite sequences having one of the forms:

$$\phi(\Delta) = \left(\left(\sum_{i=1}^{n} m(A_i)\right)^{-1} m(A_j)\right) \quad (1 \leq j \leq n),$$

$$\phi'(\Delta) = \left(m(A_j)\left(\sum_{i=1}^{n} m(A_i)\right)^{-1}\right) \quad (1 \leq j \leq n) \quad (\Delta \in \Omega).$$

In the rest of the paper, C^* denotes the smallest multiplicatively closed subset of C containing C'.

LEMMA 3.3. $\sum m(A_i) C_0^*[E] = (\sum m(A_i)) C_0^*[E] \quad ((A_i) \in \Omega)$.

Proof. By (iv) of Lemma 3.1,

$$C_0^*[E] = \sum_i \left(\sum_j m(A_j)\right)^{-1} m(A_i) C_0^*[E] = \left(\sum_j m(A_j)\right)^{-1} \sum_i m(A_i) C_0^*[E].$$

LEMMA 3.4. *If $m: \tau \to R$ is a positive measure, $C_0^*[E] \leq C_0[E]$.*

Proof. Since C_1 is multiplicatively closed and contains C', (because m is positive), and C^* is the smallest such subset of C, we have $C_1 \geq C^*$. Hence every convex subset of X is C^*-convex, and therefore $C_0^*[E] \leq C_0[E]$.

4. **Mean value theorems.** In this section the bilinear context is specialized as follows: E is a Banach space, $Y=Z=E$, $X=\mathscr{L}(E, E)$, and the continuous bilinear map is $(\phi, x) \to \phi(x)$ $(\phi \in \mathscr{L}(E, E),\ x \in E)$.

THEOREM 4.1. *Let the measure $m: \tau \to \mathscr{L}(E, E)$ be of bounded variation and satisfy the axiom of Price. For every m-integrable function $f: T \to E$, $\int f\, dm = m(T) y_0$, $y_0 \in \overline{C_0^*[f(T)]}$, and y_0 is unique if $m(T) \neq 0$.*

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Proof. It suffices to consider the nontrivial case $m(T)\neq 0$, then y_0 is unique if it exists, by the axiom of Price. If f is bounded, by Lemma 2.2, there exists a sequence of finite τ-partitions of T: $(T_i^n)_{1\le i\le k_n}$ such that

$$\lim_{n\to\infty}\sum_{i=1}^{k_n} m(T_i^n)f(t_i^n) = \int f\,dm \quad (t_i^n\in T_i^n).$$

By Lemma 3.3,

$$\sum_{i=1}^{k_n} m(T_i^n)f(t_i^n)\in \sum_{i=1}^{k_n} m(T_i^n)C_0^*[f(T)] = \left(\sum_{i=1}^{k_n} m(T_i^n)\right)C_0^*[f(T)]$$
$$= m(T)C_0^*[f(T)],$$

so that the limit $\int f\,dm$ belongs to $\overline{m(T)C_0^*[f(T)]}$. Since $x\to m(T)x$ $(x\in E)$ is a homeomorphism (axiom of Price),

$$\overline{m(T)C_0^*[f(T)]} = m(T)\overline{C_0^*[f(T)]},$$

and the proof for the bounded case is complete. There remains the case where f is not bounded. We note that, for $n=1, 2, \ldots$, $A_n=\{t \mid t\in T, |f(t)|\le n\}\in\tau$.Choose $w_0\in f(T)$ and
set

$$f_n(t) = \begin{cases} f(t), & t\in A_n\\ w_0, & t\in T-A_n.\end{cases}$$

The f_n are $|m|$-measurable ([6, p. 91]) and the sequence (f_n) converges pointwise to f. Also, for $n\ge |w_0|$, $|f_n(t)|\le |f(t)|$, for all $t\in T$. Since the Lebesgue dominated convergence theorem holds for the m-integral ([6, p. 136]), we have

$$\lim_{n\to\infty}\int f_n\,dm = \int f\,dm.$$

By the theorem for the bounded case,

$$\int f_n\,dm = m(T)y_n, \qquad y_n\in\overline{C_0^*[f_n(T)]}\le\overline{C_0^*[f(T)]}.$$

The inequality

$$|y_p-y_q| = |m(T)^{-1}(m(T)y_p-m(T)y_q)| \le |m(T)^{-1}|\,|m(T)y_p-m(T)y_q|$$

implies the convergence of (y_n) to a point $y_0\in\overline{C_0^*[f(T)]}$. By the continuity,

$$\int f\,dm = \lim_{n\to\infty} m(T)y_n = m(T)y_0.$$

REMARKS.

