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FINITE GROUPS IN WHICH SOME PROPERTY OF TWO-GENERATOR SUBGROUPS IS TRANSITIVE

COSTANTINO DELIZIA, PRIMOŽ MORAVEC AND CHIARA NICOTERA

Finite groups in which a given property of two-generator subgroups is a transitive relation are investigated. We obtain a description of such groups and prove in particular that every finite soluble-transitive group is soluble. A classification of finite nilpotent-transitive groups is also obtained.

1. INTRODUCTION

Let $\mathfrak{X}$ be a group theoretical class. A group G is said to be *$\mathfrak{X}$-transitive* (or an $\mathfrak{X}$T-*group*) if for all $x, y, z \in G\backslash\{1\}$ the relations $\langle x, y\rangle \in \mathfrak{X}$ and $\langle y, z\rangle \in \mathfrak{X}$ imply $\langle x, z\rangle \in \mathfrak{X}$. In graph theoretical terms, let $\Gamma_{\mathfrak{X}}(G)$ be the simple graph whose vertices are the nontrivial elements of G, and a and b are connected by an edge if and only if $\langle a, b\rangle \in \mathfrak{X}$. Then G is an $\mathfrak{X}$T-group precisely when all the connected components of $\Gamma_{\mathfrak{X}}(G)$ are complete graphs. Several authors have studied $\mathfrak{X}$T-groups for some special classes $\mathfrak{X}$. When $\mathfrak{X}$ is the class of all Abelian groups, these groups are also known as commutative-transitive groups or CT-*groups*. Weisner [**10**] has shown that finite CT-groups are either soluble or simple. Finite nonabelian simple CT-groups have been classified by Suzuki [**6**]. These are precisely $\mathrm{PSL}(2, 2^f)$, where $f > 1$. A characterisation of finite soluble CT-groups has been given by Wu [**11**] who has also obtained information on locally finite CT-groups and polycyclic CT-groups. When $\mathfrak{X} = \mathfrak{N}_c$, the class of all groups which are nilpotent of class $\leqslant c$, similar results have been obtained in [**1**].

The purpose of this note is to obtain a description of finite $\mathfrak{X}$T-groups for the group theoretical classes $\mathfrak{X}$ having the following properties:

($\star$) $\mathfrak{X}$ is subgroup closed, it contains all finite Abelian groups and is bigenetic in the class of all finite groups.

Here a class $\mathfrak{X}$ is said to be *bigenetic* (a terminology due to Lennox [**4**]) in the class of all finite groups when a finite group G is in $\mathfrak{X}$ if and only if all its two-generator subgroups are. Examples of classes satisfying ($\star$) are the class of all Abelian groups, all nilpotent

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groups, all supersoluble groups and all soluble groups. First we show that if $\mathfrak{X}$ is a class satisfying ($\star$), then every finite $\mathfrak{X}$T-group which does not belong to $\mathfrak{X}$ is either a Frobenius group with kernel and complement belonging to $\mathfrak{X}$, or it has no normal $\mathfrak{X}$-subgroups, that is, it is *$\mathfrak{X}$-semisimple* as defined in [**5**]. We also show that in several cases, for example, in the soluble or supersoluble case, the second possibility does not occur. As a consequence we obtain that a finite group is soluble if and only if it is soluble-transitive. In the case when $\mathfrak{X} = \mathfrak{N}$, the class of all nilpotent groups, there exist simple $\mathfrak{N}$T-groups. We obtain a complete classification of finite $\mathfrak{N}$T-groups which generalises some results of [**11**].

2. Results

Given a group theoretical class $\mathfrak{X}$, let $R_{\mathfrak{X}}(G)$ be the product of all normal $\mathfrak{X}$-subgroups of G (the *$\mathfrak{X}$-radical* of G). In general $R_{\mathfrak{X}}(G)$ does not belong to $\mathfrak{X}$. Our first result shows that this is however true within the class of all finite $\mathfrak{X}$-transitive groups when $\mathfrak{X}$ satisfies the properties ($\star$).

