THE MEASURE ALGEBRA AS AN OPERATOR ALGEBRA

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Introduction. In § I, it is shown that $M(G)^*$, the space of bounded linear functionals on $M(G)$, can be represented as a semigroup of bounded operators on $M(G)$.

Let Δ denote the non-zero multiplicative linear functionals on $M(G)$ and let P be the norm closed linear span of Δ in $M(G)^*$. In § II, it is shown that P, with the Arens multiplication, is a commutative B^*-algebra with identity. Thus $P = C(B)$, where B is a compact, Hausdorff space.

In § III, it is shown that B, with a natural multiplication, is a compact Abelian semigroup and that $M(G)$ is topologically embedded in $M(B)$. This gives a simplified construction of the Taylor structure semigroup for $M(G)$.

I am indebted to F. Birtel who directed this research.

I. Let G be a locally compact Abelian group and Γ its dual group. Let $C_0(G)$ denote the Banach algebra of continuous functions on G which vanish at infinity. Let $M(G)$ denote the Banach algebra of bounded Borel measures on G and $M(G)^*$ its topological dual space. Let $M(G)^\wedge$ denote the algebra of Fourier-Stieltjes transforms on Γ. For $\mu \in M(G)$, $\mu^\wedge$ is defined on Γ by $\mu^\wedge(\gamma) = \int_G \gamma(x)\, d\mu(x)$, $\gamma \in \Gamma$.

For $F \in M(G)^*$, let E_F denote the bounded operator on $M(G)$ defined by $(E_F\mu)^\wedge(\gamma) = F(\gamma d\mu)$, where $\gamma \in \Gamma$ and $\mu \in M(G)$. That $(E_F\mu)^\wedge \in M(G)^\wedge$ follows by Eberlein's theorem (**5**, p. 465) since

$$\left|\sum_{i=1}^{n} c_i (E_F\mu)^\wedge(\gamma_i)\right| = \left|\sum_{i=1}^{n} c_i F(\gamma_i\, d\mu)\right| \leqq$$
$$||F|| \left|\sum_{i=1}^{n} c_i\gamma_i\, d\mu\right| \leqq ||F||\, ||\mu|| \left[\sup_{x\in G}\left|\sum_{i=1}^{n} c_i\gamma_i(x)\right|\right].$$

Also, $||E_F\mu|| \leqq ||F||\,||\mu||$. Thus $||E_F|| \leqq ||F||$. Now $|F(\mu)| = |(E_F\mu)^\wedge(0)| \leqq ||E_F\mu|| \leqq ||E_F||\,||\mu||$. Thus $||F|| = ||E_F||$. Now E_F commutes with translation by $\gamma \in \Gamma$ in the sense that $\gamma dE_F(\mu) = E_F(\gamma d\mu)$ since for $\gamma_1, \gamma_2 \in \Gamma$,

$$(\gamma_1 dE_F\mu)^\wedge(\gamma_2) = (E_F\mu)^\wedge(\gamma_1 + \gamma_2) = F(\gamma_1\gamma_2 d\mu) = [E_F(\gamma_1 d\mu)]^\wedge(\gamma_2).$$

Let $\mathscr{B}$ denote the bounded operators on $M(G)$ which commute with translation by $\gamma \in \Gamma$. For $E \in \mathscr{B}$, define $F \in M(G)^*$ by $F(\mu) = (E\mu)^\wedge(0)$, $\mu \in M(G)$. Now $E_F = E$ since

$$(E_F\mu)^\wedge(\gamma) = F(\gamma d\mu) = [E(\gamma d\mu)]^\wedge(0) = (\gamma dE\mu)^\wedge(0) = (E\mu)^\wedge(\gamma)$$

for $\mu \in M(G)$ and $\gamma \in \Gamma$. Thus, we have the following results.

