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Smoothing continuous flows on two-manifolds and recurrences

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Abstract. Let $\varphi:\mathbb{R}\times M\to M$ be a continuous flow on a compact C^∞ two-manifold M. It is proved that there exists a C^1 flow ψ on M which is topologically equivalent to φ, and that the following conditions are equivalent:

(a) any minimal set of φ is trivial;
(b) φ is topologically equivalent to a C^2 flow;
(c) φ is topologically equivalent to a C^∞ flow.

Also proved is a structure and an existence theorem for continuous flows with non-trivial recurrence.

1. *Introduction*

Let M be a two-manifold and let φ, $\psi:\mathbb{R}\times M\to M$ be continuous flows on M. We say that φ and ψ are *topologically equivalent* if there is a homeomorphism of M that takes trajectories of φ onto trajectories of ψ, preserving the natural orientation of the trajectories. A non-empty compact set $\Lambda\subset M$ invariant under φ is said to be a *minimal set* (of φ) if Λ contains no compact non-empty proper subset which is invariant under φ. A subset $\Lambda\subset M$ is a *trivial minimal set* (of φ) if it is either a closed trajectory or a fixed point or else the whole manifold M, provided that $\Lambda=M$ is the torus and φ is (topologically equivalent to) an irrational flow.

Our main result is the following theorem, which can be seen as the converse of Denjoy–Schwartz theorem [**De**], [**Sch**].

SMOOTHING THEOREM. *Let $\varphi:\mathbb{R}\times M\to M$ be a continuous flow on a compact C^∞ two-manifold M. Then there exists a C^1 flow ψ on M which is topologically equivalent to φ. Furthermore, the following conditions are equivalent*:

(a) *any minimal set of φ is trivial*;
(b) *φ is topologically equivalent to a C^2 flow*;
(c) *φ is topologically equivalent to a C^∞ flow.*

The assertion that (b) implies (a) is precisely the Denjoy–Schwartz theorem. The proof of this theorem works word-for-word to give the following corollary which is useful for applications.

SMOOTHING COROLLARY. *Let $\mathcal{F}$ be a continuous one dimensional orientable foliation with singularities on a compact C^∞ boundaryless two-manifold M. If the set of* *singularities of $\mathscr{F}$ is compact, then there exists a C^1 flow ψ on M which is topologically equivalent to $\mathscr{F}$.*

The structure theorem below describes the dynamical structure of flows on two-manifolds around its *non-trivial recurrent trajectories*; that is, trajectories γ such that $\overline{\gamma-\gamma}\supset\gamma$. As a consequence of this result, we shall see that the existence theorem below provides a constructive way of obtaining 'essentially all' continuous flows with non-trivial recurrence. The smoothing properties of the flows constructed in this way will be detected at once.

We shall need some definitions to state the results more precisely.

Let $E:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ be an *interval exchange transformation*, that is, E is an injective differentiable map defined everywhere except possibly at finitely many points and, for all $x\in\text{Dom}(E)$ (its domain of definition), $|E'(x)|=1$. Let $T:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ be a continuous map that *covers* E; that is, T is injective, its domain of definition Dom (T) is an open subset of $\mathbb{R}/\mathbb{Z}$, and for some monotone continuous map $h:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ of degree one, $h(x)\in\text{Dom}(E)$ and $E\circ h(x)=h\circ T(x)$, for all $x\in\text{Dom}(T)$. Let N_E be the quotient manifold obtained from

$$\mathbb{R}/\mathbb{Z}\times[-1,1]-\{(\mathbb{R}/\mathbb{Z}-\text{Dom}(E))\times\{1\})\cup(\mathbb{R}/\mathbb{Z}-\text{Dom}(E^{-1})\times\{-1\})\}$$

by identifying $(x,1)$ and $(E(x),-1)$, for all $x\in\text{Dom}(E)$. The pair $(\mathscr{F}, N_E)$, where $\mathscr{F}$ is an oriented one-dimensional continuous foliation on N_E, is said to be a *suspension of the pair* (T, E), if the following two conditions are satisfied:

(S_1) $\mathscr{F}$ is transversal to $\mathbb{R}/\mathbb{Z}\times\{0\}$ and the set of singularities of $\mathscr{F}$ is either empty or compact.

(S_2) The forward Poincaré map $\mathbb{R}/\mathbb{Z}\times\{0\}\to\mathbb{R}/\mathbb{Z}\times\{0\}$ induced by $\mathscr{F}$ is $(x,0)\mapsto(T(x),0)$.

Given E, T and h as above, we shall also say that *T covers E via h.*

STRUCTURE THEOREM. *Let $\varphi:\mathbb{R}\times M\to M$ be a continuous flow on a compact C^∞ two-manifold M. The closure of the non-trivial recurrent trajectories of φ determine finitely many compact φ-invariant subsets $\Omega_1,\Omega_2,\ldots,\Omega_m$ of M such that any non-trivial recurrent trajectory of φ is dense in some Ω_i. Moreover, given $i,j\in\{1,2,\ldots,m\}$, there exists an open connected subset V_i of M (of finite type) such that the following conditions are verified.*

(St 1) If $i\neq j$, $V_i\cap V_j=\varnothing$. *Moreover V_i contains all non-trivial recurrent trajectories meeting Ω_i.*

(St 2) *Each V_i is a region of recurrence associated to Ω_i. That is:*

(a) *There exists a circle $C_i\subset V_i$ transversal to φ, passing through Ω_i, and such that the forward Poincaré map $T_i:C_i\to C_i$ induced by φ covers an interval exchange transformation $E_i:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ which has every orbit dense and which cannot be extended continuously to a bigger subset of $\mathbb{R}/\mathbb{Z}$.*

(b) *The pair $(\varphi|_{V_i}, V_i)$ is (topologically equivalent to) a suspension of (T_i, E_i). Also, the frontier $\mathscr{F}_r(V_i)$ of V_i can only contain fixed points, regular trajectories connecting fixed points and finitely many transversal segments that connect fixed points. Moreover, there is no arc of trajectory of φ lying in V_i and connecting two points of $\mathscr{F}_r(V_i)$.*

(St 3) *If V_i' is any other region of recurrence associated to Ω_i, then V_i and V_i' are homeomorphic. Moreover, when φ has finitely many fixed points and no other region of recurrence associated to Ω_i contains fewer fixed points than V_i (resp. V_i'), the foliations $(\varphi|_{V_i}, V_i)$ and $(\varphi|_{V_i'}, V_i')$ are topologically equivalent.*

(St 4) *The circle $C_i \subset V_i$ can be taken so that either $\Omega_i \cap C_i = C_i$ or $C_i \cap \Omega_i$ is a Cantor set.*

The next theorem needs some more definitions. Let $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an injective map *not necessarily defined everywhere*. The domain of definition of f will be denoted by Dom (f). The positive (resp. negative) semi-orbit of $x \in \mathbb{R}/\mathbb{Z}$ is the set

$$\theta^+(x) = \{f^n(x)/n \in \mathbb{Z},\ n \geq 0 \text{ and } x \in \text{Dom}\,(f^n)\}$$

(resp. $\theta^-(x) = \{f^n(x)/n \in \mathbb{Z},\ n \leq 0 \text{ and } x \in \text{Dom}\,(f^n)\}$), where f^0 denotes the identity map of $\mathbb{R}/\mathbb{Z}$. The orbit of $x \in \mathbb{R}/\mathbb{Z}$ is the set $\theta(x) = \theta^+(x) \cup \theta^-(x)$. Let $x \in \mathbb{R}/\mathbb{Z}$, we say that $y \in \{x, \theta(x)\}$ *is non-trivial recurrent* if $\overline{\theta(x)} - \theta(x) \supset \theta(x)$. When Dom (f) is an open subset of $\mathbb{R}/\mathbb{Z}$, we shall say that a non-empty compact set $\Delta \subsetneq \text{Dom}\,(f)$ is a *non-trivial minimal set* (of f) if, for all $x \in \Lambda$, $\theta(x)$ is non-trivial recurrent and $\overline{\theta(x)} = \Lambda$.

EXISTENCE THEOREM. *Let $E: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an interval exchange transformation which has every orbit dense and which cannot be extended continuously to a bigger subset of $\mathbb{R}/\mathbb{Z}$. Let $h: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a monotone continuous map of degree one. Denote by $\mathscr{C}(E, h)$ the set of continuous maps that cover E via h. Then*:

(E 1) *Suspension item*: *For all $T \in \mathscr{C}(E, h)$ there exists a suspension of (T, E).*

(E 2) *Covering item*: *There exists $\hat{T} \in \mathscr{C}(E, h)$, called maximal, such that* Dom $(\hat{T}) = h^{-1}(\text{Dom}\,(E) - \overline{\Delta(E, h)})$, *where* $\Delta(E, h) = \{x \in \text{Dom}\,(E)/h^{-1}(x)$ *and* $h^{-1}(E(x))$ *are not bijective sets*$\}$. *Moreover, for all $T \in \mathscr{C}(E, h)$, it is verified that*:

(a) Dom $(T) \subset$ Dom $(\hat{T})$;

(b) *the maps T and $\hat{T}$ are equal when restricted to the non-trivial recurrent orbits of T*;

(c) *the map $\hat{T}$ has non-trivial recurrent points if and only if $\overline{\Delta(E, h)}$ has empty interior.*

(E 3) *Recurrence item*: *Let $T \in \mathscr{C}(E, h)$. The following statements are equivalent*:

(a) *The map T (resp. any suspension of (T, E)) has a non-trivial recurrent point.*

(a′) $h(\text{Dom}\,(T))$ *contains an open and dense subset of $\mathbb{R}/\mathbb{Z}$.*

(E 4) *Minimal set item*: *Let $T \in \mathscr{C}(E, h)$. The following statements are equivalent*:

(a) *The map T (resp. any suspension of (T, E)) has a non-trivial minimal set.*

(a′) *The map h is not a homeomorphism, the set $S = \{x \in \mathbb{R}/\mathbb{Z}/h^{-1}(x)$ meets both* Dom (T) *and $\mathbb{R}/\mathbb{Z}$ − Dom $(T)\}$ is at most finite, and* Dom $(T) \supset \overline{h^{-1}(\mathbb{R}/\mathbb{Z} - S)}$. *(In particular $\Delta(E, h) = \emptyset$).*

(E 5) *Smoothing item*: *Let $T \in \mathscr{C}(E, h)$ and let $(\mathscr{F}, N_E)$ be a suspension of (T, E). There exists a C^1 flow $\varphi: \mathbb{R} \times \tilde{M} \mapsto \tilde{M}$ on a smooth compact connected boundaryless two-manifold $\tilde{M}$ such that for some empty or finite set F of fixed points of φ, the pair $(\varphi|_{\hat{M}-F}, \hat{M} - F)$ is topologically equivalent to the pair $(\mathscr{F}, N_E)$. Moreover, except when T has a non-trivial minimal set, the flow φ can be taken to be smooth.*

The existence of C^1 flows which are not topologically equivalent to C^2 flows was proved by Denjoy [**De**], [**Swe**, see appendix]. The smoothing theorem cannot be extended to higher dimensional manifolds. If M is three-dimensional then neither (b) implies (c) nor (c) implies (a). More precisely, J. Harrison [**Hr**] proved that if $r<s$ are non-negative real numbers, then there exists a C^r flow which is not equivalent to any C^s flow; besides, by suspending the 'horse-shoe' of Smale [**Sm**, § 1.5] we get a smooth flow which possesses exotic minimal sets. See also [**Chw**].

We gave in [**Gu.5**] the main lines of the proof of our results for flows with finitely many fixed points. Partial results related with our theorems are those of [**Ne.2**] and [**Gu.2**]. The structure theorem for flows with finitely many fixed points has also been proved by Veech [**Ve.2**]; see also [**Ga**], [**St**]. The extension of the structure theorem to non-orientable foliations with finitely many singularities has been done by H. Rosenberg [**Ro**], G. Levitt ([**Le.1**], [**Le.2**] and [**Le.3**]) and L.H. Mendes [**Me**].

The extension of our results to non-orientable foliations (on two-manifolds) with arbitrarily many singularities is done in [**Gu.3**].

Interval exchange transformations are studied in [**Ka**], [**Ke**], [**Ve.1**]. The existence of interval exchange transformations having derivative -1 somewhere and having every orbit dense is shown in [**Gu.1**].

The only things one needs from § 3 to read § 4 are the structure theorem, corollary (3.1) and proposition (3.2). The only things one needs from §§ 3, 4 to read § 5 are proposition (3.2) and proposition (4.3). § 6 does not depend on any of §§ 3, 4 and 5.

In § 2 we introduce some terminology and notation that will be used throughout the paper.

We wish to say something about the proof of our main result. Let φ be a flow as in the smoothing theorem and let F be the set of fixed points of φ. Suppose that any minimal set of φ is trivial. To prove that φ is topologically equivalent to a smooth flow we decompose the manifold $M-F$ into submanifolds which are almost flow boxes of φ and for which we have smooth 'models'. Disguised as 'T-sequences' these submanifolds appear in § 4 ((4.4)). They are densely distributed in $M-F$ but they may not cover it. Nevertheless, starting in proposition (4.2) and using the structure theorem, we organize these T-sequences in such a way that we are able to construct a new differentiable structure for $M-F$ with respect to which the foliation that φ induces in $M-F$ is smooth. From this point, following an idea introduced by D. Neumann [**Ne.2**] we can prove that φ is topologically equivalent to a smooth flow.

2. *Preliminaries*

As we said, we shall introduce some terminology and notation that will be used throughout the paper.

We will always denote by M a smooth compact two-manifold and by $\varphi:\mathbb{R}\times M\to M$ a continuous flow on M.

The positive *semi-trajectory* (resp. *negative semi-trajectory*) of $p\in M$ is the set

$$\gamma_p^+=\{\varphi(t,p)/t\in[0,\infty)\}$$

(resp. $\gamma_p^- = \{\varphi(t, p)/t \in (-\infty, 0]\}$). The *trajectory* $\gamma_p^+ \cup \gamma_p^-$ of p will be denoted by γ_p. A point $p \in M$ is a *regular* point of φ if p is not a fixed point of φ. We say that $x \in \{p, \gamma_p\}$ is *periodic* if p is not a fixed point and $\varphi(t, p) = p$ for some $t > 0$. A point $q \in M$ is an *ω-limit point* (resp. *α-limit point*) of either $p \in M$ or γ_p, if there is a sequence of real numbers $t_k \to \infty$ (resp. $t_k \to -\infty$) such that $\varphi(t_k; p) \to q$. The set of *ω-limit points* (resp. *α-limit points*) of $x \in \{p, \gamma_p\}$ is denoted by $\omega(x)$ (resp. $\alpha(x)$). We say that $x \in \{p, \gamma_p\}$ is *ω-recurrent* (resp. *α-recurrent*) if $\gamma_p \subset \omega(\gamma_p)$ (resp. $\gamma_p \subset \alpha(\gamma_p)$). $x \in \{p, \gamma_p\}$ is *recurrent* if it is either ω- or α-recurrent; thus $x \in \{p, \gamma_p\}$ is *non-trivial recurrent* if γ_p is neither a fixed point nor a closed trajectory.