Lemma 2.1. *Let $\mathfrak{X}$ be a class of groups satisfying ($\star$) and let G be a finite $\mathfrak{X}$T-group. Then $R_{\mathfrak{X}}(G)$ is an $\mathfrak{X}$-group.*

Proof: Let M and N be normal $\mathfrak{X}$-subgroups of G. It suffices to show that MN also belongs to $\mathfrak{X}$. Suppose first that $M \cap N \neq 1$ and let $x \in M \cap N\backslash\{1\}$. First note that for any $m \in M\backslash\{1\}$ and $n \in N\backslash\{1\}$ we have that $\langle m, x\rangle$ and $\langle x, n\rangle$ belong to $\mathfrak{X}$. As G is an $\mathfrak{X}$T-group, we conclude that $\langle m, n\rangle$ is an $\mathfrak{X}$-group. Now let $m_1, m_2 \in M\backslash\{1\}$ and $n \in N\backslash\{1\}$. We may suppose that $m_1 n \neq 1$. Then $\langle m_1 n, m_1\rangle = \langle m_1, n\rangle$ is in $\mathfrak{X}$ and $\langle m_1, m_2\rangle$ is in $\mathfrak{X}$. Thus it follows that $\langle m_1 n, m_2\rangle$ also belongs to $\mathfrak{X}$. Similarly we can prove that $\langle m n_1, n_2\rangle$ is in $\mathfrak{X}$ for every $m \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$. Now take $m_1, m_2 \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$ and suppose that $m_1 n_1 \neq 1$, $m_2 n_2 \neq 1$. Then $\langle m_1 n_1, m_2\rangle \in \mathfrak{X}$, $\langle m_2, m_2 n_2\rangle \in \mathfrak{X}$, hence $\langle m_1 n_1, m_2 n_2\rangle$ belongs to $\mathfrak{X}$. This shows that every two-generator subgroup of MN belongs to $\mathfrak{X}$. Since $\mathfrak{X}$ is bigenetic in the class of all finite groups, we get that MN is an $\mathfrak{X}$-group, as required.

Suppose now that $M \cap N = 1$. Then $[M, N] = 1$. As above it suffices to prove that every two-generator subgroup of MN is in $\mathfrak{X}$. At first let $m_1, m_2 \in M\backslash\{1\}$ and $n \in N\backslash\{1\}$. Then the groups $\langle m_1 n, n\rangle = \langle m_1, n\rangle$ and $\langle n, m_2\rangle$ are Abelian, hence they belong to $\mathfrak{X}$. By the transitivity we have that $\langle m_1 n, m_2\rangle$ belongs to $\mathfrak{X}$. Similar argument shows that $\langle m n_1, n_2\rangle \in \mathfrak{X}$ for every $m \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$. From this it follows that if $m_1, m_2 \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$, then $\langle m_1 n_1, m_2\rangle$ and $\langle m_2, m_2 n_2\rangle$ are in $\mathfrak{X}$, hence $\langle m_1 n_1, m_2 n_2\rangle$ is also in $\mathfrak{X}$. This concludes the proof. $\square$

Theorem 2.2. *Let $\mathfrak{X}$ be a class of groups satisfying ($\star$). Let G be a finite $\mathfrak{X}$T-group. Then one of the following holds.*

(i) *G belongs to $\mathfrak{X}$.*

(ii) *G is $\mathfrak{X}$-semisimple.*

(iii) *G is a Frobenius group with kernel and complement both belonging to $\mathfrak{X}$.*

PROOF: Let R be the $\mathfrak{X}$-radical of G. By Lemma 2.1, R belongs to $\mathfrak{X}$. If $R = G$, then G belongs to $\mathfrak{X}$. If $R = 1$, then G is $\mathfrak{X}$-semisimple. So from now on we assume that $1 \neq R \neq G$.