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THEOREM 1.† *The mapping $F \to E_F$ is a one-to-one, onto isometry between $M(G)^*$ and $\mathscr{B}$, the semigroup of bounded operators on $M(G)$ which commute with translation by $\gamma \in \Gamma$.*

COROLLARY. *Let Δ denote the non-zero multiplicative linear functionals on $M(G)$ and $\mathscr{E}$ the non-zero bounded endomorphisms on $M(G)$. The mapping $\pi \to E_\pi$ is a one-to-one, onto isometry between Δ and the semigroup $\mathscr{E}$.*

Remark 1. For other algebras whose topological duals can be represented as a space of operators, see (**3**).

II. Let Δ denote the non-zero multiplicative linear functionals on $M(G)$ and let P be the norm closed linear span of Δ in $M(G)^*$; i.e., $P = [\Delta]^- \subset M(G)^*$.

For $\mu \in M(G)$ and $F \in M(G)^*$, let $dF\mu \in M(G)$ be defined by $dF\mu(f) = F(fd\mu)$, where $f \in C_0(G)$. For $F, H \in M(G)^*$, let $F \times H \in M(G)^*$ be defined by $F \times H(\mu) = F(dH\mu)$, where $\mu \in M(G)$. Now "$\times$" is the Arens multiplication in $M(G)^*$ (**1**). It is known that, with the Arens multiplication, $M(G)^*$ is a commutative B^*-algebra (**4**, p. 869).

THEOREM 2. *With the Arens multiplication, P is a commutative B^*-algebra with identity.*

Proof. If $\pi_1, \pi_2 \in \Delta$, then $\pi_1 \times \pi_2 \in \Delta$ since for $\mu, \nu \in M(G)$ and $\gamma \in \Gamma$ we have that

$$[d\pi_2(\mu * \nu)]^\wedge(\gamma) = \pi_2(\gamma d\mu * \nu) = \pi_2(\gamma d\mu * \gamma d\nu) = [\pi_2(\gamma d\mu)][\pi_2(\gamma d\nu)] = (d\pi_2\mu)^\wedge(\gamma)(d\pi_2\nu)^\wedge(\gamma).$$

Let $\pi_0 \in \Delta$ be defined by $\pi_0(\mu) = \mu^\wedge(0)$, where $\mu \in M(G)$. Thus, for $\mu \in M(G)$ and $\gamma \in \Gamma$, we have that $(d\pi_0\mu)^\wedge(\gamma) = \pi_0(\gamma d\mu) = \mu^\wedge(\gamma)$. Thus $d\pi_0\mu = \mu$. Hence, π_0 is the identity for the Arens multiplication.

Since $M(G)^*$ is a commutative B^*-algebra under the Arens multiplication, it remains to show that Δ is closed under the involution, $\sim$, in $M(G)^*$. It suffices to show that if $F \in M(G)^*$, then $F^\sim(\mu) = \overline{F(\bar{\mu})}$, for $\mu \in M(G)$, where $\bar{\mu}(f) = \overline{\mu(\bar{f})}$, for $f \in C_0(G)$. To this end, we consider the proof that $M(G)^*$ is a B^*-algebra (**9**).

Let S denote the state space of $C_0(G)$; i.e., the collection of all positive linear functionals, $\mu \in M(G)$, such that $||\mu|| = 1$. For each $\mu \in S$, let H_μ denote the Hilbert space associated with μ. Let $f_\mu{}^\#$ be the standard representation of $f \in C_0(G)$ on H_μ. Let $H = \sum_{\mu \in S} H_\mu$ be the direct sum of H_μ, and let $f^\#$ be the operator on H which is on H_μ the same as $f_\mu{}^\#$. The mapping $f \to f^\#$ yields a one-to-one representation of $C_0(G)$ on H. Let $C_0(G)^\#$ denote the

†This result had been obtained earlier by I. Glicksberg in an unpublished typescript.

image under this map. Now, $M(G)^*$ can be identified with the weak closure of $C_0(G)^{\#}$, $\overline{C_0(G)^{\#}}$. Each $\mu \in M(G)$ can be represented in the form

$$\mu(f) = \sum_{k=1}^{n} [f^{\#}(h_k), n_k],$$

where $h_k, n_k \in H$. For $F \in M(G)^*$, F corresponds to $T \in \overline{C_0(G)^{\#}}$ and

$$F(\mu) = \sum_{k=1}^{n} [T(h_k), n_k].$$

Thus, $F^{\sim}$ corresponds to $T^{\sim} \in \overline{C_0(G)^{\#}}$ and

$$F^{\sim}(\mu) = \sum_{k=1}^{n} [T^{\sim}(h_k), n_k] = \sum_{k=1}^{n} [h_k, T(n_k)] = \sum_{k=1}^{n} \overline{[T(n_k), h_k]} = \overline{\sum_{k=1}^{n} [T(n_k), h_k]}.$$