The notation $\overrightarrow{pq}$ will be used for an oriented arc of trajectory (of φ) starting at p and ending at q. The orientation will be that induced by the flow.

Let N be a submanifold of M disjoint of the fixed points of φ. We will say that N is a flow box (of φ) if there exists a rectangle $A = [a, b] \times [c, d] \subset \mathbb{R}^2$ and a homeomorphism $\theta: A \to N$ such that, for all $s \in [c, d]$, $\theta([a, b] \times \{s\})$ is an arc of trajectory of φ. Such a homeomorphism $\theta: A \to N$ will also be called a *flow box*. The segments $\theta(\{a\} \times [c, d])$, and $\theta(\{b\} \times [c, d])$ (resp. $\theta([a, b] \times \{c\})$ and $\theta([a, b] \times \{d\})$) will be called *transversal* (respectively *not transversal*) *edges* of (the flow box) N. A point $p \in \{\theta(x, y)/x \in \{a, b\}, y \in \{c, d\}\}$ will be said to be a *corner* of N. If p is a regular point of φ, there exists a neighbourhood of p which is a flow box (see [**B–S**, theorem 2.9, p. 50] and [**Wt**]). A *segment* or a *circle* C is said to be *transversal* to φ if for $p \in C$ which is not an endpoint of it, there exists a flow box $\theta: [-1, 1] \times [-1, 1] \to N$ such that $\theta(0, 0) = p$ and $\theta(\{0\} \times [-1, 1]) = N \cap C$.

Let Σ_1 (resp. Σ_2) be either a segment or a circle transversal to φ. The *forward* (resp. *backward*) *Poincaré map induced by* φ is the map $f: \Sigma_1 \to \Sigma_2$ (resp. $g: \Sigma_1 \to \Sigma_2$), not necessarily defined everywhere, such that $p \in \mathrm{Dom}\,(f)$ (resp. $p \in \mathrm{Dom}\,(g)$) and $f(p) = q$ (resp. $g(p) = q$) if and only if $\varphi(\tau, p) = q \in \Sigma_2$ for some $\tau > 0$ (resp. $\tau < 0$) and $\{\varphi(t, p)/0 < t < \tau\}$ (resp. $\{\theta(t, p)/\tau < t < 0\}$) is disjoint from $\Sigma_1 \cup \Sigma_2$.

Let $f, g: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be continuous maps not necessarily defined everywhere. We say that f and g are topologically conjugate if there is a homeomorphism $h: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $h(\mathrm{Dom}\,(f)) = \mathrm{Dom}\,(g)$ and, for all $x \in \mathrm{Dom}\,(f)$, $h \circ f(x) = g \circ h(x)$.

A segment S which is an open interval and is transversal to φ will be said to be *wandering* (or φ-wandering) if any trajectory of φ intersects S at most once. A point $p \in M$ is said to be *wandering* if it is regular and there is a wandering transversal open interval containing p.

A subset $X \subset M$ is *invariant* if $x \in X$ implies that $\gamma_x \subset X$.

The set of positive integers will be denoted by $\mathbb{N}$.

3. *Structure theorem*

The main results proved in this section are the structure theorem and proposition 3.2. The proof of the following corollary of the structure theorem will be given in (3.12).

(3.1) COROLLARY. *Under the conditions and notation of the structure theorem, denote by* Rec (C_i) *the union of the arcs* $\overrightarrow{pT_i(p)}$ *such that either* $p \in \Omega_i \cap \mathrm{Dom}\,(T_i)$ *or* p *belongs to* *a connected component of* $C_i - \Omega_i$ *which is contained in* Dom (T_i). *If* Ω_i *is a non-trivial minimal set, then*:

(a) *The set* Rec (C_i) *is a (topological) compact connected two-manifold whose boundary* $\partial(\text{Rec}\,(C_i))$ *(when non-empty) is made up of circles formed by finitely many disjoint arcs of trajectory joined by the same number of subintervals of* C_i.

(b) $\Omega_i \subset \text{Rec}\,(C_i) \subset V_i$ *and there is no arc of trajectory of* φ *contained in* $\text{Rec}\,(C_i) - \partial(\text{Rec}\,(C_i))$ *and connecting two points of* $\partial(\text{Rec}\,(C_i))$.

(3.2) PROPOSITION. *Let us assume that* φ *has fixed points. Let* $\Omega_1, \Omega_2, \ldots, \Omega_m$ *and* $C_1, C_2, \ldots, C_m$ *be as in the structure theorem. Then, for all* $i \in \mathbb{N}$, *there exists a two-dimensional* C^0 *compact submanifold with boundary* M_i *of* M, *and also there exists a subset* Γ_i *of* M, *made up of finitely many pairwise disjoint compact arcs transversal to* φ, *satisfying the following*:

(i) $M_1 \cap (\bigcup_{j=1}^m C_j) = \varnothing$. *For all* $i \in \mathbb{N}$, $\text{int}\,(M_i) \supset M_{i+1}$ *and* ∂M_i *is formed by two-sided circles. These circles are made up of finitely many transversal segments connected to each other by the same number of arcs of trajectory.*

(ii) $\bigcap_{i=1}^\infty M_i = F$ *is the set of fixed points of* φ.

(iii) *Let* $M_0 = M$ *and* $\Gamma_0 = \bigcup_{i=1}^m C_i$. *For all* $i \in \mathbb{N}$, $\Gamma_i \cap \partial M_i$ *is the union of all transversal segments to* φ *contained in* ∂M_i *and* $\Gamma_i - (\Gamma_i \cap \partial M_i)$ *is contained in the interior of* $M_{i-1} - M_i$.

(iv) *If* $\Gamma = \bigcup_{i=0}^\infty \Gamma_i$, *then both the forward* $T: \Gamma \to \Gamma$ *and the backward* $T^{-1}: \Gamma \to \Gamma$ *Poincaré maps induced by* φ *are defined everywhere. Moreover, for* $\delta \in \{-1, 1\}$, T^δ *restricted to a connected component of* Γ *is discontinuous at finitely many points.*

(v) *For all* $i \in \mathbb{N}$, *there is a positive integer* λ_i *such that if* $\vec{pq}$ *is an arc of trajectory meeting* $\bigcup_{j=1}^m C_j$ *exactly at their endpoints* p *and* q *then* $\vec{pq} \cap \Gamma_i$ *has at most* λ_i *elements.*

To prove these results we shall need some definitions and lemmas.

(3.3) *An equivalence relation.* Let Σ_1 and Σ_2 be compact segments or circles which are transversal to φ. Suppose that if $\Sigma_1 \cap \Sigma_2 \neq \varnothing$, then $\Sigma_1 \cup \Sigma_2$ is either a circle or a segment. Denote by $f: \Sigma_1 \to \Sigma_2$ the forward Poincaré map induced by φ. Let $a, b \in \Sigma_1$. We say that $a \sim b$ if, and only if, there exists a closed subinterval $\overline{pq}$ of Σ_1 (with endpoints $p, q \in \text{Dom}\,(f)$) containing $\{a, b\}$ and also there exists an open disc D_{pq} disjoint of $\Sigma_1 \cup \Sigma_2$ and whose boundary is equal to

$$\overline{pq} \cup \widehat{pf(p)} \cup \widehat{qf(q)} \cup \overline{f(p)\,f(q)},$$

where $\overline{f(p)\,f(q)}$ denotes a closed subinterval of Σ_2 with endpoints $f(p)$ and $f(q)$. It may happen that $\overline{pq}$ and $\overline{f(p)\,f(q)}$ are not disjoint.

This equivalence relation $\sim$ (defined in an open subset of Σ_1 which contains Dom (f)) will be called the *relation* $\sim$ *associated to* f. Any disc D_{pq} as above will be said to be a *disc associated to the triple* $(p, q, \sim)$.

(3.4) LEMMA. *Let* f: $\Sigma_1 \to \Sigma_2$ *be as in* (3.3). *Then*

(a) *The relation* $\sim$ *associated to* f *has finitely many equivalence classes. They are open and connected subsets of* Σ_1.

(b) *Let τ be either a compact segment or a circle. If τ is transversal to φ and disjoint from $\Sigma_1 \cup \Sigma_2$, then there exists $\lambda \in \mathbb{N}$ such that for all $x \in \mathrm{Dom}(f)$ $\widehat{xf(x)} \cap \tau$ has less than λ points.*

Proof. (a) is proved in [**Ne.2**, lemmas 2.5 and 2.8].

Let A be an equivalence class of $\sim$. To simplify matters suppose that A is an interval with two endpoints p and q. Certainly there exist monotone sequences $\{p_n\}$ and $\{q_n\}$ in $A \cap \mathrm{Dom}(f)$ such that $\lim p_n = p$ and $\lim q_n = q$. Let D_n be the open disc associated to the triple $(p_n, q_n, \sim)$. We may choose D_N such that {endpoints of $\tau\} \cap (\bigcup_n D_n) \subset D_N$. Observe that τ may intersect the boundary of D_N in at most finitely many points. Thus $\tau \cap D_N$ has finitely many connected components. Since D_N is a disc, if $x \in \mathrm{Dom}(f) \cap A$ then $\widehat{xf(x)}$ may intersect a connected component of $\tau \cap D_N$ in at most one point. Therefore if B is a connected component of $A - \overline{p_N q_N}$, where $\overline{p_N q_N}$ is the closed subinterval of A with endpoints p_N and q_N, then the cardinal number of the set $\widehat{xf(x)} \cap \tau$ is finite and the same for all $x \in B \cap \mathrm{Dom}(f)$. The proof of (b) follows at once from (a) and from these remarks. □

(3.5) *The map f_C.* Let γ be a non-trivial α- or ω-recurrent trajectory of φ passing through a transversal circle C. We consider the set $\mathcal{A}(C)$ made up by all the closed intervals $[a, b]$ of C such that either $[a, b]$ is the closure of a connected component of $C - \overline{\gamma \cap C}$ or $a = b$ and a does not belong to the closure of any connected component of $C - \overline{\gamma \cap C}$. It follows from the fact that γ is non-trivial recurrent that $\mathcal{A}(C)$ is a partition of C.

Let $f: C \to C$ be the forward Poincaré map induced by φ. If $[a, b], [c, d] \in \mathcal{A}(C)$ are such that $[a, b] \cap \gamma$ is a one point set, say $\{p\}$, and $f(p) \in [c, d]$, we shall write

$$f_C([a, b]) = ([c, d]).$$

If $[a, b], [c, d] \in \mathcal{A}(C)$ are such that $[a, b] \cap \gamma$ is not a one point set, we shall write

$$f_C([a, b]) = [c, d],$$

provided that there exist sequences $\{p_n\}, \{q_n\}$ on $\mathrm{Dom}(f) \cap \gamma$ satisfying the following two conditions

(i) $\lim p_n = a$, $\lim q_n = b$, $\{\lim f(p_n), \lim f(q_n)\} = \{c, d\}$

(ii) If $\sim$ is the relation associated to f then $a \sim b \sim p_n \sim q_n$.

See figure 1.

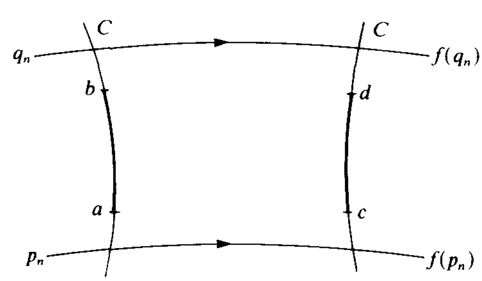

FIGURE 1

Therefore, we have established the map

$$f_C : \mathscr{A}(C) \to \mathscr{A}(C)$$

whose domain of definition is in general properly contained in $\mathscr{A}(C)$. Similarly f^{-1} induces the map

$$f_C^{-1} : \mathscr{A}(C) \to \mathscr{A}(C).$$

Since $\mathscr{A}(C)$ is a partition of C it will be provided with the quotient topology.

(3.6) LEMMA. *Under the conditions of* (3.5) *above,* $\mathscr{A}(C) - \mathrm{Dom}\,(f_C)$ *is a finite set.*

Proof. Let $[a, b] \in \mathscr{A}(C) - \mathrm{Dom}\,(f_C)$. Let $\sim$ be the relation associated to $f: C \to C$. By lemma (3.4), the equivalence classes of $\sim$ are finitely many open intervals $A_1, A_2, \ldots, A_n$. Since γ accumulates on both endpoints of $[a, b]$, and $\gamma \cap C \subset A_1 \cup A_2 \cup \cdots \cup A_n$, there exist $i, j \in \{1, 2, \ldots, n\}$ such that $a = \lim_k p_k$, $b = \lim_k q_k$ for some sequences $\{p_k\}$ in $A_i \cap \gamma$ and $\{q_k\}$ in $A_j \cap \gamma$. It must be $i \neq j$, because otherwise $p_k \sim q_k$, for all $k \in \mathbb{N}$, and therefore $[a, b]$ would belong to $\mathrm{Dom}\,(f_C)$. It is also clear that given a pair (A_i, A_j), there are at most two elements of $\mathscr{A}(C)$ intersecting both $\bar{A}_i$ and $\bar{A}_j$. Since the set $\{A_1, \ldots, A_n\}$ is finite, $\mathscr{A}(C) - \mathrm{Dom}\,(f_C)$ is also finite. □

The proof of the following two lemmas can be found in [**Gu.4**, lemma 2 of § 1, p. 312] and [**Gu.2**, lemma 5] respectively.

(3.7) LEMMA. *Let γ be a non-trivial recurrent trajectory of φ. Let Σ be an open segment transversal to φ and passing through γ. Denote by $\beta(\Sigma, \gamma)$ the set of two-sided simple closed curves of the form $\overrightarrow{pq} \cup \overline{pq}$, where $\overline{pq}$ is a subsegment of Σ and $\overrightarrow{pq}$ is a subarc of γ. Then $\beta(\Sigma, \gamma)$ is not empty and any circle $\overrightarrow{pq} \cup \overline{pq}$ of $\beta(\Sigma, \gamma)$ can be arbitrarily approximated (in the C^0 topology) by a circle which is transversal to φ.*

(3.8) LEMMA. *Let C be a circle and $T: C \to C$ be a continuous injective map defined everywhere except possibly at finitely many points. If T has dense positive semi-orbit then T is topologically conjugate to an interval exchange map.*

(3.9) LEMMA. *Let U be an open subset of M such that there is no arc of trajectory of φ contained in U and connecting two points of $\mathscr{F}_r(U)$. If γ is a non-trivial recurrent trajectory which is not contained in U, then there exist a circle C and an open set V containing C such that:*

(St 1)′ *$U \cap V = \varnothing$ and either $\bar{\gamma} \cap C = C$ or $\bar{\gamma} \cap C$ is a Cantor subset of C;*

(St 2)′ *the pair $(\varphi|_V, V)$ is a region of recurrence associated to $\bar{\gamma}$.*

Proof. Using lemma 3.7, we may construct a circle $C \subset M - U$ transversal to φ passing through γ. Let $f: C \to C$ be the forward Poincaré map induced by φ. What we shall do to prove this lemma is to show that C can be constructed so that the map $f_C : \mathscr{A}(C) \to \mathscr{A}(C)$ defined in (3.5) is (topologically conjugate to) an interval exchange transformation, and also that there exists a suspension $(\varphi|_V, V)$ of (f, f_C) as desired in this lemma. To do so, first it will be proved that (for C as at the very beginning), there are at most finitely many elements of $\mathscr{A}(C)$ which do not satisfy:

(1) If $[a, b] \in \mathscr{A}(C)$, then $\gamma \cap [a, b]$ is either empty or $\{a\}$ or else $\{b\}$.