Let $y \in R\backslash\{1\}$ and suppose that there exists $a \in C_G(y)\backslash R$. Then $\langle a, y\rangle$ is Abelian, hence it belongs to $\mathfrak{X}$. As R is an $\mathfrak{X}$-group and G is an $\mathfrak{X}$T-group, we have that $\langle a, h\rangle$ is in $\mathfrak{X}$ for every $h \in R$. By conjugation we get that $\langle a^x, h\rangle \in \mathfrak{X}$ for every $x \in G$ and $h \in R$. Since G is an $\mathfrak{X}$T-group, we get that

$$\langle a^x, a^z\rangle \in \mathfrak{X} \tag{1}$$

for every $x, z \in G$. We claim that $\langle u, v\rangle \in \mathfrak{X}$ for every $u, v \in a^G$. To prove this, we first introduce some notation. For $u \in a^G$ let r be the smallest integer such that u can be written as $a^{\pm g_1} \cdots a^{\pm g_r}$ for some $g_1, \ldots, g_r \in G$. Then we say that u is of weight r and denote $\mathrm{wt}(u) = r$. The proof of our claim goes by induction on $\mathrm{wt}(u) + \mathrm{wt}(v)$. If $\mathrm{wt}(u) + \mathrm{wt}(v) \leqslant 2$, then the claim follows from (1). Suppose that the claim holds true for all $u, v \in a^G$ with $\mathrm{wt}(u) + \mathrm{wt}(v) \leqslant l$. Let now $u, v \in a^G$ be such that $\mathrm{wt}(u) + \mathrm{wt}(v) = l + 1$. Without loss of generality we may assume that $\mathrm{wt}(u) > 1$ and $v \neq 1$. Then we can write $u = u'a^{\pm g}$ for some $g \in G$ and $u' \in a^G\backslash\{1\}$ with $\mathrm{wt}(u') = \mathrm{wt}(u) - 1$. We have that $\langle u, a^g\rangle = \langle u', a^g\rangle$ belongs to $\mathfrak{X}$ by the induction assumption. For the same reason we have that $\langle a^g, v\rangle \in \mathfrak{X}$. As G is an $\mathfrak{X}$T-group, we conclude that $\langle u, v\rangle$ belongs to $\mathfrak{X}$. This proves that every two-generator subgroup of a^G belongs to $\mathfrak{X}$. As $\mathfrak{X}$ is bigenetic in the class of all finite groups, we get $a^G \in \mathfrak{X}$, hence $a \in R$, a contradiction. By Satz 8.5 in [2] we have that G is a Frobenius group and R is its kernel. In particular, it follows from here that R is nilpotent. Let H be its complement. Then H is an $\mathfrak{X}$T-group with nontrivial centre. It follows from here that every two-generator subgroup of H belongs to $\mathfrak{X}$, hence $H \in \mathfrak{X}$. ∎

A characterisation of $\mathfrak{X}$-semisimple $\mathfrak{X}$T-groups is usually not easy and depends heavily on a choice of the class $\mathfrak{X}$; see [**1, 6, 11**]. In the case of Frobenius groups we provide a general characterisation of $\mathfrak{X}$T-groups. At first we prove the following technical result.

LEMMA 2.3. *Let $\mathfrak{X}$ satisfy $(\star)$. Let G be a finite $\mathfrak{X}$T-group and H an $\mathfrak{X}$-subgroup of G. Then*

$$C_G^{\mathfrak{X}}(H) = \{x \in G : \langle x, h\rangle \in \mathfrak{X} \text{ for some } h \in H\backslash\{1\}\}$$

is an $\mathfrak{X}$-subgroup of G containing H.