We need only note now that if $\mu(f) = \sum_{k=1}^{n}[f^{\#}(h_k), n_k]$, then

$$\bar{\mu}(f) = \overline{\sum_{k=1}^{n} [\bar{f}^{\#}(h_k), n_k]} = \sum_{k=1}^{n} \overline{[h_k, f^{\#}(n_k)]} = \sum_{k=1}^{n} [f^{\#}(n_k), h_k].$$

Thus $F^{\sim}(\mu) = \overline{F(\bar{\mu})}$.

Remark 2. We may consider $\Gamma \subset \Delta$. Let AP denote the norm closed linear span of Γ in $M(G)^*$; i.e., $AP = [\Gamma]^- \subset M(G)^*$. Then AP is also a commutative B^*-algebra with identity. In fact, it may be identified with the almost-periodic functions on G (**2**, p. 817).

Remark 3. By Theorem 1, one can define a multiplication, $\odot$, in $M(G)^*$ by $E_{F\odot H} = E_F \circ E_H$, where $F, H \in M(G)^*$; i.e., $F \odot H = F \circ E_H$. Let $f_\alpha\colon G \to [0, 1]$, $f_\alpha \in C_0(G)$, be such that $f_\alpha \to 1$ in the compact-open topology on G. Thus, for $H \in M(G)^*$, $\mu \in M(G)$, and $\gamma \in \Gamma$, we have that

$$(dH\mu)^\wedge(\gamma) = \int_G \gamma(x)\, dH\mu(x) = \lim_\alpha \int_G \gamma(x) f_\alpha(x) dH\mu(x) = \lim_\alpha H(\gamma f_\alpha d\mu) = H(\gamma d\mu) = [E_H(\gamma d\mu)]^\wedge(0) = (E_H\mu)^\wedge(\gamma).$$

Thus $dH\mu = E_H\mu$, and thus, for $F, H \in M(G)^*$, $F \odot H = F \times H$.

Remark 4. Let $F \in M(G)^*$ and let $E_F \in \mathscr{B}$ be as in § I. Let $\mu \in M(G)$ be such that $\mu \geqq 0$ and $||\mu|| = 1$. Let $f_F \in (L^1(\mu))^*$ be defined by restricting F to $L^1(\mu)$; i.e., $f_F = F|L^1(\mu)$. We consider f_F as an element of $L^\infty(\mu)$. Then, for $\nu \in L^1(\mu)$, $E_F\nu = f_F d\nu$. Define $F^{\sim} \in M(G)^*$ by $F^{\sim}(\mu) = \overline{F(\bar{\mu})}$, where $\bar{\mu}(f) = \overline{\mu(\bar{f})}$ for $f \in C_0(G)$. Then, for $\nu \in L^1(\mu)$, $E_{F^{\sim}}\nu = \bar{f}_F d\nu$. One can now show *directly* that $||E_{F^{\sim}}E_F|| = ||E_F||^2$ and that $M(G)^*$ and P are commutative B^*-algebras with identities with $\odot$ as multiplication. This alternate proof of Theorem 1 is due to I. Glicksberg.

III. Let A be a commutative semi-simple Banach algebra and A^* its topological dual space. Let A' denote the norm closed subspace of A^* spanned

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by the non-zero multiplicative linear functionals on A. Let A'' be the topological dual space of A'. Birtel (**2**) showed that A'' is a commutative Banach algebra and that A may be embedded continuously into A''. We modify his construction and apply it to the situation in § II.