Certainly, if $[a, b] \subset \mathscr{A}(C)$, with $a \neq b$, satisfies $[a, b] \cap \gamma = \{a, b\}$ then we may assume that $b = f^N(a)$ for some positive integer N. Thus, there exists $n \in \{0, 1, \ldots, N\}$ such that $(f_C)^n$ is not defined in $[a, b]$. Otherwise, $(f_C)^N([a, b]) \cap [a, b] \supset \{b\}$ would imply that $(f_C)^N([a, b]) = [a, b]$ contradicting the fact that γ is not periodic. Similarly, there exists $\hat{n} \in \{1, 2, \ldots, N\}$ such that $(g_C)^{\hat{n}}$ is not defined in $[a, b]$, where $g = f^{-1}$. By lemma (3.6), $(\mathscr{A}(C) - \text{Dom}(f_C)) \cup (\mathscr{A}(C) - \text{Dom}(g_C))$ is a finite set. These facts imply our claim above.

Let $\Sigma_1 \subset C$ be an open segment intersecting γ. We may take $\Gamma_1 \in \beta(\Sigma_1, \gamma)$ such that $\Gamma_1 \cap U = \emptyset$. If Σ_1 is disjoint from those elements of $\mathscr{A}(C)$ which do not verify (i), and $\tilde{C}$ is a circle very close to Γ_1 and transversal to φ, then any element of $\mathscr{A}(\tilde{C})$ will satisfy (1). The choices of Σ_1 and $\tilde{C}$ as above are possible by lemma (3.7). Therefore, from now on, we will assume that:

(2) there exists a circle C transversal to φ, passing through γ, such that (1) is satisfied.

It is claimed that:

(3) $\mathscr{A}(C)$ is homeomorphic to C.

In fact, when $\overline{\gamma \cap C}$ contains a subinterval of C obviously (3) is true. When $\overline{\gamma \cap C}$ has empty interior in C, then $\overline{\gamma \cap C}$ is a Cantor set. In this case, identify C with $\mathbb{R}/\mathbb{Z}$ (via a homeomorphism) and take a Cantor function **[Ha]** $\mathscr{L}: C \to C$ which is a monotone continuous map of degree one. The map $\mathscr{L}$ is constant in a closed subinterval of C if and only if this interval is the closure of a connected component of $C - \bar{\gamma}$. Certainly the quotient space $C/\mathscr{L}$ is homeomorphic to C. Therefore, since $C/\mathscr{L}$ is precisely $\mathscr{A}(C)$, (3) is verified.

Now, let $\sigma \in \mathscr{A}(C)$ intersecting γ. Because of (2), and the fact that any positive semi-trajectory of γ intersects C infinitely many times, it follows that $(f_C)^n$ is defined in σ, for all $n = 0, 1, 2, \ldots$. These facts imply that:

(4) $\{(f_C)^n(\sigma)/n = 0, 1, 2 \ldots\}$ is a semi-orbit dense in $\mathscr{A}(C)$.

By (2), f_C is injective. It is clear that f_C is continuous. Thus, lemma (3.6) and (4) permit us to apply lemma (3.8) to conclude:

(5) f_C is (topologically conjugate to) an interval exchange transformation $E: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which has a dense orbit.

Some of the properties of this map E are the following **[Ke]**: Any of its orbits is either finite or dense. It has only finitely many finite orbits. The point $x \in \mathbb{R}/\mathbb{Z}$ belongs to a finite orbit of E if and only if there exist positive integers n, m such that both E^n and E^{-m} are not defined for x.

Let $\tilde{\Sigma}_2$ be an open interval of $\mathscr{A}(C)$ disjoint of the finite orbits of f_C. Then:

(6) Any orbit of f_C meeting $\tilde{\Sigma}_2$ is dense on it. Moreover γ intersects the subinterval Σ_2 of C, generated by $\tilde{\Sigma}_2$, infinitely many times.

Let $\Gamma_2 \in \beta(\Sigma_2, \gamma)$. It follows from (6) that if $\tilde{\tilde{C}}$ is a circle transverse to φ close enough to Γ_2, then any orbit of $f_{\tilde{\tilde{C}}}$ is dense in $\mathscr{A}(\tilde{\tilde{C}})$, where $\tilde{\tilde{f}}: \tilde{\tilde{C}} \to \tilde{\tilde{C}}$ is the forward Poincaré map induced by φ.

Summarizing we may construct C so that:

(7) f_C is an interval exchange transformation having every orbit dense and $f: C \to C$ covers f_C via the quotient map $h: C \to \mathscr{A}(C)$.

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Now, either $C-\overline{\gamma\cap C}$ is dense in C or $\overline{\gamma\cap C}$ contains an open subsegment Σ_3 of C. In the second case, proceeding as above we may construct a new circle $\hat{C}$ transverse to φ and close to an element of $\beta(\Sigma_3, \gamma)$. Certainly $\overline{\gamma\cap\hat{C}}=\hat{C}$.

Therefore, we may construct C so that not only (7) is satisfied but also:

(8) $C\cap V=\emptyset$ and either $\bar{\gamma}\cap C=C$ or $\bar{\gamma}\cap C$ is a Cantor subset of C.

We remark that, by its definition,

(9) The interval exchange transformation f_C cannot be extended continuously to a bigger subset of $\mathscr{A}(C)$.

Now, we proceed to construct a suspension of (f, f_C). We shall use the concepts introduced in (3.3). Let $\sim$ (resp. $\sim'$) be the relation associated to $f: C\to C$ (resp. to $f^{-1}: C\to C$). By lemma (3.4) the relation $\sim$ (resp. $\sim'$) has finitely many equivalence classes $A_1, A_2, \ldots, A_n$ (resp. $A_1', A_2', \ldots, A_n'$). We may suppose that given $i\in\{1, 2, \ldots, n\}$ and $x\in A_i\cap\operatorname{Dom}(f)$, $f(x)\in A_i'$. It follows from (9) and from the definition of $\sim$ that:

(10) For all $i\in\{1, 2, \ldots, n\}$ there exists a connected component I_i of $\operatorname{Dom}(f_C)$ such that $I_i\subset h(A_i)\subset\bar{I}_i$ and $f_C(I_i)\subset h(A_i')\subset\overline{f_C(I_i)}$.

To simplify matters, from now on, we shall assume that $n\geq 2$. This implies that if $p, q\in A_i\cap\operatorname{Dom}(f)$ then there is a unique disc D_{pq} associated to the triple $(p, q, \sim)$. Let $V(A_i)$ be the union of $A_i\cup A_i'$ and all the open discs D_{pq} such that $p, q\in A_i\cap\operatorname{Dom}(f)$. We have that:

(11) For all $i\in\{1, 2, \ldots, n\}$, $V(A_i)-A_i\cup A_i'$ is an open disc whose frontier contains $A_i\cup A_i'$. Moreover, $V(A_i)$ is disjoint from U.

Actually, we shall only prove that $V(A_i)\cap U=\emptyset$. To do this it is enough to show that, for all $p, q\in\operatorname{Dom}(f)\cap A_i$, $(\bar{D}_{pq}-D_{pq})\cap U=\emptyset$. Observe that $\widehat{pf(p)}\cup\widehat{qf(q)}$ is disjoint from U because $\{p, q, f(p), f(q)\}\cap U=\emptyset$ and there is no arc of trajectory contained in U and connecting two points of $\mathscr{F}_r(U)$. As $C\cap U=\emptyset$, $(\bar{D}_{pq}-D_{pq})\cap U=\emptyset$ which implies (11).

Let $i\in\{1, 2, \ldots, n\}$. We claim that:

(12) If a is an endpoint of A_i (resp. A_i') then γ_a^+ (resp. γ_a^-) is contained in $\mathscr{F}_r(V(A_i))-U$ and $\omega(\gamma_a^+)$ (resp. $\alpha(\gamma_a^-)$) is a fixed point. Moreover, the complement of these semi-trajectories in $\mathscr{F}_r(V(A_i))$ can only contain fixed points and regular trajectories that connect fixed points. See figure 2.

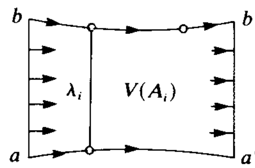

FIGURE 2

In fact, let a be an endpoint of A_i. The proof that $\gamma_a^+\subset\mathscr{F}_r(V(A_i))-U$ is similar to that of (11). Now suppose that $\omega(\gamma_a^+)$ contains a regular point p. Let Σ be a compact interval transversal to p and such that $p\in\Sigma-\{\text{endpoints of }\Sigma\}$. As $(\gamma_a^+-\{a\})\cap C=\emptyset$, γ_a^+ meets Σ infinitely many times. Using (b) of lemma 3.4 we may easily obtain a contradiction. Therefore $\omega(\gamma_a^+)$ is a fixed point. The remainder of (12) is proved similarly.

Let $\tilde{V}(A_i) = V(A_i) \cup \gamma_a^+ \cup \gamma_b^+ \cup \gamma_{a'}^- \cup \gamma_{b'}^-$ and λ_i be a continuous simple curve embedded in $V(A_i) - A_i \cup A_i'$ that connects the fixed points $\omega(\gamma_a^+)$ and $\omega(\gamma_b^+)$, where a and b (resp. a' and b') are the endpoints of A_i (resp. A_i'). See figure 2. Let $\tilde{V} = \bigcup_{i=1}^n \tilde{V}(A_i)$. We have that:

(13) If $C = \bigcup_{i=1}^n \bar{A}_i = \bigcup_{i=1}^n \overline{A_i'}$, then $\hat{V}$ is an open set and $(\varphi|_{\tilde{V}}, \tilde{V})$ is a region of recurrence.

In fact, in this case, $\tilde{V} - \bigcup_{i=1}^n \lambda_i$ is homeomorphic to $C \times (-1, 1)$ and so, using (12) and the fact that f covers f_C, $\tilde{V}$ can be seen as the manifold obtained from $C \times [-1, 1] - \{(C - \mathrm{Dom}\,(f_C)) \times \{1\} \cup (C\ \mathrm{Dom}\,(f_C^{-1}) \times \{1-1\}\}$ by identifying each interval $I_i \times \{1\}$ with the interval $f_C(I_i) \times \{-1\}$, where $i \in \{1, 2, \ldots, n\}$. Under these conditions (See (7), (9) and (12) it is easy to check that $(\varphi|_{\tilde{V}}, \tilde{V})$ is the region of recurrence desired to prove this lemma.

Finally, if $(C - \bigcup_{i=1}^n \bar{A}_i) \cup (C - \bigcup_{i=1}^n \bar{A}_i') \neq \emptyset$, then $\mathscr{F}_r(\tilde{V}) \cap \tilde{V}$ consists of finitely many segments $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ of one of the forms $B \cup \gamma_a^+ \cup \gamma_b^+$ or $B' \cup \gamma_a^- \cup \gamma_b^-$, where B (resp. B') is the closure of a connected component of $C - \bigcup_{i=1}^n \bar{A}_i$ (resp. $C - \bigcup_{i=1}^n \bar{A}_i'$) with endpoints a and b. For each Σ_j we may choose an open disc $D_j \supset \Sigma_j$ such that $\mathscr{F}_r(D_j) \cap (M - \tilde{V})$ is an arc transversal to φ connecting the endpoints of Σ_j (which are fixed points) and the foliation $\varphi|_{D_j}$ is, up to orientation, that of figure 3. Certainly we may choose these discs so small that, if $V = \tilde{V} \cup (\bigcup_{j=1}^k D_j)$, by an argument similar to that of the proof of (13), we have that $(\varphi|_V, V)$ is the region of recurrence desired to prove this lemma.

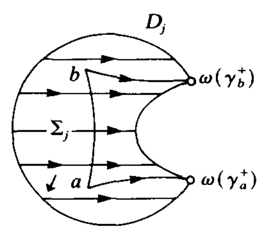

FIGURE 3

The proof of the following lemma can be found in [**Pe**, lemma 1].

(3.10) LEMMA. *There exists a positive integer N such that if* $n \geq N$ *and* $\gamma_1, \gamma_2, \ldots, \gamma_n \subset M$ *are pairwise disjoint simple closed curves, bounding no discs, then two of them enclose a cylinder of M.*

(3.11) *Proof of the structure theorem.* To obtain the pairwise disjoint regions of recurrence $V_1, V_2, \ldots, V_m$ of this theorem we may use inductively lemma 3.9. By lemma 3.10 applied to the circles $C_i \subset V_i$, there can be only finitely many of such regions of recurrence. This proves (St 1) and (St 2).

Now, given V_i' as in (St 3), let $C_i' \subset V_i'$, T_i' and E_i' be objects which are as C_i, T_i and E_i of (St 2), respectively. To prove (St 3), observe that $(T_i)_{C_i}: \mathscr{A}(C_i) \to \mathscr{A}(C_i)$(resp. $(T_i')_{C_i'}: \mathscr{A}(C_i') \to \mathscr{A}(C_i')$) and E_i (resp. E_i') are topologically conjugate.

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Using this, it is easy to see that any orbit of E_i determines a unique orbit in E_i' in such a way that:

(1) Each $p\in\mathbb{R}/\mathbb{Z}-\mathrm{Dom}\,(E_i)$ (resp. $q\in\mathbb{R}/\mathbb{Z}-\mathrm{Dom}\,(E_i^{-1})$) is in correspondence with a unique $p'\in\mathbb{R}/\mathbb{Z}-\mathrm{Dom}\,(E')$ (resp. $q'\in\mathbb{R}/\mathbb{Z}-\mathrm{Dom}\,((E')^{-1})$; and

(2) For $p\in\mathbb{R}/\mathbb{Z}-\mathrm{Dom}\,(E)$ and $q\in\mathbb{R}/\mathbb{Z}-\mathrm{Dom}\,(E^{-1})$, there exists an interval $I\in\mathrm{Dom}\,(E)$ such that $p\in\bar{I}$ and $q\in\overline{E(I)}$ if and only if, for some interval $I'\subset\mathrm{Dom}\,(E')$, $p'\in I'$ and $q'\in\overline{E'(I')}$.

By the definition of suspension, we have that (1) and (2) imply that V and V' are homeomorphic. The proof of the remainder of (St 3) can be found in [**Me**], [**Ga**]. See also [**Gu.5**, theorem B].

