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ANALYSIS OF SHALLOW SHELLS BY A COMBINATION OF FINITE ELEMENTS AND CONTOUR METHODS

D. BUCCO[1,2] and J. MAZUMDAR[1]

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Abstract

Two generalised shallow-shell bending elements are developed for the analysis of doubly-curved shallow shells having arbitrarily shaped plan-forms. Although both elements are formulated using the concept of iso-deflection contour lines, one element uses the three displacement components U, V and W as the basic unknowns, while the displacement component W and the stress function Φ serve as the unknowns in the other element. A number of illustrative examples are included to demonstrate the accuracy and relative convergence of the proposed shallow-shell elements when employed for static analysis purposes.

1. Introduction

The first application of the finite element method to the study of shallow-shell structures consisted essentially in the representation of the curved surface of the shell by flat-plate bending elements [6]. In this approach, each element is subjected to both bending and stretching deformations. However, there is no coupling of these effects within the element. This coupling, which is characteristic of shell behaviour, occurs only at nodes where successive elements not in the same plane adjoin. Consequently, a large number of such elements is required to achieve adequate accuracy.

The subsequent development of doubly-curved shallow-shell elements emanated from a desire to preserve the correct geometry of the shell, as well as to decrease the total number of elements required to model the problem. These elements have appeared in a variety of shapes, the most common being rectangular [7, 20] parallelogram [18] and triangular [8, 9, 10, 17, 11, 21].

[1]Department of Applied Mathematics, University of Adelaide, Adelaide, South Australia, 5001.
[2]Present Address: Guided Weapons Division, Weapons Systems Research Laboratory, Defence Science and Technology Organisation, Salisbury, South Australia, 5108.

A comparison of various shallow-shell elements may be found in an excellent review article by Brebbia and Deb Nath [2], where they discuss some of the difficulties encountered with the prescription of the appropriate displacement components and satisfaction of the convergence conditions.

The doubly-curved shallow-shell elements appear to have received considerable attention in the past because of their versatility in idealising arbitrary shaped regions. However, for shallow shells having a curvilinear boundary, the discretisation of the shell region into triangular elements results in a polygonal approximation to the true boundary. This can give rise to significant errors in the overall strain energy functional of the shell. The error, of course, decreases with an increase in the number of elements. Consequently, a greater demand in core storage and time is placed on the computer with a corresponding increase in effort on the part of the user to ensure convergence.

In this study it is intended to present an extension of the finite element/contour approach [5] to the analysis of shallow-shell bending problems. To this end, two generalised shallow-shell bending elements are developed for the approximate analysis of doubly-curved shallow shells having arbitrarily shaped plan forms. Although both elements are formulated using the concept of iso-deflection contour lines [16], one element uses the three displacement components U, V, and W as the basic unknowns, while the displacement component W and the stress function Φ serve as the unknowns in the other element. In the above, U and V denote the in-plane displacements of a point on the middle surface of the shell in the x and y directions, respectively.

The theory which is developed here is based on the usual assumptions of small deflection theory of thin shells. Furthermore, the "shallowness" assumptions, as given by Vlasov [23], are adopted in this study.

2. Element formulation by the displacement approach

2.1. The method of constant deflection lines

A brief outline of the method of constant deflection lines for the analysis of shallow-shell problems is now presented. Consider a thin, elastic, homogeneous, isotropic, shallow shell having an arbitrarily shaped plan-form. Let the equation of the middle surface of the shell, when referred to a system of orthogonal coordinates (x, y, z), be given by

$$z = \frac{x^2}{2R_x} + \frac{xy}{R_{xy}} + \frac{y^2}{2R_y}, \tag{1}$$

where the shell will be called shallow if $r = \sqrt{x^2 + y^2}$ is small when

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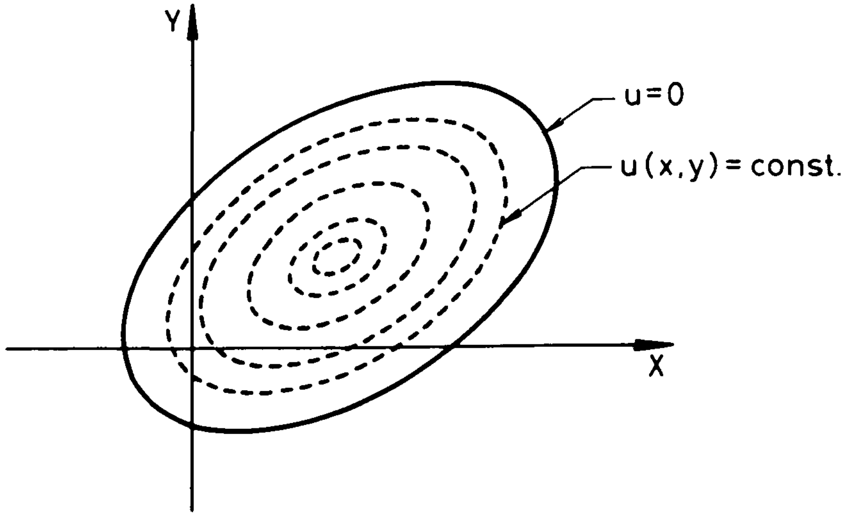

FIGURE 1. Iso-defletion contour lines.

compared to the least of R_x, R_y, and R_{xy} (the radii of curvature) everywhere in the region, and if the radii of curvature may be taken to be constants.