Let A be a commutative semi-simple Banach algebra and A^* its topological dual space. Let $D \subset A^*$ be a separating family of non-zero multiplicative linear functionals on A. Let P be the norm closed linear span of D in A^*; i.e., $P = [D]^- \subset A^*$. Now *suppose* that *P is a B^*-algebra with identity*. Thus $P = C(B)$, where B is a compact Hausdorff space. For $\pi \in D \subset P$, let $\hat{\pi} \in C(B)$ denote the Gel'fand representation of π. Define α: $\hat{D} \subset C(B) \to C(B \times B)$ by $\alpha\hat{\pi}(s, t) = \hat{\pi}(s)\hat{\pi}(t)$, where $s, t \in B$. Since the Gel'fand representation of A strongly separates the points of the maximal ideal space of A, D is linearly independent in A^*; and therefore we may extend α linearly to $[\hat{D}]$.

Let δ_s be unit point mass at $s \in B$ and $\delta_{(s,t)}$ be unit point mass at $(s, t) \in B \times B$. Now, for $a_i \in \mathbf{C}$ and $\hat{\pi}_i \in \hat{D}$, $1 \leqq i \leqq n$, we have that

$$\begin{aligned}
|\delta_{(s,t)} \circ \alpha(a_1\hat{\pi}_1 + \ldots + a_n\hat{\pi}_n)| &= |a_1\hat{\pi}_1(s)\hat{\pi}_1(t) + \ldots + a_n\hat{\pi}_n(s)\hat{\pi}_n(t)| \\
&\leqq |\delta_s[a_1\hat{\pi}_1(t)\hat{\pi}_1 + \ldots + a_n\hat{\pi}_n(t)\hat{\pi}_n]| \\
&\leqq ||\delta_s||\ ||a_1\hat{\pi}_1(t)\hat{\pi}_1 + \ldots + a_n\hat{\pi}_n(t)\hat{\pi}_n||_{C(B)} \\
&= ||a_1\hat{\pi}_1(t)\pi_1 + \ldots + a_n\hat{\pi}_n(t)\pi_n||_{A^*} \\
&= \sup_f|a_1\hat{\pi}_1(t)\pi_1(f) + \ldots + a_n\hat{\pi}_n(t)\pi_n(f)| \\
&= \sup_f|\delta_t[a_1\pi_1(f)\hat{\pi}_1 + \ldots + a_n\pi_n(f)\hat{\pi}_n]| \\
&\leqq ||\delta_t||\sup_f||a_1\pi_1(f)\hat{\pi}_1 + \ldots + a_n\pi_n(f)\hat{\pi}_n||_{C(B)} \\
&= \sup_f||a_1\pi_1(f)\pi_1 + \ldots + a_n\pi_n(f)\pi_n||_{A^*} \\
&= \sup_f\sup_g|a_1\pi_1(f)\pi_1(g) + \ldots + a_n\pi_n(f)\pi_n(g)| \\
&= \sup_{f,g}|a_1\pi_1(fg) + \ldots + a_n\pi_n(fg)| \\
&\leqq \sup_{f,g}||a_1\pi_1 + \ldots + a_n\pi_n||_{A^*}||fg||_A \\
&\leqq ||a_1\pi_1 + \ldots + a_n\pi_n||_{A^*} \\
&= ||a_1\hat{\pi}_1 + \ldots + a_n\hat{\pi}_n||_{C(B)},
\end{aligned}$$

where, for example, $\sup_f$ is the supremum over all elements f with $||f|| \leqq 1$. Thus, $\delta_{(s,t)} \circ \alpha$ is bounded on $[\hat{D}]$. Also, α is bounded on $[\hat{D}]$ and may be extended to all of $C(B)$. Call the extension β. Thus, β: $C(B) \to C(B \times B)$, $||\beta|| = 1$, and $\beta\hat{\pi}(s, t) = \hat{\pi}(s)\hat{\pi}(t)$, where $\hat{\pi} \in \hat{D}$.

Using the map β, we define a multiplication in B. The map $\delta_{(s,t)} \circ \beta$: $C(B) \to \mathbf{C}$ is a non-zero multiplicative linear functional, and thus there is an $r \in B$ such that $\delta_{(s,t)} \circ \beta = \delta_r$. Define m: $B \times B \to B$ by $m(s, t) = r$. We write st for $m(s, t)$.