PROOF: Clearly $C_G^{\mathfrak{X}}(H)$ contains H. Let $x, y \in C_G^{\mathfrak{X}}(H)\backslash\{1\}$. Then there exist $h, k \in H\backslash\{1\}$ such that $\langle x, h\rangle \in \mathfrak{X}$ and $\langle y, k\rangle \in \mathfrak{X}$. Since $\langle h, k\rangle \in \mathfrak{X}$, we get that $\langle x, y\rangle$ also belongs to $\mathfrak{X}$. If $xy \neq 1$, then $\langle xy, y\rangle = \langle x, y\rangle$ belongs to $\mathfrak{X}$, hence also $\langle xy, k\rangle \in \mathfrak{X}$. Thus $xy \in C_G^{\mathfrak{X}}(H)$. Note also that every two-generator subgroup of $C_G^{\mathfrak{X}}(H)$ is in $\mathfrak{X}$, hence $C_G^{\mathfrak{X}}(H)$ also belongs to $\mathfrak{X}$. ∎

PROPOSITION 2.4. *Let $\mathfrak{X}$ be a group theoretical class satisfying $(\star)$. Let G be a Frobenius group with kernel F and complement H. Then G is an $\mathfrak{X}$T-group if and only if $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ are $\mathfrak{X}$-groups.*

PROOF: Let $\mathfrak{X}$ and G be as above. If G is an $\mathfrak{X}$T-group, then it follows from Theorem 2.2 that F and H belong to $\mathfrak{X}$. Consequently $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ are also $\mathfrak{X}$-groups by Lemma 2.3. Conversely, suppose that $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ are $\mathfrak{X}$-groups. Let $x, y, z \in G\backslash\{1\}$ and suppose that $\langle x, y\rangle \in \mathfrak{X}$ and $\langle y, z\rangle \in \mathfrak{X}$. Assume first that $y \in F$. Then $x, z \in C_G^{\mathfrak{X}}(F)$ and consequently $\langle x, z\rangle \in \mathfrak{X}$. If $y \notin F$, then $y \in H^g$ for some $g \in G$. But then $x, z \in C_G^{\mathfrak{X}}(H^g) = \left(C_G^{\mathfrak{X}}(H)\right)^g$, thus $\langle x, z\rangle$ belongs to $\mathfrak{X}$. Thus G is an $\mathfrak{X}$T-group. ∎

When $\mathfrak{X}$ is the class of all Abelian groups, then all three possibilities of Theorem 2.2 can occur [**6, 11**]. In some cases, however, we can exclude the existence of $\mathfrak{X}$-semisimple $\mathfrak{X}$T-groups.

THEOREM 2.5. *Let $\mathfrak{X}$ be a class of groups satisfying $(\star)$, and suppose that $\mathfrak{X}$ contains all finite dihedral groups and that every finite $\mathfrak{X}$-group is soluble. If G is a finite $\mathfrak{X}$T-group which is not in $\mathfrak{X}$, then G is a Frobenius group with complement belonging to $\mathfrak{X}$. In particular, G is soluble.*

Before proving this result we mention here the well known Thompson's classification of minimal simple groups; that is, finite nonabelian simple groups all whose proper subgroups are soluble. It turns out [**9**] that every such group is isomorphic to one of the following groups.

(i) $\mathrm{PSL}(2, p)$, where p is a prime, $p > 3$ and $p^2 - 1 \neq 0 \mod 5$.
(ii) $\mathrm{PSL}(2, 2^f)$, where f is a prime.
(iii) $\mathrm{PSL}(2, 3^f)$, where f is an odd prime.
(iv) $\mathrm{PSL}(3, 3)$.
(v) $\mathrm{Sz}(q)$, where $q = 2^{2n+1}$ and $2n + 1$ is a prime.