The proof of (St 4) is the same as that of (St 1)′ of lemma (3.9). □

(3.12) *Proof of corollary* (3.1). Because of the structure of $\mathscr{F}_r(V_i)$ and since $\Omega_i\subset V_i$, we may easily see that if the interval $I\subset\mathrm{Dom}\,(T_i)$ is the closure of a connected component of $C_i-\Omega_i$, then $\{\widehat{pT_i(p)}/p\in I\}\subset V_i$. Thus $\mathrm{Rec}\,(C_i)\subset V_i$. We observe that as $\Omega_i\cap C_i$ is compact, it is contained in finitely many connected components of $\mathrm{Dom}\,(T_i)$. Therefore there is a set Σ made up of finitely many closed subintervals of $C_i\cap\mathrm{Dom}\,(T_i)$ such that $\mathrm{Rec}\,(C_i)=\bigcup_{p\in\Sigma}\widehat{pT_i(p)}$. Using this fact and the structure theorem, the proof can easily be completed. □

(3.13) *Proof of proposition* (3.2). First we shall prove that:

(1) There exists a denumerable family $\{V_i/i=1,2,\ldots\}$ of open subsets of M such that

(1a) $M-\bigcup_{i=1}^m C_i\supset\bar{V}_1\supset V_1\supset\bar{V}_2\supset V_2\supset\bar{V}_3\supset\cdots\supset V_n\supset\bar{V}_{n+1}\supset\cdots\supset F$.

(1b) For all $i=1,2,\ldots$, $d(F,M-V_{i+1})\le\frac{1}{2}\min\{1/i,d(F,M-V_i)\}$ and $\partial V_i=\bar{V}_i-V_i$ is made up of finitely many pairwise disjoint two-sided circles.

In fact, let $2\varepsilon_1\le\min\{1,d(F,\bigcup_{i=1}^m C_i)\}$. Since F is compact, if ε_1 is small enough, there are finitely many open geodesic balls $B_1,B_2,\ldots,B_{k_1}$ strongly convex with the same radius ε_1 centred at points of F and such that $F\subset\bigcup_{i=1}^{k_1}B_i=V_1$. Certainly $\partial V_1=\bar{V}_1-V_1$ is made up of continuous two-sided simple closed curves intersecting between each other only tangentially and in at most finitely many points. Keep the centre of the balls $B_1,B_2,\ldots,B_{k_1}$ fixed but reduce their radii by an appropriately small amount so that the new balls, still denoted by $B_1,B_2,\ldots,B_{k_1}$, continue covering F but now ∂V_1 is made up of pairwise disjoint two-sided circles. With this procedure it is easy to construct inductively the family $\{V_i/i=1,2,\ldots\}$ required to prove (1).

Using tubular flow boxes centred at points of ∂V_i, $i=1,2,\ldots$, approximate each circle of ∂V_i by a circle contained in $V_i-\bar{V}_{i+1}$ and made up of finitely many segments, transversal to φ, connected to each other by arcs of trajectories. The union of all of these new circles form the boundary of a compact bidimensional manifold M_i satisfying $\bar{V}_{i+1}\subset\mathrm{Int}\,(M_i)\subset\bar{M}_i\subset V_i$.

Using (1) we can see that the family $\{M_i/i=1,2,\ldots\}$ satisfies (i) and (ii) of this proposition. Now, we claim that:

(2) For all $i\in\mathbb{N}$, the union of all closed orbits of φ contained in $\overline{M_{i-1}-M_i}$ is compact.

In fact, let $\{\gamma_n\}$ be a sequence of closed orbits in $M_{i-1}-M_i$ accumulating on an orbit γ. Since, by the structure theorem, $\bigcup_n \gamma_n$ is disjoint of $\bigcup_{i=1}^m C_i$, γ is also disjoint from $\bigcup_{i=1}^m C_i$. Therefore, the minimal sets of $\omega(\gamma)$, being contained in $\overline{M_{i-1}-M_i}$, must be closed orbits. This implies that $\omega(\gamma)$ is a closed orbit. Since γ is accumulated by closed orbits, we must have that $\omega(\gamma)=\gamma$ is a closed orbit. This proves (2).

Given $i\in\mathbb{N}$ and a closed orbit $\gamma\subset\overline{M_{i-1}-M_i}$, we may choose a pair $(V_\gamma, \Sigma_\gamma)$, formed by an open neighbourhood V_γ of γ and a compact interval Σ_γ transversal to φ passing through γ, satisfying:

(3) Σ_γ is contained in the interior of $M_{i-1}-M_i$. Moreover, for all $(p,\delta)\in V_\gamma\times\{-1,1\}$ there exists $t\in(0,\infty)$ such that $\varphi(\delta t,p)\in\Sigma_\gamma$.

By compactness, given $i\in\mathbb{N}$, there exist finitely many pairs (V_{i1},Σ_{i1}), $(V_{i2},\Sigma_{i2}),\ldots,(V_{in_i},\Sigma_{in_i})$ as above such that $\bigcup_{j=1}^{n_i} V_{ij}$ contains the set of closed orbits of φ contained in $\overline{M_{i-1}-M_i}$. Define Γ_i as the union of $\bigcup_{j=1}^{n_i}\Sigma_{ij}$ and the transversal segments to φ contained in ∂M_i. Given $p\in\Gamma$, the ω-limit set of p (resp. α-limit set of p) contains either a fixed point or a closed orbit or else a non-trivial recurrent point. In any case, by the construction of Γ, the fact that the positive (resp. negative) semi-trajectory starting at p has to return to Γ. This implies (iii) and (iv) of this proposition. Item (v) follows from lemma (3.4) and from the fact that each Γ_i has finitely many connected components.

4. *Decomposition in smoothable flow boxes*

The main results of this section are the following two propositions. To state them we shall need some terminology.

(4.1) *μ-coordinates.* Let σ be an oriented closed segment starting at a and ending at b. A measure μ on the Borel algebra of σ will be said to be a *distinguished measure on σ* if $0<\mu(\sigma)<\infty$ and the map $h:\sigma\to[0,\mu(\sigma)]$ given by $h(x)=\mu(\overline{ax})$ is a homeomorphism, where $\overline{ax}$ is the subsegment of σ with endpoints a and x. This map h will be said to be the *μ-homeomorphism of σ*. Let $\Sigma=\{\sigma_i\}_{i\in I}$ be such that the connected components of each σ_i, $i\in I$, are either circles or segments. μ is said to be a *distinguished measure on (the family)* Σ if, for all $i\in I$, and for all oriented closed segments $\lambda\subset\sigma_i$, $\mu|_\lambda$ is a distinguished measure on λ. When the family has only one term we shall identify the family with the term.

Let B be a flow box of φ having transversal edges A_1 and A_2. Let $\tau: A_1\to A_2$ be the Poincaré map induced by $\varphi|_B$ (the restriction of φ to B). Suppose that A_1 and A_2 are oriented so that τ preserves orientation. Suppose also that $\{A_1, A_2\}$ is provided with a distinguished measure μ. Denote by h_1 and h_2 the μ-homeomorphisms of A_1 and A_2, respectively. Either the map τ of the flow box B is said to be μ-smooth (resp. μ-C^1) if the *μ-coordinate expression of τ*,

$$\tilde{\tau}=h_2\circ\tau\circ h_1^{-1}:[0,\mu(A_1)]\to[0,\mu(A_2)],$$

is smooth (resp. C^1).

Let I be an interval of $\mathbb{R}$. Let $f, g: I\to\mathbb{R}$ be smooth maps. We define

$$\|f\|_0=\sup\{|f(x)|/x\in I\},$$

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and for $k \in \mathbb{N}$,

$$\|f\|_k = \max\{\|f\|_0, \|f'\|_0, \ldots, \|f^{(k)}\|_0\}.$$

Let $\varepsilon > 0$. We say that f is *ε-close to g in the C^k-topology* if $\|f-g\|_k \leq \varepsilon$.

(4.2). PROPOSITION. *Under the considerations of proposition* (3.2), *there exists a distinguished measure μ* on Γ *and a non-negative integer n_0 such that: for each segment Σ contained in Γ whose image $T(\Sigma)$ is also a segment it is established that:*

(a) *$T|_\Sigma : \Sigma \to T(\Sigma)$ is μ-C^1;*

(b) if *$\Sigma \subset \Gamma_i$ and $i \geq n_0$, then $T|_\Sigma : \Sigma \to T(\Sigma)$ is μ-smooth;*

(c) *if the minimal sets of φ are trivial then $n_0 = 0$.*

(4.3) PROPOSITION. *Suppose that the set F of fixed points of φ is not empty. Then $M-F$ can be expressed as the union of flow boxes θ_i, with $i \in \mathbb{N}$, such that:*

(a) *each compact set of $M-F$ is contained in finitely many flow boxes θ_i;*

(b) *if $i \neq j$, then $\theta_i \cap \theta_j = \partial\theta_i \cap \partial\theta_j$;*

(c) *there exists $n_0 \in \mathbb{N}$ and a distinguished measure μ on the transversal edges of the flow boxes θ_i such that, for all $i \in \mathbb{N}$, θ_i is μ-C^1 and, for all $i \geq n_0$, θ_i is μ-smooth;*

(d) *if the minimal sets of φ are trivial then $n_0 = 1$.*

(4.4) *T-sequences.* Let $\tau \in \{T, T^{-1}\}$. A finite sequence $\Sigma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ of open segments of Γ, pairwise disjoint, is said to be a *τ-sequence* if any two consecutive terms Σ_i and Σ_{i+1} of it satisfy $\tau(\Sigma_i) = \Sigma_{i+1}$.

Let $\Sigma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ be a τ-sequence with $\tau \in \{T, T^{-1}\}$, then:

(a) Σ is said to be *attracting* if $\tau^n(\Sigma_1) \subset \Sigma_1$, $\tau^n|_{\Sigma_1}$ admits continuous extension to $\bar{\Sigma}_1$ and this extension has a unique fixed point which is an attractor and is situated at one of the endpoints of Σ_1.

(b) Σ is said to be *periodic* if $\tau(\Sigma_n) = \Sigma_1$ and each $p \in \Sigma_1$ is a fixed point of τ^n.

(c) Σ is said to be *wandering* if Σ_1 is φ-wandering.

(d) When Σ is attracting, the half-open segment $\Sigma_1 - \tau(\Sigma_n)$ will be called a *fundamental domain* of Σ.

(e) The union of all terms of Σ will be denoted by span (Σ).

(4.5) LEMMA. *Under the conditions and notation of the structure theorem and corollary* (3.1), *suppose that $\Omega_{l+1}, \Omega_{l+2}, \ldots, \Omega_{l+v}$ are the non-trivial minimal sets of φ. Given $i, j \in \{l+1, l+2, \ldots, l+v\}$ it is verified that: If V_i^1 (resp. $V_i^{(-1)}$) denotes the union of the transversal segments of $\partial(\mathrm{Rec}\,(C_i))$ from where the trajectories enter (resp. exit)* Rec (C_i) *and $\tilde{\mu}$ is a distinguished measure on $V_i^1 \cup V_i^{(-1)}$, the gate of* Rec (C_i), *then there exists a distinguished measure μ on C_i extending $\tilde{\mu}$ and such that, in μ-coordinates,*

$$T_i|_{(C_i - V_i^{(-1)})} : (C_i - V_i^{(-1)}) \to C_i$$

is C^1 and has derivative equal to 1 *at all the endpoints of the connected components of $C_i - V_i^{(-1)}$.*

Proof. It follows from the structure theorem and corollary (3.1) that:

(1) V_i^1 and $V_i^{(-1)}$ have the same number of (closed) connected components. Also, if $V_{i1}^\delta, V_{i2}^\delta, \ldots, V_{is_i}^\delta$ denote those of V_i^δ, $\delta \in \{-1, 1\}$, then, for all $(n, j) \in \mathbb{N} \times \{1, 2, \ldots, s_i\}$, $T^{n\delta}$ is defined in V_{ij}^δ.

(2) There exists $J\subset\mathbb{N}$ and a family of closed subintervals $\{Z_j\}_{i\in J}$ of $C_i-(V_i^1\cup V_i^{-1})$ not reduced to points such that for all $(n,j)\in\mathbb{Z}\times J$, T^n is defined in Z_j and the family $\{T^n(Z_j), T^{m\delta}(V_{ik}^\delta)|n\in\mathbb{Z},\ j\in J,\ m\in\mathbb{N}\cup\{0\},\ \delta\in\{1,-1\},\ k\in\{1,2,\ldots,s_i\}\}$ is made up of pairwise disjoint intervals whose union is a dense subset of C_i.

Suppose now that M is the torus, that $\operatorname{Rec}(C_i)=M$ (in particular $\operatorname{Dom}(T_i)=C_i$) and that $J=\{1\}$. In this case, by Denjoy [**De**] there is a measure μ on C_i such that

(3) $\mu(C_i)=1$, $\mu(C_i-\bigcup_{n\in\mathbb{Z}}T^n(Z_1))=0$ and $T_i: C_i\to C_i$ *is* μ-C^1.

It is not difficult to see that, even when $J\subset N$ is arbitrary (because of (1) and (2)), the same idea can be used to construct a measure μ on C_i extending $\tilde{\mu}$ and satisfying this lemma. □

The proof of the following lemma can be found in [**S–T**], [**Ne.1**].

(4.7) LEMMA. *The interior of the set of non-wandering points of φ consists of recurrent points.*

(4.8) LEMMA. *With the conditions and notation of the structure theorem and proposition* (3.2), *suppose that $i\in\{1,2,\ldots,l\}$ (resp. $i\in\{l+1,l+2,\ldots,l+v\}$) if and only if $\Omega_i\cap C_i=C_i$ (resp. Ω_i is a non trivial minimal set of φ). Given $\sigma\in\{A,P,W\}$ there is a set $\mathbb{A}_\sigma=\{\sigma_1,\sigma_2,\ldots,\sigma_n,\ldots\}$, at most denumerable, such that*

(i) *Each A_i (resp. P_i) is an attracting (resp. periodic) T^{δ_i}-sequence (T^{ε_i}-sequence) with $\delta_i\in\{-1,1\}$ (resp. $\varepsilon_i\in\{-1,1\}$). Each W_i is a wandering open interval.*

(ii) *If $i\neq j$, then* $\operatorname{span}(\sigma_i)\cap\operatorname{span}(\sigma_j)=\varnothing$ *and*

$$\left[\bigcup_n T^n(W_i)\right]\cap\left[\bigcup_n T^n(\dot{W}_j)\right]=\varnothing.$$

(iii) *The closure of the set* $\bigcup_{i,j}(\operatorname{span}(A_i)\cup\operatorname{span}(P_j))$ *is a neighbourhood (in Γ) of the set of points of Γ which belong to closed orbits of φ.*

(iv) $\bigcup_k W_k$ *is disjoint from* $\bigcup_{i,j}(\operatorname{span}(A_i)-\bar{D}_i)\cup(\operatorname{span}(P_j)), \bigcup_{t=1}^{l}\bigcup_n T^n(C_t)$ *and* $\bigcup_{r=l+1}^{l+v}[\operatorname{Rec}(C_r)-G_r]\cap\Gamma$. *Here D_i denotes the fundamental domain of A_i and G_r denotes the gate of* $\operatorname{Rec}(C_r)$.

(v) *The union of the sets* $\bigcup_{i,j}(\operatorname{span}(A_i)\cup\operatorname{span}(P_j))$, $\bigcup_k\bigcup_n T^n(W_k)$, $\bigcup_{t=1}^{l}\bigcup_n T^n(C_t)$ and $\bigcup_{r=l+1}^{l+v}(\operatorname{Rec}(C_r)\cap\Gamma)$ *is an open and dense subset of* Γ.