Remark 5. The multiplication in B defined by m agrees with the convolution of point measures in $M(B)$; i.e., $\delta_{m(s,t)} = \delta_s * \delta_t$, where convolution in $M(B)$ is the restricted Arens multiplication as defined by Birtel (**2**, p. 816).

THEOREM 3. *Let A be a commutative semi-simple Banach algebra with topological dual space A^*. Let $D \subset A^*$ be a separating family of non-zero multiplicative linear functionals on A. Let P be the norm closed linear span of D in A^*; i.e., $P = [D]^- \subset A^*$. Suppose that P is a commutative B^*-algebra with identity. Then there exists a compact Abelian topological semigroup, B, such that A is continuously algebraically isomorphic to a subalgebra of $M(B)$. If $D \subset P$ is a group, then B is also a group.*

Proof. We first show that $m: B \times B \to B$ is continuous. Let $V = \{s \in B: |p(s)| < \delta\}$ for some $p \in C(B)$ and $\delta > 0$. Now V is a typical sub-basic neighbourhood in B. Let $U = \{(s, t) \in B \times B: |\beta p(s, t)| < \delta\}$. Thus, U is a neighbourhood in $B \times B$ and $m(U) \subset V$ since $\beta p(s, t) = p(m(s, t)) = p(st)$.

If $\hat{\pi}(s) = \hat{\pi}(t)$, $s, t \in B$, for all $\hat{\pi} \in \hat{D}$, then $s = t$, since $[\hat{D}]$ is dense in $C(B)$; in particular, $\hat{D}$ separates the points of B. For $\hat{\pi} \in \hat{D}$, $\hat{\pi}(st) = \hat{\pi}(s)\hat{\pi}(t) = \hat{\pi}(t)\hat{\pi}(s) = \hat{\pi}(ts)$. Thus $st = ts$. For $\hat{\pi} \in \hat{D}$, we have that

$$\hat{\pi}[(st)u] = \hat{\pi}(st)\hat{\pi}(u) = \hat{\pi}(s)\hat{\pi}(t)\hat{\pi}(u) = \hat{\pi}(s)\hat{\pi}(tu) = \hat{\pi}[s(tu)].$$

Thus $(st)u = s(tu)$. Thus, (B, m) is a compact Abelian topological semigroup.

Suppose now that D is a group. It suffices to show that B satisfies the cancellation laws (**6**, p. 99). Let $st = sr$. Now for $\hat{\pi} \in \hat{D}$, $\hat{\pi}(s)\hat{\pi}(t) = \hat{\pi}(st) = \hat{\pi}(sr) = \hat{\pi}(s)\hat{\pi}(r)$. Since $\pi^\wedge(\pi^{-1})^\wedge = 1$, $\hat{\pi}(s) \neq 0$. Thus, $\hat{\pi}(t) = \hat{\pi}(r)$ for all $\hat{\pi} \in \hat{D}$. Hence $t = r$.

Let $f \in A$. Let $f^{**} \in A^{**}$ be defined by $f^{**}(F) = F(f)$, for $F \in A^*$. Let f^P be defined on $P \subset A^*$ by restricting f^{**} to P; i.e., $f^P = f^{**}|P$. For $F \in P$, let $\hat{F}$ denote its Gel′fand representation in $C(B)$. Let $f^B \in C(B)^* = M(B)$ be defined by $f^B(\hat{F}) = f^P(F)$. Let λ denote the map from A to $M(B)$ defined by $\lambda(f) = f^B$.