If $G = \mathrm{PSL}(2, F)$ where F is a Galois field of odd characteristic and $|F| > 5$, then G can be generated by an involution and an element of even order. This can be easily seen as follows. Let $q = |F|$. By Dickson's theorem [**2**], G contains elements a and b with $|a| = (q-1)/2$ and $|b| = (q+1)/2$. Note that precisely one of $|a|$, $|b|$ is even, without loss of generality we may assume that this is true for $|a|$. Then $N_G(a) = D_{q-1}$ and this is the only maximal subgroup of G containing a; this follows from the proof of Dickson's theorem [**2**]. So if we choose any involution u from $G\backslash N_G(a)$, we have $\langle a, u\rangle = G$, as required. A similar result holds true for $\mathrm{PSL}(3, 3)$ and $\mathrm{Sz}(q)$. In the first case note that $\mathrm{PSL}(3, 3)$ can be generated by the canonical projections of matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix},$$

which are of orders 2 and 8 in PSL(3, 3), respectively. For the Suzuki groups $\mathrm{Sz}(q)$ it follows from [8] that they can always be generated by an involution and an element of order 4. We summarise this in the following lemma.

Lemma 2.6. *Let G be one of the following groups:* $\mathrm{PSL}(2, F)$ *where F is a Galois field of odd characteristic and $|F| > 5$,* $\mathrm{PSL}(3,3)$ *or* $\mathrm{Sz}(q)$. *Then G can be generated by an involution and an element of even order.*

Note that for the groups $\mathrm{PSL}(2, 2^f)$ the conclusion of the above lemma does not hold. In this case we have the following result that can be proved by straightforward calculation.

Lemma 2.7. *Let $G = \mathrm{PSL}(2, 2^f)$, $f > 1$. Denote by ζ a generator of* $\mathrm{GF}(2^f)$ *and let a, b and c be the elements of G which are projections of*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & \zeta \end{pmatrix},$$

respectively. Then $\langle a, b\rangle$ and $\langle b, c\rangle$ are dihedral groups and $\langle a, c\rangle = G$.

Proof of Theorem 2.5: We may suppose that G does not belong to $\mathfrak{X}$, hence $R_{\mathfrak{X}}(G) \neq G$. If we prove that G is soluble, then $R_{\mathfrak{X}}(G) \neq 1$ and our claim follows from Theorem 2.2. So suppose that there exist finite insoluble $\mathfrak{X}$T-groups, and let G be a counterexample of minimal order. Then every proper subgroup of G is soluble. By Theorem 2.2 we have that $R_{\mathfrak{X}}(G) = 1$. Let R be the soluble radical of G. Since $\mathfrak{X}$ contains all finite Abelian groups, we have that $R = 1$. It is now easy to see that G has to be simple. By Thompson's classification of minimal simple groups [**9**], G is isomorphic to one of the groups in the above mentioned list. By Lemma 2.7, G is not isomorphic to any of $\mathrm{PSL}(2, 2^f)$, where f is a prime. If G is one of the groups of Lemma 2.6, then $G = \langle a, b\rangle$, where $|a| = 2$ and $|b| = 2k$, $k > 1$. We have that $\langle a, b^k\rangle$ is a dihedral group and $\langle b^k, b\rangle$ is a cyclic group, hence G is in $\mathfrak{X}$ by the $\mathfrak{X}$T-property, a contradiction. This concludes the proof. $\square$

Using Theorem 2.5, we obtain a rather surprising characterisation of finite soluble groups.

Corollary 2.8. *Every finite soluble-transitive group is soluble.*

Note that the class of all supersoluble groups also satisfies all the assumptions of Theorem 2.5. Thus we have the following.

Corollary 2.9. *Let G be a finite supersoluble-transitive group. If G is not supersoluble, then G is a Frobenius group with supersoluble complement. In particular, G is always soluble.*

In view of Corollary 2.8 we may ask if every finite supersoluble-transitive group is supersoluble. This is not true however, as the group A_4 shows. It is also not difficult

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to find an example of a Frobenius group with supersoluble complement which is not supersoluble-transitive. This example also shows that Proposition 2.4 is in a certain sense best possible. Indeed, it is not possible to replace $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ by F and H, respectively.