(vi) *For all $A_i\in\mathbb{A}_A$ and all $r\in\{l+1,l+2,\ldots,l+v\}$, $\bigcup_k\bigcup_n T^n(W_k)$ intersects neither {endpoints of D_i} nor {endpoints of connected components of G_r}.*

Proof. Let $\mathscr{A}_P$ (resp. $\mathscr{A}_A$) be the set characterized as follows: $\lambda\in\mathscr{A}_P$ (resp. $\lambda\in\mathscr{A}_A$) if and only if:

(1) Any element of λ is a periodic (resp. attracting) T^δ-sequence, with $\delta\in\{-1,1\}$. Moreover any pair $\lambda_1,\lambda_2\in\lambda$ satisfies $\operatorname{span}(\lambda_1)\cap\operatorname{span}(\lambda_2)=\varnothing$.

Let $\sigma\in\{P,A\}$. If $\mathscr{A}_\sigma\neq\varnothing$, the inclusion of sets '$\subset$' determines a partial order relation in $\mathscr{A}_\sigma$. It is clear that Zorn's lemma can be used to find out a maximal element $\mathbb{A}_\sigma\in\mathscr{A}_\sigma$ for this partial order relation.

To continue, suppose that

(2) $\mathbb{A}_A=\{A_i/i=1,2,\ldots\}$ and $\mathbb{A}_P=\{P_j/j=1,2,\ldots\}$.

Since $\mathbb{A}_\sigma$, with $\sigma \in \{A, P\}$, is a maximal element of the partial order relation '$\subset$' in $\mathscr{A}_\sigma$, (iii) of this lemma holds.

Let ψ be the set of all trajectories of φ which pass through the endpoints of the fundamental domain of elements of $\mathbb{A}_P$. It is clear that

(3) $\psi \cap \Gamma$ is a discrete subset of Γ.

Let $\tilde{\Lambda}$ be the complement in Γ of the union of $\bigcup_{i,j} (\text{span}(A_i) - \bar{D}_i) \cup \text{span}(P_j)]$, $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$ and $\bigcup_{r=l+1}^{l+v} [\text{Rec}(C_r) - G_r] \cap \Gamma$, where D_i is the fundamental domain of A_i and G_r is the gate of Rec (C_r). Let Λ be the interior of the wandering points of $\tilde{\Lambda} - \psi$. Since (iii) is true, it follows from lemma (4.7) and (3) that

(4) Λ is (open) and dense in $\tilde{\Lambda}$.

Let $\mathscr{A}_W$ be the set characterized as follows: $\lambda \in \mathscr{A}_W$ if and only if:

(5) Any element of λ is a φ-wandering open segment contained in Λ. Moreover, any pair $\lambda_1, \lambda_2 \in \lambda$ satisfies

$$\left(\bigcup_n T^n(\lambda_1)\right) \cap \left(\bigcup_n T^n(\lambda_2)\right) = \varnothing.$$

If Λ contains some φ-wandering segment, then $\mathscr{A}_W \neq \varnothing$. Proceeding as above, there is a maximal element $\mathbb{A}_W = \{W_k | k = 1, 2, \ldots\} \in \mathscr{A}_W$ for the partial order relation determined by the inclusion of sets '$\subset$'. Certainly $\mathbb{A}_W$ is at most denumerable. Maximality of $\mathbb{A}_W$ and the fact that each W_i is open imply that

(6) $\bigcup_k \bigcup_n T^n(W_k)$ is open and dense in Λ.

It follows, from the construction of the sequences $\mathbb{A}_A$, $\mathbb{A}_P$ and $\{W_k\}$ that (i)–(v) of this lemma are verified. Since ψ is disjoint from Λ, each W_k is a wandering interval and also (for $r \in \{l+1, l+2, \ldots, l+v\}$) each endpoint of any connected components of G_r meets Ω_r, we conclude that (vi) of this lemma is true. □

(4.9) LEMMA. *Let $\tau \in \{T, T^{-1}\}$ and $\Sigma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ be an attracting τ-sequence. For any $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}$, there exists $\rho > 0$ such that if $\tilde{\mu}$ is a distinguished measure on the fundamental domain D of Σ such that $\tilde{\mu}(D) \leq \rho$, then there exists a canonical distinguished measure μ on Σ, <u>extending</u> $\tilde{\mu}$, <u>having order</u> k <u>and size</u> ε. That is, if Σ_1 is also denoted by Σ_{n+1}, then*:

(a) *For all $i \in \{1, 2, \ldots, n\}$ and for appropriate orientations of Σ_i and Σ_{i+1}, the μ-coordinate expression $\tau_i: [0, \mu(\Sigma_i)] \to [0, \mu(\Sigma_{i+1})]$ of $\tau|_{\bar{\Sigma}_i}: \bar{\Sigma}_i \to \bar{\Sigma}_{i+1}$ is smooth, 2^{-k}-close to the identity map of $[0, \mu(\Sigma_i)]$ in the C^k-topology and its derivative τ_i' has infinite order contact at $x \in \{0, \mu(\Sigma_i)\}$ with the constant map $\equiv 1$.*

(b) $\sum_{i=1}^{n} \mu(\Sigma_i) \leq \varepsilon$.

Proof. Let $\theta: \mathbb{R} \to [0, 1]$ be a smooth function such that $\theta^{-1}(0) = (-\infty, 0]$, $\theta^{-1}(1) = [1, \infty)$ and, for all $t \in (0, 1)$, $\theta'(t) > 0$. Given $\varepsilon > 0$ and $k \in \mathbb{N}$, we define

$$\rho = \frac{\varepsilon^{k+1}}{n^{k+1}(1 + \|\theta\|_k)}. \tag{1}$$

Let $\tilde{\mu}$ be a distinguished measure on the fundamental domain D of Σ such that $\tilde{\mu}(D) = \bar{\rho} < \rho$. We define $\sigma: \mathbb{R} \to \mathbb{R}$ by $\sigma(t) = t - \bar{\rho}\theta(t/(\varepsilon/n))$. The following facts are easily verified.

(2) The map σ is a diffeomorphism which is ε-close to the identity map of $\mathbb{R}$ in the C^k-topology. Also, σ has infinite order contact at 0 (resp. at ε/n) with the map $t\to t$ (resp. $t\to t-\bar{\rho}$) defined in $\mathbb{R}$. Moreover $\sigma|_{[0,\varepsilon/n)}$ has 0 as a unique fixed point which is an attractor. Finally $\sigma((0,\varepsilon/n)=(0,\varepsilon/n-\bar{\rho})$.

Certainly $\tau^n:\Sigma_1\to\Sigma_1$ and $\sigma|_{(0,\varepsilon/n)}:(0,\varepsilon/n)\to(0,\varepsilon/n-\bar{\rho})$ are topologically conjugate. Let $h:\Sigma_1\to(0,\varepsilon/n)$ be the unique homeomorphism which conjugates τ^n and $\sigma|_{(0,\varepsilon/n)}$ and such that $h|_D-\varepsilon/n+\bar{\rho}$ is a $\tilde{\mu}$-homeomorphism of D (where D is a fundamental domain of τ^n).

The Lebesgue measure of $(0,\varepsilon/n)$ (resp. $(0,\varepsilon/n-\bar{\rho})$ induces, via h (resp. via $\sigma\circ h\circ\tau^{-1}|_{\Sigma_2}:\Sigma_2\to(0,\varepsilon/n-\bar{\rho})$) a distinguished measure on Σ_1 (resp. Σ_2). Extend μ to a distinguished measure on Σ by defining $\mu|_{\Sigma_{i+2}}=\mu\cdot\tau^{-i}|_{\Sigma_{i+2}}$, $i=1,2,\ldots,n-2$. It follows from the construction of μ that the μ-coordinate expression of $\tau|_{\Sigma_1}:\Sigma_1\to\Sigma_2$ is precisely $\sigma|_{(0,\varepsilon/n)}:(0,\varepsilon/n)\to(0,\varepsilon/n-\bar{\rho})$ and moreover that the μ-coordinate expression of $\tau|_{\Sigma_i}:\Sigma_i\to\Sigma_{i+1}$, $i=2,3,\ldots,n$, is the identity map of $(0,\varepsilon/n-\bar{\rho})$. This together with (2) implies (a) and (b) of this lemma. □

The proof of the following lemma is similar to that of (4.9).

(4.10) LEMMA. *Let $\tau\in\{T,T^{-1}\}$ and $\Sigma=\{\Sigma_1,\Sigma_2,\ldots,\Sigma_n,\Sigma_{n+1}\}$ be a τ-sequence. Let $\tilde{\mu}$ be a distinguished measure on Σ_1. Then, for any $\varepsilon>0$, there exists a canonical distinguished measure μ on Σ extending $\tilde{\mu}$ and having size ε. That is if any term of Σ is oriented so that, for all $i=1,2,\ldots,n$, $\tau|_{\Sigma_i}:\Sigma_i\to\Sigma_{i+1}$ is orientation preserving, then*

(a) *The continuous extension $\tau_1:[0,\mu(\Sigma_1)]\to[0,\mu(\Sigma_2)]$ of the μ-coordinate expression of $\tau|_{\Sigma_1}$ (defined in $(0,\mu(\Sigma_1))$ is smooth and its derivative τ_1' has infinite order contact with the constant map $\equiv 1$.*

(b) *For all $i=2,3,\ldots,n$, the μ-coordinate expression of $\tau|_{\Sigma_i}$ is the identity map of $(0,\varepsilon/n)$.*

(4.11) LEMMA. *Let $\Sigma=\{\Sigma_1,\Sigma_2,\ldots,\Sigma_i,\ldots\}$ be an infinite sequence, such that for all $n\in\mathbb{N}$, $\{\Sigma_1,\Sigma_2,\ldots,\Sigma_n\}$ is a T-sequence (resp. T^{-1}-sequence). Suppose that for some $p\in\Sigma_1$ the ω-limit set of p (resp. α-limit set of p) contains a fixed point of φ. Then for any $k\in\mathbb{N}$ there exist j, $l\in\mathbb{N}$ such that $\Sigma_{2+j}\subset\Gamma_{l+k}$.*

Proof. Consider only the case in which $\Sigma_{i+1}=T(\Sigma_i)$ for $i\in\mathbb{N}$. Let $k\in\mathbb{N}$ be given. Certainly $\Sigma_3\subset\Gamma_N$ for some $N\in\mathbb{N}$. If $N>k$, this lemma follows immediately by taking $j=1$ and $l=N-k$. So assume that $N\le k$. Since $\omega(p)$ contains a fixed point of φ, there exists $j\in\mathbb{N}$ such that $T^{j+1}(p)\in\Gamma_{k+1}$. Therefore, using the fact that $\{\Sigma_1,\Sigma_2,\ldots,\Sigma_{j+2}\}$ is a T-sequence, it follows that $T^{j+1}(\Sigma_1)=\Sigma_{j+2}\subset\Gamma_{k+1}$. The lemma is proved. □

The proof of the following lemma of calculus will be omitted.

(4.12) LEMMA. *Let $f:[a,b]\to[c,d]$ be an increasing homeomorphism. Then f is smooth (in $[a,b]$) and has infinite order contact at $x\in\{a,b\}$ with the map $t\to t-x+f(x)$ if for some sequence $\{[a_i,b_i]/i\in\mathbb{N}\}$ of pairwise disjoint closed subintervals of $[a,b]$, the following conditions are satisfied*:

(A) $[a,b]=\bigcup_{i=1}^{\infty}[a_i,b_i]$, $\sum_{i=1}^{\infty}(b_i-a_i)=b-a$ *and* $\sum_{i=1}^{\infty}(f(b_i)-f(a_i))=d-c$.

(B) *For all* $i \in \mathbb{N}$ *and for* $x \in \{a_i, b_i\}$, $f|_{[a_i,b_i]}$ *is smooth and has infinite order contact at* x *with the map* $t \mapsto t - x + f(x)$ *defined on* $[a_i, b_i]$.

(C) *There exists* N *such that for all positive integers* $i \geq N$,

$$\|f|_{[a_i,b_i]} - (\mathrm{Id}|_{[a_i,b_i]} + f(a_i) - a_i)\|_i \leq 1/i,$$

where Id *is the identity map of* $\mathbb{R}$.

(4.13) *Proof of proposition* (4.2). Consider the notations and facts stated in lemma (4.8). It is clear (by such lemma) that

(1) The sets $\bigcup_j \mathrm{span}\,(P_j)$, $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$, $\bigcup_i (\mathrm{span}\,(A_i) - \bar{D}_i)$, $\bigcup_k \mathscr{B}_k$ and $\bigcup_{r=l+1}^{l+v} \{\mathrm{Rec}\,(C_r) \cap \Gamma - G_r\}$ are pairwise disjoint and their union is open and dense in Γ, where

$$\mathscr{B}_k = \bigcup_n T^n(W_k) - \left[\left\{\bigcup_i (\mathrm{span}\,(A_i) - \bar{D}_i)\right\} \cup \left\{\bigcup_{r=l+1}^{l+v} [\mathrm{Rec}\,(C_r) \cap \Gamma - G_r]\right\}\right].$$

Start defining a distinguished measure μ on $\bigcup_j \mathrm{span}\,(P_j)$. Let $P_j = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ be an arbitrary element of $\mathbb{A}_P$. By definition P_j is a T^δ-sequence, for some $\delta \in \{-1, 1\}$. Define μ on P_j so that for $i = 1, 2, \ldots, n$ (and for appropriate orientations of Σ_i and $T^\delta(\Sigma_i)$) the μ-coordinate expression of $T^\delta|_{\Sigma_i}: \Sigma_i \to T^\delta(\Sigma_i)$ is the identity map of $(0, 2^{-i}/n)$. This is possible because each $p \in \Sigma_1$ is a fixed point of T^n. Hence:

(2) μ is a distinguished measure on $\bigcup_j \mathrm{span}\,(P_j)$ such that $\mu(\bigcup_j \mathrm{span}\,(P_j)) \leq 1$ and if I and $T(I)$ are open intervals of $\bigcup_j \mathrm{span}\,(P_j)$ then the μ-coordinate expression of $T|_I: I \to T(I)$ is the identity map of $(0, \mu(I))$.

Proceed to define μ on $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$. By assumption (lemma (4.8)), when $i \in \{1, 2, \ldots, l\}$, there is a non-trivial recurrent trajectory dense in C_i. Therefore, by the structure theorem, there is a homeomorphism $h_i: C_i \to \mathbb{R}/\mathbb{Z}$ conjugating the Poincaré map $T_i: C_i \to C_i$ induced by φ with an interval exchange transformation. The Lebesgue measure of $\mathbb{R}/\mathbb{Z}$ induces, via h_i, a distinguished measure μ on C_i, $i = 1, 2, \ldots, l$, which is T_i-invariant. Let I be an open interval of $\bigcup_{t=1}^{l} C_t$. If for $\delta \in \{-1, 1\}$ and $n \in \mathbb{N}$, $T^{n\delta}(I)$ is an interval, define $\mu|_{T^{n\delta}(I)} = \mu \circ T^{-n\delta}|_{T^{n\delta}(I)}$. This is well defined because each $\mu|_{C_i}$ is T_i-invariant when $i = 1, 2, \ldots, l$. Also by (iv) of proposition (3.2), given $n \in \mathbb{N}$, $\delta \in \{-1, 1\}$ and $i \in \{1, 2, \ldots, l\}$, $T^{n\delta}|_{C_i}: C_i \to \Gamma$ is discontinuous at finitely many points. Thus, μ is already defined in $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$ except for a denumerable subset which only accumulates outside $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$. Take the μ measure of this denumerable set to be zero. To show that μ is a distinguished measure on $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$ it will be seen that

(3) For all $j \in \mathbb{N}$, $\mu([\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)] \cap \Gamma_j) \leq l\lambda_j$, where λ_j is as in (v) of proposition (3.2).