When the shell is deflected by external loads, e.g. transverse or inertial, then at any instant of time τ, the intersections between the deflection surface and the parallels $z = constant$ yield contours which, after projection onto the base plane $z = H$, are a set of level curves, $u(x, y) = constant$ (Figure 1).

These curves represent the iso-deflection contour lines for the problem. If the boundary of the shell is subjected to any combination of clamping and simple support, so that it is restrained from moving in the z-direction, then clearly the boundary, assumed to be a simple closed curve C, will belong to the family of equal deflection lines and, without loss in generality, one may consider that $u = 0$ on the boundary.

In general, this family of contour lines form a system of non-intersecting closed curves, starting with the outer boundary as one of the curves. The system of curves $u(x, y) = constant$, may be denoted by C_u, $0 \leq u \leq u^*$, so that $C_0 = C$ is the boundary of the shell, while C_{u^*} coincides with the point at which the maximum $u = u^*$ is attained.

2.2. Contour elements excluding the origin

Let the point at which the maximum value u^* is attained be represented by (x^*, y^*). Then an appropriate set of axes x', y' can be defined so that the origin is located at this point. An element e is now considered as that region, Ω_e, of the shell bounded by any two contour lines, $u = u_e$ and

$u = u_{e+1}$, such that $u_e < u_{e+1}$ and $u_{e+1} \neq u^*$. These lines constitute the two nodes of the element, as shown in Figure 2. It is assumed that the value of u increases inwardly.

Denote by $\vec{n}$, $\vec{s}$, respectively, the outward normal and tangent to the curve $u(x, y) = constant$ (Figure 2). Then the deformation at a point on the middle surface of the shell can be completely determined from a knowledge of U^n, U^s and W. Here U^n and U^s are the in-plane displacements of the point in the $\vec{n}$, $\vec{s}$ directions, respectively, while W represents the transverse displacement. Since W is a function of u, the appropriate conformity requirements are immediately satisfied by using the shape functions of the corresponding flat plate element [5]. Consequently, the variation of W over the shallow-shell element may be expressed in the form

$$W = [N]\{\delta\}, \tag{2}$$

where

$$[N] = [N_1 \; N_2 \; N_3 \; N_4], \tag{3}$$

and

$$\{\delta\} = \{W_e \; \left(\frac{dW}{du}\right)_e \; W_{e+1} \; \left(\frac{dW}{du}\right)_{e+1}\}^T, \tag{4}$$

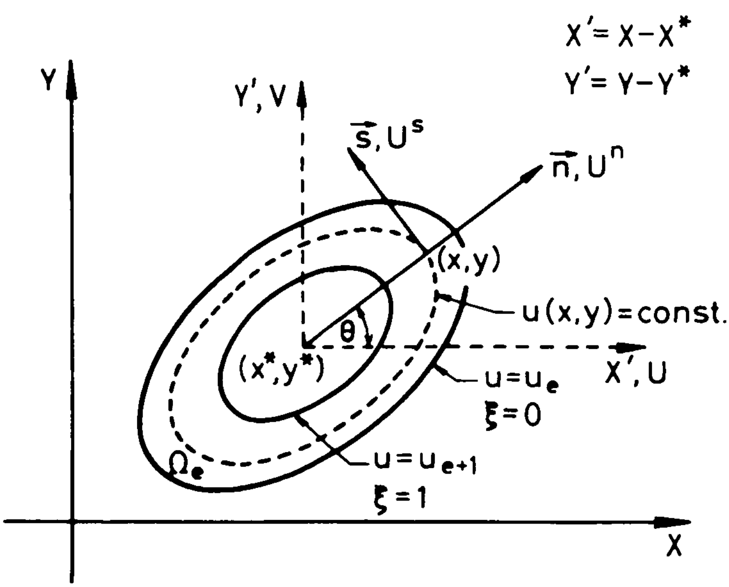

FIGURE 2. Shallow shell element with $u_{e+1} \neq u^*$.