Example 2.10. Let $A = \langle x \rangle \oplus \langle y \rangle$ be an elementary group of order 9 and let α be the automorphism of A given by the matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $\langle \alpha \rangle$ acts fixed-point-freely on A. Let $G = A \rtimes \langle \alpha \rangle$. This is a group of order 36 which is not supersoluble-transitive. To see this, note that $\langle \alpha^2, (\alpha y)^2 \rangle$ is a dihedral group, $\langle (\alpha y)^2, \alpha y \rangle$ is cyclic, whereas $\langle \alpha^2, \alpha y \rangle = G$ is not supersoluble. Denoting by $\mathfrak{S}$ the class of all supersoluble groups, note that $C_G^{\mathfrak{S}}(\langle \alpha \rangle)$ has 20 elements and it is thus not a subgroup of G. On the other hand, $C_G^{\mathfrak{S}}(A)$ is a subgroup of index 2 in G.

Theorem 2.5 cannot be applied in the case of $\mathfrak{N}$T-groups, where $\mathfrak{N}$ denotes the class of all nilpotent groups. Thus it is to be expected that there exist finite insoluble $\mathfrak{N}$T-groups. This is confirmed by the following characterisation of finite $\mathfrak{N}$T-groups which is essentially contained in [**1**]. We include a proof for the sake of completeness.

Theorem 2.11. *Let G be a finite $\mathfrak{N}$T-group. Then one of the following holds.*

(i) *G is nilpotent.*

(ii) *G is a Frobenius group with nilpotent complement.*

(iii) *$G \cong \mathrm{PSL}(2, 2^f)$ for some $f > 1$.*

(iv) *$G \cong \mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.*

Conversely, every finite group under (i)–(iv) *is an $\mathfrak{N}$T-group.*

Proof: If G is soluble and not nilpotent, then the Fitting subgroup F of G is a proper nontrivial subgroup of G. By Theorem 2.2, G is a Frobenius group with nilpotent complement. So suppose that G is not soluble. It is easy to see that in every finite $\mathfrak{N}$T-group G the centralisers of nontrivial elements are nilpotent, that is, G is an CN-group. By a result of Suzuki [**7**, Part I, Theorem 4], the centraliser of any involution in G is a 2-group. Let P and Q be any Sylow p-subgroups of G and suppose that $P \cap Q \neq 1$. Since P and Q are nilpotent and G is an $\mathfrak{N}$T-group, we conclude that $\langle P, Q \rangle$ is nilpotent. This shows that the Sylow subgroups of G are independent. Combining Theorem 1 in Part I and Theorem 3 in Part II of [**7**], we conclude that G has to be simple. Additionally, it follows from [**8**] that G is isomorphic either to $\mathrm{PSL}(2, 2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.

Let G be a finite Frobenius group with the kernel N and a complement H and suppose that H nilpotent. Let $x, y, z \in G \backslash \{1\}$ and let the groups $\langle x, y \rangle$ and $\langle y, z \rangle$ be nilpotent. Let c be the nilpotency class of $\langle x, y \rangle$. First suppose that $x \in N$ and $y \notin N$.

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Then $[x, {}_c y] = 1$, which implies $[x, {}_{c-1} y] = 1$, since H acts fixed-point-freely on N. By the same argument we get $x = 1$, which is not possible. This shows that if $x \in N$ then $y \in N$ and similarly also $z \in N$. But in this case $\langle x, z\rangle$ is clearly nilpotent, since N is nilpotent. Thus we may assume that $x, y, z \notin N$. Let $x \in H^g$ and $y \in H^k$ for some $g, k \in G$ and suppose $H^g \neq H^k$. We clearly have $C_G(x) \leqslant H^g$ and $C_G(y) \leqslant H^k$. Let ω be any commutator of weight c with entries in $\{x, y\}$. Then $\omega \in C_G(x) \cap C_G(y) = 1$ implies that $\langle x, y\rangle$ is nilpotent of class $\leqslant c - 1$, a contradiction. Hence we conclude that $\langle x, y\rangle \leqslant H^g$ and similarly also $\langle y, z\rangle \leqslant H^g$. Therefore we have $\langle x, z\rangle \leqslant H^g$. But H^g is nilpotent, hence the group $\langle x, z\rangle$ is also nilpotent. This shows that the groups under (ii) are $\mathfrak{N}$T-groups.