In fact, from (v) of proposition (3.2) and from the existence of a trajectory of φ which is dense in any circle C_i with $i \in \{1, 2, \ldots, l\}$, it follows that: Given $i \in \{1, 2, \ldots, l\}$ and $j \in \mathbb{N}$, any arc of trajectory, intersecting C_i at one of its endpoints and having its other endpoint at Γ_j, meets Γ_j at most λ_j times. This implies (3).

Under these conditions

(4) μ is a distinguished measure on $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$ such that if I and $T(I)$ are open subintervals of this set, then the μ-coordinate expression of $T|_I: I \to T(I)$ is the identity map of $(0, \mu(I))$.

Now, we will define a distinguished measure μ on $\bigcup_k \mathcal{B}_k$. By lemma (4.9), given any term A_i of $\mathbb{A}_A$ there exists a positive real number $\sigma_i \in (0, 1)$ such that

(5) If $\tilde{\mu}$ is a distinguished measure in the fundamental domain D_i of A_i satisfying $0 < \tilde{\mu}(D_i) \leq \sigma_i$; then there exists a canonical distinguished measure $\hat{\mu}$ on A_i extending $\tilde{\mu}$, having order i and size 2^{-i}.

Given $(\delta, s) \in \{-1, 1\} \times \mathbb{N}$ and an open interval λ contained in $\bigcup_k \bigcup_n T^n(W_k)$ define $G(\delta, s, \lambda)$ as the maximal sequence of the form $\{\lambda, T^\delta(\lambda), \ldots, T^{n\delta}(\lambda), \ldots\}$ that satisfy:

(6) Any one of its terms is an open interval (of $\bigcup_k \bigcup_n T^n(W_k)$) disjoint from the union of $\bigcup_{i,j} (\text{span}(A_i) - \bar{D}_i)$ and $\bigcup_{r=l+1}^{l+v} [\text{Rec}(C_r) \cap \Gamma - G_r]$. Moreover, except possibly for λ and $T^\delta(\lambda)$, any term of Σ is contained in $\bigcup_{j=0}^{s} \Gamma_j$.

Let λ be an open interval of Γ. An open non-empty subinterval I of λ is said to be a T^δ-derived subinterval of λ if it is maximal with respect to the following property: $T^\delta(I)$ is an open segment of Γ. Since, given $i \in \mathbb{N} \cup \{0\}$ and $\delta \in \{-1, 1\}$, $T^\delta|_{\Gamma_i}: \Gamma_i \to \Gamma$ is discontinuous at finitely many points ((iv) of proposition (3.2)), it is verified that:

(7) Given $\delta \in \{-1, 1\}$, the closure of any interval $\lambda \subset \Gamma$ is the finite union of the closure of its T^δ-derived subintervals.

For each $W_k \in \mathbb{A}_W$ and $s \in \mathbb{Z} - \{0\}$, define inductively on s finite or empty families $\sum_k^s = \{\sum_{k1}^s, \sum_{k2}^s, \ldots, \Sigma_{ks_k}^s\}$ of wandering T^δ-sequences, with $\delta = s/|s|$, as follows.

(8) When $s = \delta \in \{-1, 1\}$, then $s_k = 1$ and $\Sigma_{k1}^s = G(\delta, k, W_k)$. Suppose that Σ_k^{s-1} has been defined then $\Sigma_k^s = \{G(\delta, k|s|, \lambda)/\lambda$ is an open interval T^δ-derived of the last term of Σ_{kj}^{s-1}, with $j \in \{1, 2, \ldots, (s-1)_k\}\}$.

Observe that by lemma (4.11) each $G(\delta, k|s|, \lambda)$ above has finitely many terms, i.e. $G(\delta, k|s|, \lambda)$ is in fact a T^δ-sequence. Now:

(9) For a given Γ_j, the family $\{\Sigma_{kr}^s / s \in \mathbb{Z} - \{0\}, k \in \{2, 3, \ldots, n, \ldots\}, r \in \{1, 2, \ldots, s_k\}$ and its first term is contained in $\Gamma_j\}$ is finite.

To prove this, first notice that, since j is fixed, there are only finitely many terms of the form Σ_{kr}^s with $(k-1)|s| \leq j+1$. Now if some Σ_{kr}^s, with $k \geq 2$ and $(k-1)|s| > j+1$ has its first term I in Γ_j, then for $\delta = |s|/s$, I is not only T^δ-derived of the last term of some sequence of Σ_k^{k-1} but also is properly contained in such a last term. This implies that T^δ is discontinuous at one of the endpoints of I. (9) follows from the fact that $T^\delta|_{\Gamma_j}$ has finitely many discontinuity points (see (iv) of proposition (3.2)).

Observe that by (vi) of lemma (4.8), the last term of any non-empty sequence Σ_{k1}^s as above is either contained in some D_{i_0} or is disjoint from $\bigcup_i D_i$. Therefore if $n(s)$ denotes the number of all of the terms of the sequences forming Σ_k^s and Σ_k^{-s} and if $\Sigma_{kj}^s \in \Sigma_k^s$ is non-empty, we may define:

(10) $\sigma(\Sigma_{kj}^s)$ is either $[2^{-k-|s|}/n(s)] \cdot \sigma_{i_0}$ or $2^{-k-|s|}/n(s)$ according to whether the last term of Σ_{kj}^s is contained in some D_{i_0} or is disjoint from $\bigcup_i D_i$, respectively.

Now, proceed to define μ on $\bigcup_{j=1}^{s_k} \bigcup_{s \in \mathbb{Z}-\{0\}} \text{span}(\Sigma_{kj}^s)$.

Let $W_k \in \mathbb{A}_W$. Let μ be a distinguished measure on W_k such that $\mu(W_k) = \sigma(\Sigma_{k1}^1) \cdot \sigma(\Sigma_{k1}^{-1})$. If $\delta \in \{-1, 1\}$ and $T^{\delta j}(W_k)$ is a term of Σ_{k1}^δ, define μ on $T^{\delta j}(W_k)$ as $\mu \circ T^{-\delta j}|_{T^{\delta j}(W_k)}$. This implies that:

(11) If λ_1 and λ_2 are two consecutive terms of Σ_{k1}^{δ}, then the μ-coordinate expression of $T^{\delta}: \lambda_1 \to \lambda_2$ is the identity map of $(0, \mu(W_k))$.

Let $s \in \mathbb{N}$. Suppose that the distinguished measure μ has been defined on all $\Sigma_{kj}^{t} \in \Sigma_k^t$, when $t \in \{-s+1, -s+2, \ldots, -2, -1, 1, 2, \ldots, s-2, s-1\}$ and $j \in \{1, 2, \ldots, t_k\}$. Proceed inductively to define μ on all sequences of $\Sigma_k^{-s} \cup \Sigma_k^s$ as follows: Let $\delta \in \{-1, 1\}$ and $\Sigma_{kj}^{\delta s} \in \Sigma_k^{-s} \cup \Sigma_k^s$. Certainly, μ has already been defined in the first term of $\Sigma_{kj}^{\delta s}$. Using lemma (4.10) there exists a canonical distinguished measure on $\Sigma_{kj}^{\delta s}$ extending μ and having size $\sigma(\Sigma_{kj}^{\delta s})$. In this way we may assume that:

(12) μ has been defined in the terms of all sequences of Σ_k^s for $k \in \mathbb{N}$ and $s \in \mathbb{Z} - \{0\}$. Moreover if $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+1}\} \in \Sigma_k^s$, with $k \geq 2$; then for $i = 2, 3, \ldots, n$ and $\delta = |s|/s$, the μ-coordinate expression of $T^{\delta}|_{\Sigma_i}: \Sigma_i \to \Sigma_{i+1}$ is the identity map of $(0, \mu(\Sigma_i))$.

Since, for $j \in \mathbb{N} \cup \{0\}$, $T|_{\Sigma_j}$ is discontinuous at finitely many points, and by the way that the sequences Σ_k^s, with $k \in \mathbb{N}$ and $s \in \mathbb{Z} - \{0\}$, were constructed, it follows that

(13) For all $k \in \mathbb{N}$, $\bigcup_{j=1}^{s_k} \bigcup_{s \in \mathbb{Z}-\{0\}} \text{span}(\Sigma_{kj}^s)$ is contained in $\mathcal{B}_k$ and fails to be equal to it by a set which is at most denumerable.

Extend μ to a distinguished measure on each $\mathcal{B}_k$ by defining

(14) $$\mu\left(\mathcal{B}_k - \left[\bigcup_{j=1}^{s_k} \bigcup_{s \in \mathbb{Z}-\{0\}} \text{span}(\Sigma_{kj}^s)\right]\right) = 0.$$

Calculations show that

(15) $\mu(\bigcup_k \mathcal{B}_k) = \sum_k \sum_{j=1}^{s_k} \sum_{s \in \mathbb{Z}-\{0\}} \mu[\text{span}(\Sigma_{kj}^s)] \leq 1$.

Next, μ will be extended to a distinguished measure on $\bigcup_i (\text{span}(A_i))$.

By (v) of lemma (4.8), for all D_i, $(\bigcup_k \mathcal{B}_k) \cap D_i = \tilde{D}_i$ is open and dense in D_i. Since μ is defined in $\bigcup_k \mathcal{B}_k$ (see (13) and (14)), μ can be extended to D_i by defining $\mu(D_i - \tilde{D}_i) = 0$. An easy computation, using (10), implies that $0 < \mu(D_i) \leq \sigma_i$. Therefore, by (5):

(16) For all $A_i \in \mathbb{A}_A$, μ can be extended to a canonical distinguished measure on A_i having order i and size 2^{-i}.

It follows from this construction that

(17) $\mu(\bigcup_i \text{span}(A_i)) = \sum_i \mu(\text{span}(A_i)) \leq 1$.

Finally, μ will be extended to $\bigcup_{r=l+1}^{l+v} \text{Rec}(C_r) \cap \Gamma$.

By (iv) and (v) of lemma (4.8), for all $r \in \{l+1, l+2, \ldots, l+v\}$, $(\bigcup_k \mathcal{B}_k) \cap G_r = \tilde{G}_r$ is open and dense in G_r. Since μ is defined in $\bigcup_k \mathcal{B}_k$, extend μ to G_r by defining $\mu(G_r - \tilde{G}_r) = 0$. Using lemma (4.6) μ can be extended to $\bigcup_{r=l+1}^{l+v} C_r$. Let I be an open interval of $\bigcup_{r=l+1}^{l+v} C_r$. If for some $n \in \mathbb{N}$, $T^n(I)$ is an interval and $(\bigcup_{i=1}^{n} T^i(I)) \cap (\bigcup_{r=l+1}^{l+v} C_r) = \emptyset$, define $\mu|_{T^n(I)} = \mu \circ T^{-n}|_{T^n(I)}$. Thus again, as in (3), by (iv) of proposition (3.2), μ is already defined in $\bigcup_{r=l+1}^{l+v} \text{Rec}(C_r) \cap \Gamma$ except for a denumerable subset. Take the μ measure of this denumerable set to be zero. As in the proof of (3) it can be seen that

(18) For all $j \in \mathbb{N}$, $\mu(\bigcup_{r=l+1}^{l+v} \text{Rec}(C_r) \cap \Gamma_j) < \infty$.

Under these circumstances, by (1), μ is defined in an open and dense subset of Γ. Extend μ to the whole of Γ by making of measure zero the remainder set where μ has not yet been defined.

Now it will be proved that T is μ-smooth. It follows by (2), (3), (15), (17) and (18) that for all $j \in \mathbb{N}$, $\mu(\Gamma_j) < \infty$. By this and by the manner that μ was constructed, it is verified that:

(19) μ is a distinguished measure on Γ.

Let $\Sigma \subset \Gamma$ be such that both Σ and $T(\Sigma)$ are segments. Orient Σ and $T(\Sigma)$ in such a way that $T|_\Sigma: \Sigma \to T(\Sigma)$ is orientation preserving. By the construction of μ, the connected components of the intersection of Σ with the union of $\bigcup_j \operatorname{span}(P_j)$, $\bigcup_{t=1}^{l} \bigcup_n T^n(C_t)$, $\bigcup_i (\operatorname{span}(A_i) - \bar{D}_i)$, $\bigcup_k \mathcal{B}_k$ and $\bigcup_{r=l+1}^{l+v} \{\operatorname{Rec}(C_r) \cap \Gamma - G_r\}$ form a family $\{I_s\}_{s\in S}$ of subintervals of Σ such that

(20) $\sum_{s\in S} \mu(I_s) = \mu(\Sigma)$.

By enlarging Σ if necessary we may assume that

(21) The endpoints of Σ are disjoint from $\bigcup_{s\in S} I_s$.

It follows from (9), (12), (13) and (14) that there are at most finitely many terms $I_{\bar{s}}$ of the family $\{I_s\}_{s\in S}$ such that $I_{\bar{s}} \subset \bigcup_k \mathcal{B}_k$ and the μ-coordinate expression of $T|_{\bar{I}_s}: \bar{I}_{\bar{s}} \to T(\bar{I}_{\bar{s}})$ is not the identity map of $[0, \mu(I_{\bar{s}})]$. Similarly, proceeding to check the μ-coordinate expression of $T|_{\bar{I}_s}$ for every interval I_s of $\{I_s\}_{s\in S}$, it can be seen, by (20) and (21), that lemma (4.12) can be used to prove that the μ-coordinate expression $\tilde{T}$ of $T|_{\bar{\Sigma}}: \bar{\Sigma} \to T(\bar{\Sigma})$ is C^1 (resp. smooth) when Σ meets (resp. does not meet) $\bigcup_{r=l+1}^{l+v} \operatorname{Rec}(C_r)$, and also that if $x \in \{0, \mu(\Sigma)\}$, the derivative $\tilde{T}'$ of $\tilde{T}$ satisfies $\tilde{T}'(x) = 1$ (resp. $\tilde{T}'$ has infinite order contact at x with the constant map $\equiv 1$). This finishes the proof of the proposition because $\bigcup_{r=l+1}^{l+v} \operatorname{Rec}(C_r)$ is far away from the fixed points of φ which implies, by (ii) and (iii) of proposition (3.2), the existence of a non-negative integer n_0 such that, for all $i \geq n_0$,

$$\Gamma_i \cap \left(\bigcup_{r=l+1}^{l+v} \operatorname{Rec}(C_r) \right) = \varnothing. \qquad \square$$

(4.14) *Proof of proposition* (4.3). Let us assume all considerations and notation of propositions (3.2) and (4.2). Let Δ be the union of the set of arcs of trajectory $\overrightarrow{pq}$ of φ such that: $p, q \in \Gamma$, $T(p) = q$ and either T is discontinuous at p or T^{-1} is discontinuous at q. Let $\mathcal{Y}$ be the set of connected components of $M - (F \cup \Delta \cup \Gamma)$. The closure $\bar{Y}$ of $Y \in \mathcal{Y}$ may fail to be a flow box only because the 'transversal edges' meet each other. However $\bar{Y}$ can be expressed as the union of two flow boxes Y_1 and Y_2 such that $Y_1 \cap Y_2$ is a global cross-section for $\varphi|_{\bar{Y}}$ and also each Y_i, $i = 1, 2$, shares with Y exactly one transversal edge. Let $\mathcal{C}$ be the union of all transversal edges of Y_i, with $i \in \{1, 2\}$ and $Y \in \mathcal{Y}$. From this construction:

(1) $\mathcal{C} \supset \Gamma$.