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and the components of $[N]$ are defined by

$$N_1 = 1 - 3\xi^2 + 2\xi^3, \qquad N_2 = \ell(\xi - 2\xi^2 + \xi^3),$$
$$N_3 = 3\xi^2 - 2\xi^3, \qquad N_4 = \ell(-\xi^2 + \xi^3). \tag{5}$$

In the above, ξ is a local coordinate of the element defined by

$$\xi = (u - u_e)/\ell, \qquad \ell = u_{e+1} - u_e. \tag{6}$$

The in-plane displacements U^n, U^s, in general, are not constant along the nodal contour lines $u = u_e$ and $u = u_{e+1}$. If the region bounded by any closed curve $u(x, y) = \mathit{constant}$, is starshaped, then it is permissible to express U^n and U^s as single-valued functions of θ along this curve, where θ is defined by

$$\sin\theta = y' / \left(x'^2 + y'^2\right)^{1/2}, \quad \cos\theta = x' / \left(x'^2 + y'^2\right)^{1/2}. \tag{7}$$

Hence, proceeding as for the transverse displacement W, the in-plane displacements are expressed in the form

$$U^n = [N]\{\alpha^e\}, \tag{8}$$

$$U^s = [N]\{\beta^e\}, \tag{9}$$

where the shape function matrix $[N]$ is given by (3), while $\{\alpha^e\}$ and $\{\beta^e\}$ are defined by

$$\{\alpha^e\} = \{U_e^n \ \left(\frac{\partial U^n}{\partial u}\right)_e \ U_{e+1}^n \ \left(\frac{\partial U^n}{\partial u}\right)_{e+1}\}^{\mathrm{T}}, \tag{10}$$

$$\{\beta^e\} = \{U_e^s \ \left(\frac{\partial U^s}{\partial u}\right)_e \ U_{e+1}^s \ \left(\frac{\partial U^s}{\partial u}\right)_{e+1}\}^{\mathrm{T}}. \tag{11}$$

In the above, the components of $\{\alpha^e\}$ and $\{\beta^e\}$ denote the nodal unknowns of U^n and U^s respectively, and are functions of the variable θ. Since, along any contour $u(x, y) = \mathit{const.}$, both U^n and U^s are periodic, the corresponding nodal unknowns may be expanded in a Fourier series form, according to

$$U_i^n = U_{i0}^n + \sum_{k=1}^{L}\{U_{ik}^n \cos k\theta + \hat{U}_{ik}^n \sin k\theta\}, \tag{12}$$

$$\left(\frac{\partial U^n}{\partial u}\right)_i = \bar{U}_{i0}^n + \sum_{k=1}^{L}\{\bar{U}_{ik}^n \cos k\theta + \bar{\hat{U}}_{ik}^n \sin k\theta\}, \tag{13}$$

$$U_i^s = U_{i0}^s + \sum_{k=1}^{L}\{U_{ik}^s \cos k\theta + \hat{U}_{ik}^s \sin k\theta\}, \tag{14}$$

$$\left(\frac{\partial U^s}{\partial u}\right)_i = \bar{U}^s_{i0} + \sum_{k=1}^{L}\{\bar{U}^s_{ik}\cos k\theta + \bar{\bar{U}}^s_{ik}\sin k\theta\}, \tag{15}$$

where $i = e, e+1$. If a row vector $[T]$ is defined such that

$$[T] = [1 \ \cos\theta \ \sin\theta \ \ldots \ \cos L\theta \ \sin L\theta], \tag{16}$$

then equations (12) to (15) may be expressed compactly as

$$U^n = [\bar{N}]\{\bar{\alpha}^e\}, \tag{17}$$

and

$$U^s = [\bar{N}]\{\bar{\beta}^e\}, \tag{18}$$

where

$$[\bar{N}] = [\bar{N}_1 \ \bar{N}_2 \ \bar{N}_3 \ \bar{N}_4], \tag{19}$$

$$\{\bar{\alpha}^e\} = \begin{Bmatrix} U_{ne} \\ \bar{U}_{ne} \\ U_{ne+1} \\ \bar{U}_{ne+1} \end{Bmatrix}, \qquad \{\bar{\beta}^e\} = \begin{Bmatrix} U_{se} \\ \bar{U}_{se} \\ U_{se+1} \\ \bar{U}_{se+1} \end{Bmatrix}, \tag{20}$$

and each $[\bar{N}_i]$, $i = 1, 2, 3, 4$, is a row vector of the form

$$[\bar{N}_i] = N_i[T]. \tag{21}$$

Hence, the variation of the three orthogonal displacements, U^n, U^s and W, over the element, is prescribed according to (2), (17) and (18), respectively.

2.3. Closure element

In the limiting case, when u_{e+1} coincides with u^*, the inner contour of the foregoing element degenerates into a nodal point, thus giving rise to a "closure" element, that is, an element enclosing the origin, as shown in Figure 3. Although the variation of W, as prescribed by (2), poses no inconsistencies in this case, differentiation of either U^n or U^s, defined by (17) and (18) respectively, give rise to infinite in-plane strains, and hence stresses, at the origin. This is physically unacceptable, and may be circumvented by judiciously redefining the variation of U^n and U^s over this element.