It remains to prove that the groups under (iii) and (iv) are $\mathfrak{N}$T-groups. If $G = \mathrm{PSL}(2, 2^f)$, $f > 1$, then every centraliser of a nontrivial element of G is Abelian by [**6**]. It follows from here that G is an $\mathfrak{N}$T-group. Now let $G = \mathrm{Sz}(q)$ where $q = 2^{2n+1} > 2$. By Theorem 3.10 (c) in [**3**], G has a nontrivial partition $(G_i)_{i\in I}$, where for every $i \in I$ the group G_i is nilpotent and contains centralisers of each of its nontrivial elements. Let $x, y, z \in G\backslash\{1\}$ and suppose that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent. Let a and b be nontrivial elements in $Z(\langle x, y\rangle)$ and $Z(\langle y, z\rangle)$, respectively, and suppose that $a \in G_i$ and $b \in G_j$ for some $i, j \in I$. Then $y \in C_G(a) \cap C_G(b) \leqslant G_i \cap G_j$, hence $i = j$. But now we get $x, z \in G_i$ and since G_i is nilpotent, the same is true for the group $\langle x, z\rangle$. Hence G is an $\mathfrak{N}$T-group. $\square$

References

[1] C. Delizia, P. Moravec, and C. Nicotera, 'Groups in which the bounded nilpotency of two-generator subgroups is a transitive relation', *Beiträge Algebra Geom.* (to appear).

[2] B. Huppert, *Endliche gruppen* (Springer–Verlag, Berlin, 1967).

[3] B. Huppert and N. Blackburn, *Finite groups III* (Springer–Verlag, New York, 1982).

[4] J.C. Lennox, 'Bigenetic properties of finitely generated hyper-(abelian-by-finite) groups', *J. Austral. Math. Soc.* **16** (1973), 309–315.

[5] D.J.S. Robinson, (Part 1), *Finiteness conditions and generalized soluble groups* (Springer–Verlag, Berlin, 1972).

[6] M. Suzuki, 'The nonexistence of certain type of simple groups of odd order', *Proc. Amer. Math. Soc.* **8** (1957), 686–695.

[7] M. Suzuki, 'Finite groups with nilpotent centralizers', *Trans. Amer. Math. Soc.* **99** (1961), 425–470.

[8] M. Suzuki, 'On a class of doubly transitive groups', *Ann. of Math.* **75** (1962), 105–145.

[9] J.G. Thompson, 'Non-solvable finite groups all whose local subgroups are solvable', *Bull. Amer. Math. Soc.* **74** (1968), 383–437.

[10] L. Weisner, 'Groups in which the normaliser of every element except identity is abelian', *Bull. Amer. Math. Soc.* **31** (1925), 413–416.

[11] Y.F. Wu, 'Groups in which commutativity is a transitive relation', *J. Algebra* **207** (1998), 165–181.

Dipartimento di Matematica e Informatica
Università di Salerno
Via Ponte don Melillo
84084 - Fisciano (SA)
Italy
e-mail: cdelizia@unisa.it

Fakulteta za Matematiko in Fiziko
Univerza v Ljubljani
Jadranska 19
1000 Ljubljana
Slovenia
e-mail: primoz.moravec@fmf.uni-lj.si

Dipartimento di Matematica e Informatica
Università di Salerno
Via Ponte don Melillo
84084 - Fisciano (SA)
Italy
e-mail: cnicotera@unisa.it