1. Let $m\colon\tau\to E$ be a measure of bounded variation with values in a Banach

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algebra E, such that $m(A)$ has an inverse whenever $m(A)\neq 0$. Then for every m-integrable function $f: T \to E$,

$$\int f\,dm = m(T)y_0, \quad y_0 \in \overline{C_0^*[f(T)]}$$

(y_0 unique for $m(T)\neq 0$). In fact, considering E as a subspace of $\mathscr{L}(E, E)$, this is a special case of 4.1.

2. Let E be a real Banach space and let $\mu: \tau \to R \leq \mathscr{L}(E, E)$ be a positive measure. Applying 4.1, with scalar multiplication playing the role of the continuous bilinear map, we have, for every μ-integrable function $f: T \to E$,

$$\int f\,d\mu = \mu(T)y_0, \quad y_0 \in \overline{C_0[f(T)]},$$

because $C_0^*[f(T)] \leq C_0[f(T)]$.

3. The Remark 2 contains the following theorem of Choquet ([3, p. 38]): If f is a *continuous application* of a compact interval $[a, b]$ into a Banach space,

$$\int_a^b f(t)\,dt = (b-a)y_0, \quad y_0 \in \overline{C_0[f([a, b])]}.$$

(Since $f([a, b])$ is compact, therefore separable, f is Lebesgue measurable.)

Let $\mathscr{A}$ be an algebra of subsets of T and let $\mathscr{N}$ be a hereditary subring of $\mathscr{A}$. Let E be a Banach space and let $m: \mathscr{A} \to \mathscr{L}(E, E)$ be an additive function of bounded semi-variation, absolutely continuous ($\mathscr{N}$). Consider the space of $\mathscr{N}$-almost totally measurable functions ([6, p. 154]) $f: T \to E$. Such functions admit the Riemann representation of 2.2, by construction, and the integral of an almost totally measurable function vanishing on $\mathscr{N}$-negligible sets is the integral of an $\mathscr{N}$-equivalent totally measurable function; therefore we have what is needed to carry out the first part of the proof of Theorem 4.1 for such functions; so we have, in this context:

THEOREM 4.2. *Let $m: \mathscr{A} \to \mathscr{L}(E, E)$ be an additive function of bounded semi-variation, absolutely continuous ($\mathscr{N}$), satisfying the axiom of Price. For every almost totally measurable function $f: T \to E$, vanishing on $\mathscr{N}$-negligible sets, $\int f\,dm = m(T)y_0$, $y_0 \in \overline{C_0^*[f(T)]}$ (y_0 unique for $m(T)\neq 0$).*

5. **Application to vector differentiation.** Two points a, b of a Banach space define the *segment* $[a, b] = \{\lambda a + \mu b \mid \lambda, \mu \geq 0, \lambda + \mu = 1\}$.

THEOREM 5.1. *Let $f: U \to Y$ be a continuously differentiable application of an open set U of a Banach space X into a Banach space Y, and let $[a, b] < U$. Given $\epsilon > 0$,*

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there is a finite sequence $(x_i)_{1\le i\le n}$ *of points of* $[a, b]$, *and numbers* $\lambda_i \ge 0$ $(1\le i\le n)$ *such that* $\sum_1^n \lambda_i = 1$, *and*

$$|f(b)-f(a)-\sum_1^n \lambda_i f'(x_i)\cdot(b-a)| < \epsilon.$$

Proof. For the case $X=R$, $[a, b]=[0, 1]$,

$$f(1)-f(0) = \int_0^1 f'(t)\, dt \quad ([9, p. 171])$$

and

$$\int_0^1 f'(t)\, dt = y_0 \in \overline{C_0[f'([0, 1])]} \quad \text{(Remark 2 of 4.1)}.$$