The closure element is envisaged as that region Ω_c of the shell bounded by the closed curve, $u = u_c$, and enclosing the point (x^*, y^*) where the maximum, $u = u_{c+1} = u^*$, occurs. Again the element contains two nodes.

In order to overcome the difficulties encountered at the origin, and to ensure finite values for the strains, the closure element in-plane displacements U^n and U^s are assumed to have the following respective forms [4],

$$U^n = [P]\{\alpha^c\}, \tag{22}$$

$$U^s = [P]\{\beta^c\}, \tag{23}$$

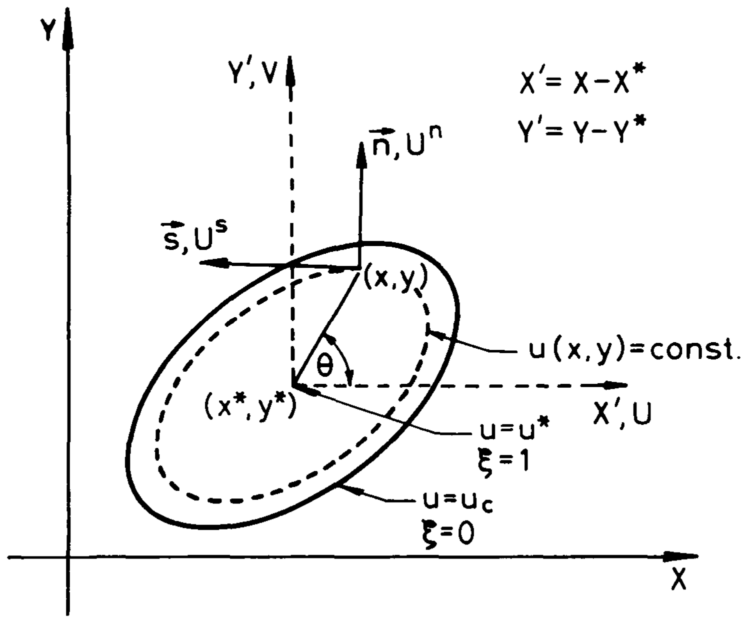

FIGURE 3. Closure element.

where

$$[P] = [P_1 \; P_2], \tag{24}$$

$$\{\alpha^c\} = \{U_c^n \; \left(\frac{\partial U^n}{\partial u}\right)_c\}^{\mathrm{T}}, \tag{25}$$

$$\{\beta^c\} = \{U_c^s \; \left(\frac{\partial U^s}{\partial u}\right)_c\}^{\mathrm{T}}, \tag{26}$$

and the shape function components of $[P]$ are expressed by

$$\begin{aligned} P_1 &= (1+\xi/2)\,(1-\xi)^{1/2}, \\ P_2 &= \ell\xi\,(1-\xi)^{1/2}. \end{aligned} \tag{27}$$

The in-plane displacements for this element now read

$$U^n = [\bar{P}]\{\bar{\alpha}^c\}, \tag{28}$$

$$U^s = [\bar{P}]\{\bar{\beta}^c\}, \tag{29}$$

where

$$[\bar{P}] = [\bar{P}_1 \; \bar{P}_2], \tag{30}$$

and

$$\{\bar{\alpha}^c\} = \begin{Bmatrix} U_{nc} \\ \bar{U}_{nc} \end{Bmatrix}, \qquad \{\bar{\beta}^c\} = \begin{Bmatrix} U_{sc} \\ \bar{U}_{sc} \end{Bmatrix}. \tag{31}$$

Further, $[\bar{P}_i]$, $i = 1, 2$, is defined by

$$[\bar{P}_i] = P_i[T]. \tag{32}$$

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Consequently, the variation of the three displacement components, W, U^n and U^s, over the closure element is expressed according to (2), (28) and (29), respectively.

3. Derivation of the element equations for shallow shell bending by the displacement approach

3.1. Elements excluding the origin

The total potential energy of the shallow shell in bending may be written as

$$\Pi_p = \Pi_u - \iint_{\Omega_e} \{f\}^{\mathsf{T}}\{Q\} d\Omega_e, \tag{33}$$

where

$$\{f\} = \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}, \qquad \{\Omega\} = \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}, \tag{34}$$

and Π_u is the strain energy given by

$$\Pi_u = 1/2 \iint_{\Omega_e} \{\varepsilon\}^{\mathsf{T}}\{\sigma\} d\Omega_e. \tag{35}$$

In the above relations, $\{\sigma\}$ and $\{\varepsilon\}$ denote vectors of generalised stresses and strains respectively, while q_x, q_y and q_z represent the external loads acting in the x, y and z directions, respectively.The functions U and V are related to the prescribed displacements U^n and U^s via

$$\begin{Bmatrix} U \\ V \end{Bmatrix} = -\frac{1}{\sqrt{t}} \begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix} \begin{Bmatrix} U^n \\ U^s \end{Bmatrix}, \tag{36}$$

where $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$ and $t = u_x^2 + u_y^2$. Thus all the displacement components U, V and W, are known in terms of the nodal variables of the element.