By (iv) of proposition (3.2), given $i \in \mathbb{N}$, there are only finitely many connected components of $M - (F \cup \Delta \cup \Gamma)$ intersecting $M - M_i$. Therefore the family $\{Y_i / i \in \{1, 2\}$ and $y \in \mathcal{Y}\}$ of the closure of the connected components of $M - (F \cup \Delta \cup \mathcal{C})$ can be enumerated as $\theta_1, \theta_2, \ldots, \theta_j, \ldots$ so that for all $i \in \mathbb{N}$, there exists $k(i) \in \mathbb{N}$ such that if $j \geq k(i)$ then $\theta_j \cap (M - M_i) = \varnothing$. Certainly

(2) $M - F = \bigcup_j \theta_j$.

Since any compact subset of $M - F$ is contained in some $M - M_i$, (a) of this proposition is true. (b) is verified by construction of $\{\theta_j\}_j$.

The distinguished measure μ on Γ will be extended to $\mathcal{C}$. Denote by $\tilde{T}: \mathcal{C} \to \mathcal{C}$ the forward Poincaré map induced by φ. Given $Y \in \mathcal{Y}$ assume that the flow box crosses $Y_1 \cap Y_2$ from Y_1 to Y_2. Define $\mu|_{Y_1 \cap Y_2} = \mu \circ \tilde{T}^{-1}|_{Y_1 \cap Y_2}$. It follows from this and from proposition (4.2), that (c) and (d) of this proposition are verified. □

5. *Proof of the smoothing theorem*

It is obvious that (c)⇒(b). From the Denjoy & Schwartz theorem we have (b)⇒(a), so we will only prove (a)⇒(c). To do this, we shall begin by supposing that

(1) All minimal sets of φ are trivial.

First, assume that φ has no fixed point. Then M is either the torus or the Klein bottle. If φ has no non-trivial recurrent trajectories, then (as φ has no fixed point) by Neumann's smoothing theorem [**Ne.2**], φ is topologically equivalent to a smooth flow. If φ has a non-trivial recurrent trajectory, by the Poincaré-Bendixson theorem for the Klein bottle [**Ma**], M is the torus; this fact and (1) imply that φ is topologically equivalent to an irrational flow which - in particular - is smooth. Therefore we shall proceed with the proof assuming that:

(2) The set F of fixed points of φ is not empty.

Also, we shall assume that

(3) Any subinterval of $\mathbb{R}$ is provided with the canonical positive orientation.

Let $M - F = \bigcup_{i=1}^{\infty} \theta_i$, where $\theta_1, \theta_2, \ldots, \theta_i, \ldots$ are the flow boxes of φ satisfying all the conditions of proposition (4.3). A closed arc of trajectory σ with the positive orientation induced by φ will be called a tangent elementary arc if it is contained in the boundary of some θ_i and intersects the set formed by all the corners of the flow boxes $\theta_1, \theta_2, \ldots, \theta_i, \ldots$ exactly at their endpoints. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be all the tangent elementary arcs forming one of the edges of a flow box θ_i. Choose positive real numbers $\sigma_1(i), \sigma_2(i), \ldots, \sigma_n(i)$ such that $\sum_{j=1}^{n} \sigma_j(i) = 1$.

Let σ be a tangent elementary arc. Since σ is in the boundary of two flow boxes, say θ_i and θ_j, we have chosen two real numbers $\sigma(i)$ and $\sigma(j)$. Now, proceed to choose for each $k \in \{i, j\}$ an orientation preserving homeomorphism $h_{\sigma(k)}: \sigma \to [0, \sigma(k)]$ such that $h_{\sigma(i)} \circ h_{\sigma(j)}^{-1}$ is a smooth difeomorphism having derivative $\equiv 1$ in a neighbourhood of 0 and $\sigma(j)$.

The smooth submanifold $M - F$ of M considered as a set of points without its differentiable structure will be denoted by $\widehat{M - F}$. A new smooth structure on $\widehat{M - F}$ will be constructed. Given θ_i denote by A_{i1} and A_{i2} the transversal edges of θ_i. Assume that the flow goes from A_{i1} to A_{i2}. Given $j \in \{1, 2\}$, there are two μ-homeomorphisms $g_{ijk}: A_{ij} \to [0, \mu(A_{ij})]$, $k = 1, 2$, determined by the two possible orientations of A_{ij}. For each g_{ijk} choose a surjective continuous flow box of φ:

$$\alpha_{ijk} : [1, 2] \times [0, \mu(A_{ij})] \to \theta_i,$$

such that

(4 a) For all $t \in [0, \mu(A_{ij})]$, $\alpha_{ijk}([1, 2] \times \{t\})$ is an arc of trajectory of $\varphi|_{\theta_i}$.

(4 b) $(\alpha_{ijk})^{-1}$ restricted to each (oriented) tangent elementary arc $\sigma = \overline{ab}$ of θ_i, starting at a, coincides with $h_{\sigma(i)}$ + first coordinate of $\alpha_{ijk}^{-1}(a)$.

(4 c) $(\alpha_{ijk})^{-1}|_{A_{ij}}$ is the μ-homeomorphism $g_{ijk}: A_{ij} \to \{j\} \times [0, \mu(A_{ij})]$.

Let $\mathscr{A}_1 = \{\hat{\alpha}_{ijk} = \alpha_{ijk}|_{(1,2) \times (0, \mu(A_{ij}))} : (1, 2) \times (0, \mu(A_{ij})) \to \widehat{M - F}$, such that $j, k \in \{1, 2\}$ and $i \in \mathbb{N}\}$. This set will form part of the new coordinate system for $\widehat{M - F}$. Notice

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that all changes of coordinates $\hat{\alpha}_{ijk} \circ \hat{\alpha}^{-1}_{i\tilde{j}\tilde{k}}$, with $j, k, \tilde{j}, \tilde{k} \in \{1, 2\}$, $i = 1, 2, \ldots$ are smooth because, by proposition (4.3), the flow box θ_i is μ-smooth.

Let σ be a tangent elementary arc. Suppose that σ is contained in $\partial\theta_i \cap \partial\theta_l$. Choose orientations on A_{i1} and A_{l1}, that is choose $k, \tilde{k} \in \{1, 2\}$, so that

$$\begin{aligned}\sigma &= \alpha_{i1k}([\varepsilon_{i1}, \varepsilon_{i2}] \times \{\mu(A_{i1})\}) \\ &= \alpha_{l1\tilde{k}}([\varepsilon_{l1}, \varepsilon_{l2}] \times \{0\}),\end{aligned}$$

where $[\varepsilon_{i1}, \varepsilon_{i2}]$ and $[\varepsilon_{l1}, \varepsilon_{l2}]$ are subintervals of $[1, 2]$. Let $\varepsilon = \min\{\mu(A_{i1}), \mu(A_{l1})\}$. Define $\beta(\sigma)\colon (\varepsilon_{l1}, \varepsilon_{l2}) \times (-\varepsilon, \varepsilon) \to \widehat{M - F}$ as follows

$$(5) \qquad \beta(\sigma)(s, t) = \begin{cases} \alpha_{l1\tilde{k}}(s, t) & \text{if } t \geq 0 \\ \alpha_{i1k}(h_{\sigma(i)} \circ h^{-1}_{\sigma(l)}(s - \varepsilon_{l1}) + \varepsilon_{i1}, \mu(A_{i1}) + t), & \text{if } t \leq 0. \end{cases}$$

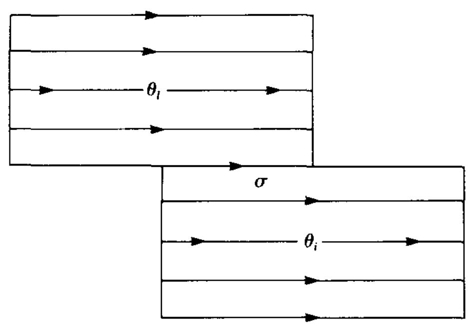

FIGURE 4

Notice that $h_{\sigma(i)}\colon \sigma \to [0, \varepsilon_{i2} - \varepsilon_{i1}]$ and $h_{\sigma(l)}\colon \sigma \to [0, \varepsilon_{l2} - \varepsilon_{l1}]$. It follows from (4b) that $\beta(\sigma)$ is well defined. All possible changes of coordinates involving $\beta(\sigma)$, $\hat{\alpha}_{l1\tilde{k}}$, $\hat{\alpha}_{i1k}$ are smooth because $h_{\sigma(i)} \circ h^{-1}_{\sigma(l)}$ is smooth. This implies that all possible changes of coordinates involving elements of $\mathscr{A}_2 = \{\beta(\sigma)/\sigma$ is a tangent elementary arc$\} \cup \mathscr{A}_1$ are smooth.

An oriented closed segment transverse to the flow φ will be called a transversal elementary arc if it is contained in the boundary of some θ_i and intersects the set formed by all the corners of the flow boxes $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ exactly at their endpoints. Let Σ be a transversal elementary arc. Suppose that $\Sigma \subset A_{i2} \cap A_{l1}$, where A_{i2} and A_{l1} are transversal edges of θ_i and θ_l, respectively, and moreover that the flow φ crosses Σ from θ_i to θ_l. Extend the orientation of Σ to an orientation of both A_{i2} and A_{l1}. Let $k, \tilde{k} \in \{1, 2\}$ be such that $g_{i2k}\colon A_{i2} \to [0, \mu(A_{i2})]$ and $g_{l1\tilde{k}}\colon A_{l1} \to [0, \mu(A_{l1})]$ preserve orientations. Let $[\delta_1, \delta_2] \subset [0, \mu(A_{i2})]$ and $[\varepsilon_1, \varepsilon_2] \subset [0, \mu(A_{l1})]$ be such that

$$\Sigma = \alpha_{i2k}(\{2\} \times [\delta_1, \delta_2]) = \alpha_{l1\tilde{k}}(\{1\} \times [\varepsilon_1, \varepsilon_2]).$$

It follows from (4c) that for all $t \in \Sigma$

(6) $g_{i2k}(t) - \delta_1 = g_{l1\tilde{k}}(t) - \varepsilon_1$. In particular $\varepsilon_2 - \varepsilon_1 = \delta_2 - \delta_1$.

Define $\beta(\Sigma)\colon (1, 3) \times (\delta_1, \delta_2) \to \widehat{M - F}$ as follows

$$\beta(\Sigma)(s, t) = \begin{cases} \alpha_{i2k}(s, t), & \text{if } s \in (1, 2], \\ \alpha_{l1\tilde{k}}(s - 1, t + \varepsilon_1 - \delta_1), & \text{if } s \in [2, 3). \end{cases}$$

Because of (6), $\beta(\Sigma)$ is well defined. It is clear that all possible changes of coordinates involving $\beta(\Sigma)$, α_{i2k} and $\alpha_{l1\tilde{k}}$ are smooth. This implies that all possible changes of coordinates involving elements of $\mathscr{A}_3 = \mathscr{A}_2 \cup \{\beta(\Sigma)/\Sigma$ is a transversal elementary arc$\}$ are smooth.

Now, we will construct coordinate systems around the corners of the flow boxes $\theta_1, \theta_2, \ldots, \theta_n, \ldots$. Let p be a corner of an arbitrary θ_i. There are three cases to be considered

(7 a) The set $\{\theta_r / r \in \mathbb{N}, \theta_r$ meets $p\} = \{\theta_i, \theta_l, \theta_n, \theta_s\}$ and p is a corner of all of them.

(7 b) (resp. (7 c)). The set $\{\theta_r / r \in \mathbb{N}, \theta_r$ meets $p\} = \{\theta_i, \theta_l, \theta_n\}$, p is an endpoint of two consecutive tangent (resp. transversal) elementary arcs of θ_l and p is a corner of θ_n and θ_i. See figure 5.

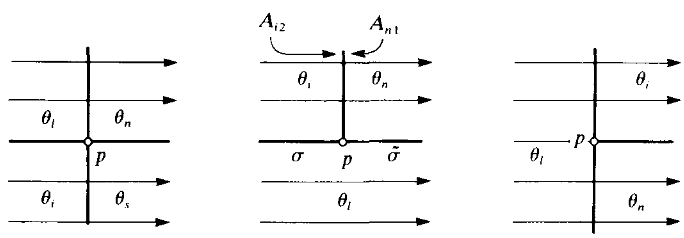

FIGURE 5

Suppose that p is as in case (7b). Let σ (resp. $\tilde{\sigma}$) be the elementary arc of θ_l and θ_i (resp. θ_n) having p as an endpoint. Suppose that the flow φ passes by σ before passing $\tilde{\sigma}$. See figure 5. Choose orientations of A_{i2}, A_{n1} and A_{l1}, that is choose k, $\tilde{k}$, $\hat{k} \in \{1, 2\}$, so that for some subintervals $[\varepsilon_{l1}, \varepsilon_{l2}]$, $[\varepsilon_{l2}, \varepsilon_{l3}]$, $[\varepsilon_{i1}, \varepsilon_{i2}]$, $[\varepsilon_{n1}, \varepsilon_{n2}]$ of $[1, 2]$ it holds that

$$\sigma = \alpha_{l1k}([\varepsilon_{l1}, \varepsilon_{l2}] \times \{\mu(A_{l1})\}) = \alpha_{i2\tilde{k}}([\varepsilon_{i1}, \varepsilon_{i2}] \times \{0\})$$
$$\tilde{\sigma} = \alpha_{l1k}([\varepsilon_{l2}, \varepsilon_{l3}] \times \{\mu(A_{l1})\}) = \alpha_{n1\hat{k}}([\varepsilon_{n1}, \varepsilon_{n2}] \times \{0\}).$$

Let Σ be the transversal elementary arc contained in A_{2i} and having p as an endpoint.

(8) Take $\varepsilon \in (0, \mu(\Sigma))$ so small that for all $s \in [0, \varepsilon]$,

$$(h_{\sigma(l)} \circ h^{-1}_{\sigma(i)})'(s) = h_{\tilde{\sigma}(l)} \circ h^{-1}_{\tilde{\sigma}(n)})'(s) = 1.$$

Define $\beta(p): (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to \widehat{M - F}$ as follows

$$\beta(p)(s, t) = \begin{cases} \alpha_{l1k}(s - \varepsilon_{l2}, t + \mu(A_{l1})), & t \leq 0 \\ \alpha_{i2\tilde{k}}(s + \varepsilon_{i2}, t), & s \leq 0 \text{ and } t \geq 0 \\ \alpha_{n1\hat{k}}(s + \varepsilon_{n1}, t), & s \geq 0 \text{ and } t \geq 0. \end{cases} \tag{9}$$

It follows from (4b), (4c) and (8) that $\beta(p)$ is not only well defined but also that all change of coordinates involving $\beta(p)$, $\beta(\sigma)$, $\beta(\tilde{\sigma})$ and $\beta(\Sigma)$ are smooth.