Since B is a compact Abelian semigroup, $M(B)$ is a commutative Banach algebra under convolution (**8**); i.e., for $\mu, \nu \in M(B)$, $\mu * \nu \in M(B)$ is defined by $\mu * \nu(f) = \int_B\int_B f(st)d\mu(s)d\nu(t)$, $f \in C(B)$. Let $f, g \in A$ and $\pi \in D$. Then $[\lambda(fg)](\hat{\pi}) = (fg)^B(\hat{\pi}) = \pi(fg) = \pi(f)\pi(g) = f^B(\hat{\pi})g^B(\hat{\pi}) =$

$$\int_B \hat{\pi}(s)df^B(s)\int_B\hat{\pi}(t)dg^B(t) = \int_B\int_B\hat{\pi}(s)\hat{\pi}(t)df^B(s)dg^B(t) =$$
$$\int_B\int_B\hat{\pi}(st)df^B(s)dg^B(t) = (f^B * g^B)(\hat{\pi}).$$

Since $[\hat{D}]$ is dense in $C(B)$, it follows that λ preserves multiplication. Now

$$||\lambda(f)|| = ||f^B||_{M(B)} = ||f^P||_{P^*} \leqq ||f^{**}||_{A^{**}} = ||f||_A.$$

Hence $||\lambda|| \leqq 1$. Finally, we note that λ is one-to-one since D is separating.

Remark 6. In Theorem 3, let $A = M(G)$ and $D = \Delta$. Then B is the Taylor structure semigroup for $M(G)$ (**10**, p. 158).

Let B be a compact (semi) group. If $p \in C(B)$, $p \not\equiv 0$, is such that $p(st) = p(s)p(t)$, $s, t \in B$, then p is called a (semi) character. Let $\hat{B}$ denote the collection of all (semi) characters.

THEOREM 4. *With the notation of Theorem* 3, *$M(B)$ is semi-simple, $\lambda(A) = A^B$ is weak* dense in $M(B)$, and $\hat{B} = \hat{D}$.*

Proof. Since D is a separating family of multiplicative linear functionals, it follows that $M(B)$ is semi-simple and that $\lambda(A) = A^B$ is weak* dense in $M(B)$.

It follows, from the definition of the multiplication in B, that $\hat{D} \subset \hat{B}$. Let $\hat{F} \in \hat{B}$ such that $\hat{F}(st) = \hat{F}(s)\hat{F}(t)$, $0 \not\equiv \hat{F} \in C(B)$. Let $F \in P$ be such that the Gel'fand representation of F is $\hat{F}$. Then for $f, g \in A$,

$$F(fg) = \int_B \hat{F}(s)d(fg)^B(s) = \int_B \hat{F}(s)d(f^B * g^B)(s) = \int_B\int_B \hat{F}(st)df^B(s)dg^B(t) = \int_B \hat{F}(s)df^B(s)\int_B \hat{F}(t)dg^B(t) = F(f)F(g).$$

Thus $F \in D$ and $\hat{F} \in \hat{D}$.

THEOREM 5. *There is a compact Abelian semigroup, B, such that $M(G)$ is isometric isomorphic to a closed subalgebra of $M(B)$ such that the maximal ideal space of $M(G)$ is identified with the semi-characters on B.*

Proof. With the previous notation, it remains only to show that the map $\mu \rightarrow \mu^B$ is an isometry from $M(G)$ into $M(B)$. We know that $||\mu^B|| \leqq ||\mu||$. Now, for $\mu \in M(G)$,

$$||\mu^B|| \geqq \sup\{|(a_1\pi_1 + \ldots + a_n\pi_n)(\mu)|: ||a_1\pi_1 + \ldots + a_n\pi_n||_{M(G)^*} \leqq 1, \text{ where } \pi_i \in \Gamma\} \geqq ||\mu^\beta||,$$

where μ^β is the extension of μ to the Bohr group. That $||\mu^\beta|| \geqq ||\mu||$ is well known (see, e.g., **2**, p. 817). Thus $||\mu^\beta|| = ||\mu||$.

Remark 7. M. Rieffel (**7**, p. 64) has characterized measure algebras on locally compact Abelian groups. His proof is also based on the construction of Birtel (**2**). Following (**7**, p. 47), one could show that $||\mu|| = ||\mu^B||$ by the Kaplansky density theorem; i.e., $P = [\Delta]^- \subset M(G)^*$ is a weak* dense C^*-subalgebra of the W^*-algebra $M(G)^*$, and thus the unit ball of P is weak* dense in the unit ball of $M(G)^*$.

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