For an elastic, isotropic material, the generalised stresses and strains are related by Hooke's law in the form

$$\{\sigma\} = [D]\{\varepsilon\}, \tag{37}$$

where

$$\{\sigma\} = \{N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\}^{\mathsf{T}}, \tag{38}$$

$$\{\varepsilon\} = \{\varepsilon_x\ \varepsilon_y\ \varepsilon_{xy}\ \kappa_x\ \kappa_y\ \kappa_{xy}\}^{\mathsf{T}}, \tag{39}$$

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and the elasticity matrix $[D]$ can be written in the form

$$[D] = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \tag{40}$$

where the submatrices $[D_i]$, $i = 1, 2$, are defined by

$$[D_1] = \frac{Eh}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \qquad [D_2] = \frac{h^2}{12}[D_1]. \tag{41}$$

In the above relations, N_x, N_y and N_{xy} represent the usual membrane stresses corresponding to the in-plane strains ε_x, ε_y and ε_{xy}, respectively, while M_x, M_y and M_{xy} are the stress resultants associated with the respective curvatures κ_x, κ_y and κ_{xy}. Furthermore, E, ν and h denote respectively Young's modulus, Poisson's ratio and the thickness of the shell.

Following the shallow-shell theory of Vlasov, the in-plane strains are related to the displacements in the form

$$\varepsilon_x = \frac{\partial U}{\partial x} + \frac{W}{R_x}, \quad \varepsilon_y = \frac{\partial V}{\partial y} + \frac{W}{R_y}, \quad \varepsilon_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{2W}{R_{xy}}, \tag{42}$$

while the curvatures are given by

$$\kappa_x = -\frac{\partial^2 W}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 W}{\partial y^2}, \quad \kappa_{xy} = \frac{2\partial^2 W}{\partial x \partial y}. \tag{43}$$

Upon substitution for U, V and W in the above relationships and after minimising the total potential energy functional as expressed by (33), the following set of element equations for the nodal unknowns is obtained [4],

$$[\hat{K}^e]\{\hat{\delta}^e\} = \{\hat{F}^e\}, \tag{44}$$

where the element stiffness matrix $[\hat{K}^e]$ and force vector $\{\hat{F}^e\}$ are written in the forms

$$[\hat{K}^e] = \iint_{\Omega_e} [\hat{B}^e]^{\mathrm{T}}[D][\hat{B}^e]\,d\Omega_e, \tag{45}$$

and

$$\{\hat{F}^e\} = \iint_{\Omega_e} [\hat{N}^e]^{\mathrm{T}}\{Q\}\,d\Omega_e, \tag{46}$$

respectively. In the above, $[\hat{K}^e]$ is a symmetric square matrix of dimension $(16L + 12)$ where L denotes the number of the final term in the Fourier series approximation of both U^n and U^s. Thus the present element is essentially a "variable order" shallow-shell element in which the total number of degrees of freedom depends on the number of terms retained in the series.

3.2. Closure element

Proceeding as before, the equations associated with the closure element are again derived on the basis of the potential energy functional given by (33). In this case, the nodal unknowns can be obtained from the relation [4]

$$[\hat{K}^c]\{\hat{\delta}^c\} = \{\hat{F}^c\}, \tag{47}$$

where $[\hat{K}^c]$ is the stiffness matrix of the element given by

$$[\hat{K}^c] = \iint_{\Omega_c} [\hat{B}^c]^{\mathrm{T}}[D][\hat{B}^c]d\Omega_c\ , \tag{48}$$

and $\{\hat{F}^c\}$ is the consistent force vector defined by

$$\{\hat{F}^c\} = \iint_{\Omega_c} [\hat{N}^c]^{\mathrm{T}}\{Q\}d\Omega_c\ . \tag{49}$$

Again, the square matrix $[\hat{K}^c]$, as given in (48), is symmetric and positive definite. However, the dimension in this case is only $(8L + 8)$. A complete listing of all the components of $[\hat{K}^c]$ and $\{\hat{F}^c\}$ is given by Bucco [4].