The construction of coordinate systems when p is either as in case (7a) or (7c) is similar to the case above. Therefore, $\mathscr{A} = \mathscr{A}_3 \cup \{\beta(p)/p$ is a corner of some θ_i, $i = 1, 2, \ldots\}$ gives a smooth system of coordinates for $\widehat{M - F}$ which provides it with a smooth manifold structure denoted by $\widetilde{M - F}$. It is claimed that

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(10) The foliation $\varphi|_{\widetilde{M-F}}$ on $\widetilde{M-F}$ is smooth and topologically equivalent to the continuous foliation $\varphi|_{M-F}$ on $M-F$.
In fact, the elements of $\mathscr{A}$ are also smooth flow boxes for $\varphi|_{\widetilde{M-F}}$, this implies that $\varphi|_{\widetilde{M-F}}$ is a smooth foliation on $\widetilde{M-F}$. Since each element of $\mathscr{A}$ is a continuous flow box of $\varphi:\mathbb{R}\times M\to M$, the identity map $\mathrm{Id}: M-F\to\widetilde{M-F}$ is a homeomorphism which provides the required topological equivalence. This proves (10).

Proceeding as D. Neumann [**Ne.2**, theorem 5.1], by [**Mu**, theorem 6.3], there is a C^∞ diffeomorphism k of $M-F$ onto $\widetilde{M-F}$ (theorem (6.3) is stated for 3-manifolds but the theorem and its proof are valid for arbitrary manifolds of dimension $m\le 3$), in fact k can be chosen within any pre-assigned continuous function $\delta: M-F\to(0,\infty)$ of the identity. Observe that $k: M-F\to M-F$ is a homeomorphism because, as we said above, the identity $\widetilde{M-F}\to M-F$ is a homeomorphism. It may be assumed that $\delta(x)$ tends to zero as x approaches any point of F and hence that k extends to a homeomorphism of M that fixes each rest point of φ.

Let $\mathscr{F}$ be the smooth orientable foliation on $M-F$ such that $k(\mathscr{F})=\varphi|_{\widetilde{M-F}}$. It will be seen that:

(11) There is a smooth vector field $Y\in\mathfrak{X}^\infty(M)$ whose set of singularities is precisely F and such that $Y|_{M-F}$ and $\mathscr{F}$ have the same phase portrait.
In fact, let $X\in\mathfrak{X}^\infty(M-F)$ be such that the foliation that it induces is $\mathscr{F}$. Using proposition (3.2), there is a family $\{M_i/i=2,\ldots,n,\ldots\}$ of compact subsets of M such that $\bigcap_{i=1}^\infty M_i=F$ and for all $i=1,2,\ldots,n,\ldots$, $M_{i+1}\subset\mathrm{Int}(M_i)$. It may be assumed that $M_1=M$. Define $V_i=\mathrm{Int}(M_i)-M_{i+2}$. Certainly $\{V_i/i=1,2,\ldots,n,\ldots\}$ is a locally finite open covering of M. Let $\{\psi_i: M\to[0,1]/i=1,2,\ldots,n,\ldots\}$ be a partition of unity strictly subordinate to this covering. Thus, the support of ψ_i is contained in V_i. For $r=0,1,2,\ldots$, give a norm $\|\ \|_r$ on $\mathfrak{X}^\infty(M)$ compatible with its C^r-topology and such that, for all $Z\in\mathfrak{X}^\infty(M)$, $\|Z\|_{r+1}\ge\|Z\|_r$. Extend X to a vector field $\hat{X}$ on M by defining $\hat{X}(p)=0$ for all $p\in F$. Certainly each $\psi_i\cdot\hat{X}\in\mathfrak{X}^\infty(M)$. Given $i=1,2,\ldots$ let c_i be a real positive number such that $\|c_i\psi_i\hat{X}\|_i\le 1/2^i$. Since, for all $r=1,2,3,\ldots$,

$$\sum_{i=1}^{\infty}\|c_i\psi_i\hat{X}\|_r\le\sum_{i=1}^{r}\|c_i\psi_i\hat{X}\|_r+\sum_{j>r}^{\infty}\frac{1}{2^j}<\infty,$$

the series $(\sum_{i=1}^\infty c_i\psi_i)\hat{X}=\sum_{i=1}^\infty c_i\psi_i\hat{X}$ converges to a smooth vector field Y as required to prove (11).

The induced flow ψ of the vector field Y, obtained in (11), is smooth and topologically equivalent, under k, to φ. This proves the theorem under the assumption (1).

Now suppose that φ has non-trivial minimal sets. Proceeding as above and using the same notation in the corresponding cases, $\widehat{M-F}$ can be provided with a C^1 manifold structure denoted by $\widetilde{M-F}$ so that (see (10)):

(10′) The foliation $\varphi|_{\widetilde{M-F}}$ on $\widetilde{M-F}$ is C^1 and topologically equivalent to the continuous foliation $\varphi|_{M-F}$ on $M-F$. Moreover there exists a compact subset $K\subset\widetilde{M-F}$ such that the submanifold $\widetilde{M-(F\cup K)}$ of $M-F$ is smooth and $\varphi|_{\widetilde{M-(F\cup K)}}$ is smooth.

Taking an appropriate subatlas, we may assume that $\widetilde{M-F}$ is smooth. Certainly, even in this new structure $\varphi|_{\widetilde{M-F}}$ is only C^1. However, under these conditions, by (10′), the proof that φ is topologically equivalent to a C^1 flow, can be done in the same way as the one above under the assumption (1).

6. *The existence theorem*

In the following proof, given a topological space X, a subset $A \subset X$ and an injective function $f: A \to X$ such that $f(A) \cap A = \varnothing$, we shall denote by X/f the quotient space obtained from X by identifying x with $f(x)$, for all $x \in A$.

(6.1) *Proof of the existence theorem.* First, let us prove (E 1). Let $\mathcal{R}$ be the equivalence relation on $\mathbb{R}/\mathbb{Z} \times [-1, 1]$ such that the equivalence class containing a point p is the union of $\{p\}$ and the connected component of

$$\{(\mathbb{R}/\mathbb{Z} - \operatorname{Dom}(T)) \times \{1\}\} \cup \{(\mathbb{R}/\mathbb{Z} - \operatorname{Dom}(T^{-1})) \times \{-1\}\}$$

which contains p.

This relation $\mathcal{R}$ has been defined so that the map $(x, 1) \mapsto (T(x) - 1)$ (defined in $\operatorname{Dom}(T) \times \{1\})$), admits an extension to a homeomorphism

$$\tau: [(\mathbb{R}/\mathbb{Z} \times \{1\})/\mathcal{R} - \{P_1, P_2, \ldots, P_n\}] \to [(\mathbb{R}/\mathbb{Z} \times \{-1\})/\mathcal{R} - \{P_{-1}, P_{-2}, \ldots, P_{-n}\}],$$

where $p_{\delta \cdot 1}, P_{\delta \cdot 2}, \ldots, P_{\delta \cdot n}$, with $\delta \in \{-1, 1\}$ are the points of $[(\mathbb{R}/\mathbb{Z} - \operatorname{Dom}(T^\delta)) \times \{\delta\}]/\mathcal{R}$ which are not contained in $[h^{-1}(\operatorname{Dom}(E^\delta)) \times \{\delta\}]/\mathcal{R}$. Let N be the quotient manifold

$$[(\mathbb{R}/\mathbb{Z} \times [-1, 1])/\mathcal{R} - \{P_1, \ldots, P_n, P_{-1}, \ldots, P_{-n}\}]/\tau.$$

It is clear that the constant vector field $(0, 1)$ induces an orientable foliation $\mathcal{F}$ on N (see figure 6) whose set of singularities is

$$[(\mathbb{R}/\mathbb{Z} - \operatorname{Dom}(T)) \times \{1\}]/\mathcal{R} - \{P_1, \ldots, P_n, P_{-1}, \ldots, P_{-n}\}.$$

Certainly there exists a C^∞ compact manifold $\tilde{M}$ and a finite subset $F \subset \tilde{M}$ such that, up to a homeomorphism, $N = \tilde{M} - F$. Therefore, using the smoothing corollary, up to topological equivalence, there exists a C^1 flow $\psi: \mathbb{R} \times \tilde{M} \to \tilde{M}$ such that $\psi|_{\tilde{M}-F} = \mathcal{F}$. As $(\psi|_{\tilde{M}-F}, \tilde{M} - F)$ is a suspension of (T, E), (E 1) is proved.

It is easy to see that (E 3a) implies (E 3a′). Let us show that (E 3a′) implies (E 3a). Let $\mathcal{A}$ be the family of residual subsets of $\mathbb{R}/\mathbb{Z}$. Let $A = \{x \in \bigcap_{n \in \mathbb{Z}} \operatorname{Dom}(E^n)/h^{-1}(x)$

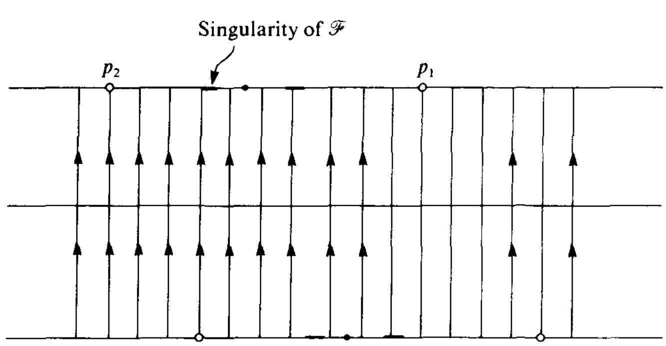

FIGURE 6

is a one point set} and $\tau = T|_{h^{-1}(A)}$. The injectivity of $h|_{h^{-1}(A)}$ and τ will be used to prove that

(1) For all $n \in \mathbb{Z}$, $h(\tau^n(\mathrm{Dom}\,(\tau^n)) \in \mathscr{A}$.

In fact, by assumption $\mathrm{Dom}\,(T) \cap h^{-1}(A) = \mathrm{Dom}\,(\tau)$ and so $h(\mathrm{Dom}\,(T)) \cap A = h(\mathrm{Dom}\,(\tau)$. Therefore $h(\mathrm{Dom}\,(\tau)) \in \mathscr{A}$. Suppose inductively that for some $n \in \mathbb{N}$, $h(\tau^n(\mathrm{Dom}\,(\tau^n)) \in \mathscr{A}$. This implies that

$$h[\tau^n(\mathrm{Dom}\,(\tau^n)) \cap \mathrm{Dom}\,(\tau)] = h(\tau^n(\mathrm{Dom}\,(\tau^n)) \cap h(\mathrm{Dom}(\tau) \in \mathscr{A}.$$

Therefore

$$Eh[\tau^n(\mathrm{Dom}\,(\tau^n)) \cap \mathrm{Dom}(\tau)] = h\tau[\tau^n(\mathrm{Dom}\,(\tau^n)) \cap \mathrm{Dom}\,(\tau)] \in \mathscr{A}$$

but

$$\tau[\tau^n(\mathrm{Dom}\,(\tau^n)) \cap \mathrm{Dom}\,(\tau)] \subset \tau^{n+1}(\mathrm{Dom}\,(\tau^{n+1})).$$

Therefore $h(\tau^{n+1}(\mathrm{Dom}\,\tau^{n+1})) \in \mathscr{A}$.

Now observe that $h(\mathrm{Dom}\,(T)) \in \mathscr{A}$ implies that $Eh(\mathrm{Dom}\,(T)) = h(T(\mathrm{Dom}\,(T)) = h(\mathrm{Dom}\,(T^{-1}) \in \mathscr{A}$. Therefore, proceeding as above, for all $n \in \mathbb{N}$, $h(\tau^{-n}(\mathrm{Dom}\,\tau^{-n})) \in \mathscr{A}$. This finishes the proof of (1).

It follows from (1) that $\bigcap_{n\in\mathbb{Z}} h(\tau^n(\mathrm{Dom}\,(\tau^n)) \in \mathscr{A}$. If $y \in \bigcap_{n\in\mathbb{Z}} h(\tau^n(\mathrm{Dom}\,(\tau^n))$, it is clear that, for all $n \in \mathbb{Z}$, $y \in \mathrm{Dom}\,(T^n)$. Moreover, as T covers an interval exchange transformation (via h), y is a non-trivial recurrent point of T. Therefore (E 3a′) implies (E 3a).

Using (E 3), it is easy to see that (E 4a′) implies (E 4a). Let us prove that (E 4a) implies (E 4a′). It is clear that h cannot be a homeomorphism. Since T has a non-trivial minimal set Ω, $\Omega \cap \mathbb{R}/\mathbb{Z}$ is a compact subset of $\mathrm{Dom}\,(T)$. It follows from this that there exists finitely many connected components $V_1, V_2, \ldots, V_n$ of $\mathrm{Dom}\,(T)$ whose union contains $\Omega \cap \mathbb{R}/\mathbb{Z}$. Therefore, as $T \in \mathscr{C}(E, h)$, if $U_1, U_2, \ldots, U_n$ are the connected components of $\mathbb{R}/\mathbb{Z} - \bigcup_{i=1}^n V_i$ then $h(U_i)$ is a one point set but $h^{-1}(h(U_i))$ is not a one point set. By defining $S = \{h(U_1), h(U_2), \ldots, h(U_n)\}$, we may easily check that (E 4a) implies (E 4a′).

The existence of $\hat{T}$, as in (E 2), is obvious. The remainder of the proof of (E 2) is trivial.

(E 5) is a corollary of the smoothing theorem and of the smoothing corollary.

□

(6.2) *Examples.* Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a non-periodic geometric rotation defined everywhere. Let h_1 (resp. h_2)$: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a monotone continuous map of degree one such that, for some $x_0 \in \mathbb{R}/\mathbb{Z}$, the set $\{x \in \mathbb{R}/\mathbb{Z}$ such that $h_1^{-1}(x)$ (resp. $h_2^{-1}(x)$) is not a one point set} is precisely the orbit $\mathcal{O}(x)$ (resp. the half-orbit $\mathcal{O}^+(x)$) of R_α. Let $\hat{T}_1 \in \mathscr{C}(E, h_1)$ (resp. $\hat{T}_2 \in \mathscr{C}(E, h_2)$) be maximal as in (E 2) (of the existence theorem). Let $y_0 \in \mathbb{R}/\mathbb{Z}$ be such that $h^{-1}(h(y_0)) = y_0$ and define $T_3 = \hat{T}_1|_{\mathbb{R}/\mathbb{Z}-\{y_0\}}$. Given $T \in \{\hat{T}_1, \hat{T}_2$ and $T_3\}$ denote by φ_T the differentiable flow obtained by applying (E 5) to the suspension of (T, E) constructed in the proof of (E 1). Then:

(a) $\varphi_{\hat{T}_1}$ is the *Denjoy* example **[De]**.

(b) $\varphi_{\hat{T}_2}$ is ‘essentially’ a smooth version of the *Cherry* example **[Chr]**.

(c) φ_{T_3} is the *Katok* example.

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