This means that there is a vector

$$y = \sum_1^n \lambda_i f'(t_i), \qquad t_i \in [0, 1], \quad \lambda_i \ge 0, \quad \sum_1^n \lambda_i = 1$$

such that $|y-y_0| < \epsilon$. The special case established, let $[a, b] < U$, and introduce the function $g(t)=f(a+t(b-a))$, $0\le t\le 1$. Then

$$|g(1)-g(0)-\sum_1^n \lambda_i g'(t_i)| < \epsilon, \qquad t_i \in [0, 1], \quad \lambda_i \ge 0, \quad \sum_1^n \lambda_i = 1.$$

Since $g'(t)=f'(a+t(b-a))\cdot(b-a)$, putting $x_i=a+t_i\,(b-a)$, we prove the general case.

REMARK. The theorem implies the inequality

$$|f(b)-f(a)| \le \sup_{x\in[a,b]} |f'(x)|\; |b-a|,$$

which appears in ([5, p. 155]) under a weaker hypothesis: the continuity of the derivative is not assumed.

THEOREM 5.2. *Let* $f: U \to R^n$ *be a continuously differentiable application of an open set U of a real Banach space X into the real Euclidean n-space, and let* $[a, b] < U$. *There is a sequence of n points* x_i *belonging to* $[a, b]$ *and n numbers* $\lambda_i \ge 0$ *such that*

$$\sum_1^n \lambda_i = 1 \quad \text{and} \quad f(b)-f(a) = \sum_1^n \lambda_i f'(x_i)\cdot(b-a).$$

Proof. As in the proof of 5.1, it suffices to treat the case $X=R$, $[a, b]=[0, 1]$, and we have

$$f(1)-f(0) = \int_0^1 f'(t)\, dt = y_0 \in \overline{C_0[f'([0, 1])]}$$

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Since $f'([0, 1])$ is a compact subset of R^n, $C_0[f'([0, 1])]$ is compact ([4, p. 115]), therefore closed, so that $y_0 \in C_0[f'([0, 1])]$. Since $f'([0, 1])$ is connected, the Fenchel–Bunt theorem ([10, p. 36]) gives the required expression:

$$y_0 = \sum_1^n \lambda_i f'(t_i), \qquad t_i \in [0, 1], \quad \lambda_i \geq 0, \quad \sum_1^n \lambda_i = 1.$$

COROLLARY. *Let f: $U \to C^n$ be a continuously differentiable application of an open set U of a complex Banach space X into the complex Euclidean n-space, and let $[a, b] < U$. There is a sequence of $2n$ points x_i belonging to $[a, b]$ and $2n$ numbers $\lambda_i \geq 0$ such that*

$$\sum_1^{2n} \lambda_i = 1 \quad \text{and} \quad f(b)-f(a) = \sum_1^{2n} \lambda_i f'(x_i)\cdot(b-a).$$

Proof. It suffices to replace X by the underlying real Banach space and C^n by R^{2n} ([5, p. 145]).

REMARKS.

1. For $n=1$, $X=R^m$, 5.2 reduces to the classical mean value theorem for continuously differentiable functions ([14, p. 121], [12, p. 304]).

2. McLeod ([13, p. 203]), applying real value and convexity techniques, obtains a mean value differentiation theorem which contains 5.2, for $X=R$.

3. There are other mean value theorems in the context of McLeod's theorem 10, for example, the following result of Dotson ([7, p. 144]): *Let z_1, z_2 be distinct points of an open set U in the complex plane such that U contains the segment $[z_1, z_2]$. If f is a complex holomorphic function defined on U, there exists points w_1, w_2 of $[z_1, z_2]$ such that*

$$\frac{f(z_1)-f(z_2)}{z_1-z_2} = \operatorname{Re} f'(w_1)+i \operatorname{Im} f'(w_2).$$

This may be deduced by applying the classical mean value differentiation (or integral) theorem to the auxiliary function

$$F(t) = \frac{f(z_2+t(z_1-z_2))}{z_1-z_2} \quad (0 \leq t \leq 1).$$

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