4. Derivation of the element equations for shallow shell bending by the stress function approach

According to Vlasov's theory [23], the coupled differential equations governing the bending of shallow shells under transverse load, $q_z(x, y)$, are given by

$$D_f\nabla^4 W + \nabla_k^2\Phi = q_z, \tag{50}$$

$$\nabla_k^2 W - \frac{1}{Eh}\nabla^4\Phi = 0, \tag{51}$$

where the differential operators are defined by

$$\nabla^4(\ ) \equiv \frac{\partial^4(\ )}{\partial x^4} + 2\frac{\partial^4(\ )}{\partial x^2\partial y^2} + \frac{\partial^4(\ )}{\partial y^4}, \tag{52}$$

and

$$\nabla_k^2(\ ) \equiv \frac{1}{R_x}\frac{\partial^2(\ )}{\partial y^2} + \frac{1}{R_y}\frac{\partial^2(\ )}{\partial x^2} - \frac{2}{R_{xy}}\frac{\partial^2(\ )}{\partial x\partial y}. \tag{53}$$

and D_f denotes the flexural rigidity of the shell defined by

$$D_f = \frac{Eh^3}{12\left(1 - \nu^2\right)}. \tag{54}$$

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As previously, consider a shallow-shell element to be the region, Ω_e, enclosed by the two contour lines $u = u_e$ and $u = u_{e+1}$, such that $u_e < u_{e+1}$, as depicted in Figure 4. These lines represent the two nodes of the element. Since the coupled differential equations (50) and (51) must be satisfied over the entire region of the shell, they must also be satisfied over any sub-region Ω_e. Consequently, attention here is confined to a particular element e with W and Φ as the nodal unknowns.

The appropriate stiffness properties of the element are derived with the aid of the method of weighted residuals, in particular using Galerkin's criterion [12, 22]. Firstly, however, the variation of both W and Φ over the element must be prescribed. All the continuity requirements associated with W may be satisfied immediately by using the shape functions $[N]$ of the corresponding flat-plate element [5]. Furthermore, the apparent similarity that exists in the role of both W and Φ in the governing shallow-shell equations given by (50) and (51), suggests representing the variation of Φ analogously to that of W over the element.

Thus, as a first approximation, assuming that Φ can be expressed as a suitable function of u, it is permissible to write

$$\Phi = [N]\{\Delta^e\}, \tag{55}$$

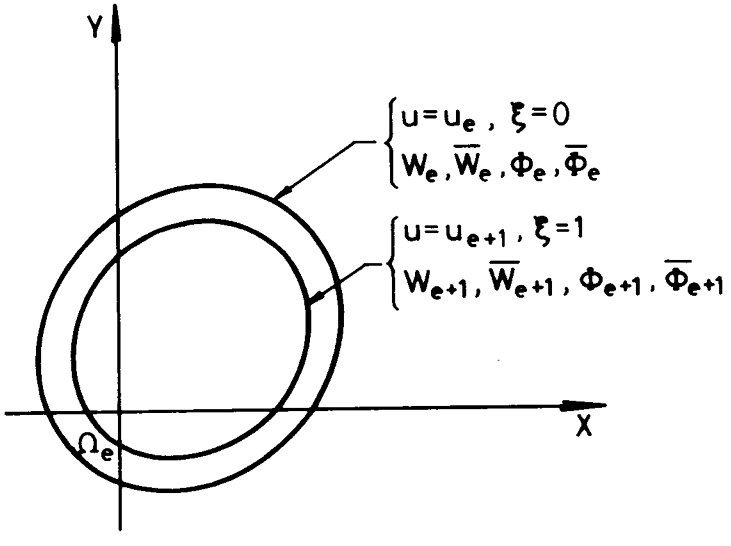

FIGURE 4. Shallow-shell element using the stress-function approach.

where

$$\{\Delta^e\} = \left\{\Phi_e \; \left(\frac{d\Phi}{du}\right)_e \; \Phi_{e+1} \; \left(\frac{d\Phi}{du}\right)_{e+1}\right\}^{\mathrm{T}}. \tag{56}$$

Consequently, the total number of degrees of freedom for this element is only eight: W, Φ, dW/du and $d\Phi/du$ at each node. The application of Galerkin's method to equations (50) and (51) with the shape functions $[N]$ serving as the appropriate weighting functions, yields the following set of eight equations,

$$\iint_{\Omega_e} [D_f \nabla^4 W + \nabla_k^2 \Phi - q_z] N_i \, d\Omega_e = 0, \qquad i = 1, 2, 3, 4, \tag{57}$$

$$\iint_{\Omega_e} [\nabla_k^2 W - \frac{1}{Eh} \nabla^4 \Phi] N_i \, d\Omega_e = 0, \qquad i = 1, 2, 3, 4, \tag{58}$$

On application of Green's theorem, and after some algebraic manipulations, the above equations transform to [4]

$$\begin{aligned} D_f \iint_{\Omega_e} \Bigg(& \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 W}{\partial x^2} + 2(1-\nu)\frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 W}{\partial y^2} \\ & + \nu \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 W}{\partial x^2}\Bigg) d\Omega_e + \iint_{\Omega_e} N_i \nabla_k^2 \Phi \, d\Omega_e \\ = & \iint_{\Omega_e} N_i q_z \, d\Omega_e + \oint_{C_{u_e}} \left(N_i V_n + \frac{\sqrt{t}}{\ell}\frac{dN_i}{d\xi} M_n\right) ds \\ & - \oint_{C_{u_{e+1}}} \left(N_i V_n + \frac{\sqrt{t}}{\ell}\frac{dN_i}{d\xi} M_n\right) ds, \qquad i = 1, 2, 3, 4, \end{aligned} \tag{59}$$

$$\begin{aligned} \iint_{\Omega_e} N_i \nabla_k^2 W \, d\Omega_e - \frac{1}{Eh} \iint_{\Omega_e} \Bigg(& \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 \Phi}{\partial x^2} + 2(1+\nu)\frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 \Phi}{\partial x \partial y} \\ & + \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 \Phi}{\partial y^2} - \nu \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} - \nu \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2}\Bigg) d\Omega_e \\ = \frac{1}{Eh} \oint_{C_{u_e}} & \left(N_i \frac{\partial(\nabla^2 \Phi)}{\partial n} + Eh\frac{\sqrt{t}}{\ell}\frac{dN_i}{d\xi}\varepsilon_s\right) ds \\ - \frac{1}{Eh} \oint_{C_{u_{e+1}}} & \left(N_i \frac{\partial(\nabla^2 \Phi)}{\partial n} + Eh\frac{\sqrt{t}}{\ell}\frac{dN_i}{d\xi}\varepsilon_s\right) ds, \qquad i = 1, 2, 3, 4, \end{aligned} \tag{60}$$

where

$$M_n = M_x \cos^2 \alpha + M_y \sin^2 \alpha - 2M_{xy} \sin \alpha \cos \alpha ,$$
$$M_{ns} = (M_x - M_y) \sin \alpha \cos \alpha + M_{xy}(\cos^2 \alpha - \sin^2 \alpha),$$
$$Q_n = Q_x \cos \alpha + Q_y \sin \alpha ,$$
$$V_n = Q_n - \partial M_{ns}/\partial s ,$$
$$\varepsilon_s = \varepsilon_x \sin^2 \alpha + \varepsilon_y \cos^2 \alpha - 2\varepsilon_{xy} \sin \alpha \cos \alpha ,$$
$$\cos \alpha = -u_x/\sqrt{t}, \quad \sin \alpha = -u_y/\sqrt{t}, \quad t = u_x^2 + u_y^2.$$

After substitution for W and Φ, (59) and (60) reduce to

$$D_f[K_1^e]\{\delta^e\} + [K_2^e]\{\Delta^e\} = \{F^e\} + \{P^e\} - \{P^{e+1}\}, \tag{61}$$

$$[K_2^e]^T\{\delta^e\} - \frac{1}{Eh}[K_3^e]\{\Delta^e\} = \frac{1}{Eh}\left(\{Q^e\} - \{Q^{e+1}\}\right), \tag{62}$$

where $[K_k^e]$, $k = 1, 2, 3$ are 4 x 4 generalised stiffness matrices in which the (i, j) components are given by

$$[K_1^e]_{ij} = \iint_{\Omega_e} \left\{ S_1 \frac{dN_i}{d\xi}\frac{dN_j}{d\xi} + S_2\left(\frac{dN_i}{d\xi}\frac{d^2N_j}{d\xi^2} + \frac{d^2N_i}{d\xi^2}\frac{dN_j}{d\xi}\right) + S_3 \frac{d^2N_i}{d\xi^2}\frac{d^2N_j}{d\xi^2} \right\} d\Omega_e , \tag{63}$$

$$[K_2^e]_{ij} = \iint_{\Omega_e} N_i \left\{ S_4 \frac{dN_j}{d\xi} + S_5 \frac{d^2N_j}{d\xi^2} \right\} d\Omega_e , \tag{64}$$

$$[K_3^e]_{ij} = \iint_{\Omega_e} \left\{ S_6 \frac{dN_i}{d\xi}\frac{dN_j}{d\xi} + S_7\left(\frac{dN_i}{d\xi}\frac{d^2N_j}{d\xi^2} + \frac{d^2N_i}{d\xi^2}\frac{dN_j}{d\xi}\right) + S_3 \frac{d^2N_i}{d\xi^2}\frac{d^2N_j}{d\xi^2} \right\} d\Omega_e , \quad i, j = 1, 2, 3, 4, \tag{65}$$

and the quantities S_k, $k = 1, \cdots, 7$, are defined by

$$\begin{aligned}
S_1 &= \{(\nabla^2 u)^2 - 2(1-\nu)(u_{xx}u_{yy} - u_{xy}^2)\}/\ell^2, \\
S_2 &= \{t\nabla^2 u - (1-\nu)(u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy})\}/\ell^3, \\
S_3 &= t^2/\ell^4, \\
S_4 &= \left\{\frac{u_{xx}}{R_y} + \frac{u_{yy}}{R_x} - \frac{2u_{xy}}{R_{xy}}\right\}/\ell, \\
S_5 &= \left\{\frac{u_x^2}{R_y} + \frac{u_y^2}{R_x} - \frac{2u_x u_y}{R_{xy}}\right\}/\ell^2, \\
S_6 &= \{(\nabla^2 u)^2 - 2(1+\nu)(u_{xx}u_{yy} - u_{xy}^2)\}/\ell^2, \\
S_7 &= \{t\nabla^2 u - (1+\nu)(u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy})\}/\ell^3.
\end{aligned} \tag{66}$$

The components of the column vectors $\{F^e\}$, $\{P^e\}$, $\{Q^e\}$, etc., may be determined from the following expressions

$$F_i^k = \iint_{\Omega_e} N_i q_z \, d\Omega_e, \tag{67}$$

$$P_i^k = \oint_{C_{u_k}} \left(N_i V_n + \frac{\sqrt{t}}{\ell}\frac{dN_i}{d\xi} M_n\right) ds, \tag{68}$$

and

$$Q_i^k = \oint_{C_{u_k}} \left(N_i \frac{\partial(\nabla^2\Phi)}{\partial n} + \frac{Eh\sqrt{t}}{\ell}\frac{dN_i}{d\xi}\varepsilon_s\right) ds, \qquad k = e, e+1, \; i = 1, 2, 3, 4, \tag{69}$$

Grouping together all element unknowns, the element equations can be conveniently re-arranged to yield the standard form

$$[\tilde{K}^e]\{\tilde{\delta}^e\} = \{\tilde{F}^e\} + \{\tilde{P}^e\}. \tag{70}$$

5. Illustrative examples

5.1. Bending of doubly-curved shallow shells elliptical in plan

(a) *Clamped Boundary.* Consider the deflection of a clamped shallow shell, elliptical in plan, and under a uniformly distributed load q_z. The surface of the shell is described by

$$z(x, y) = \frac{x^2}{2R_x} + \frac{y^2}{2R_y}, \tag{71}$$

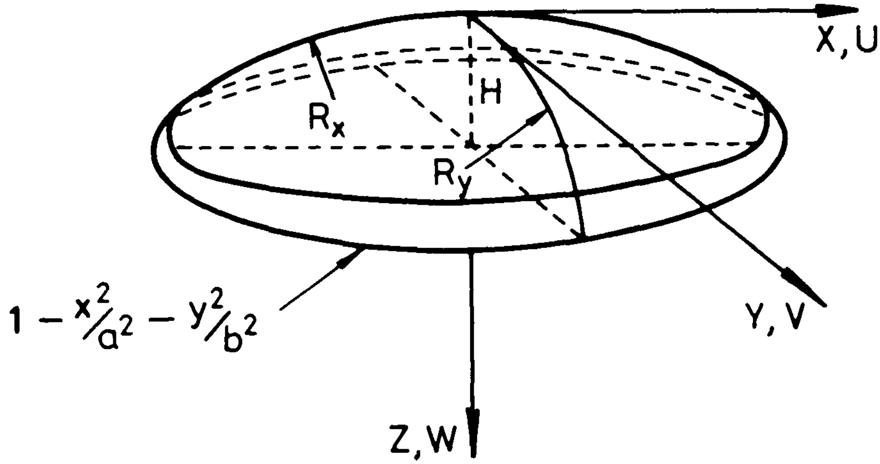

FIGURE 5. Shallow shell elliptical in plan-form.

where R_x, R_y are the constant radii of curvature in the x, y directions, respectively (Figure 5).

This problem is of major technical importance, since shells of this type are frequently encountered in the aerospace industry, and widely used in the design and construction of concrete shell roofs [19], e.g. football stadiums. As a consequence, a number of papers on the approximate analytical solution to the problem have emerged in the literature [3, 13, 14].

In view of the symmetrical nature of the problem, the equation for the contour lines has the form

$$u(x, y) = 1 - x^2/a^2 - y^2/b^2, \tag{72}$$

where a, b are respectively the semi-major and semi-minor axes of the elliptic base, as shown in the figure.

For this example, the elliptical region of the shell is discretised into four contour elements and the results obtained via the displacement approach and the stress function approach are summarised in Tables 1 and 2. The tables also include corresponding quantities computed from the approximate closed-form solutions to the problem that exist in the literature [13, 14].

(b) *Simply-Supported Boundary.* Consider now the same shallow shell as in the previous example, but with a simply supported boundary, and under the influence of a concentrated load (P) at the apex. As before, because of symmetry, the constant-deflection lines may be approximated by a family of similar and similarly situated ellipses, described by the expression in (72).

The deflection under the load, obtained by both the displacement and stress function approaches to this problem, is presented in Table 3 for various values of the aspect ratio of the elliptical base.

TABLE 1. Central deflection (W^*) of a shallow shell upon an elliptical base. $(K^* = 1.3218,\ R_x/R_y = a^2/b^2)$

$W^* \times 10^3$					
a/b	1.1	1.3	1.5	2.0	5.0
Displacement Approach	12.3157	8.0844	5.3265	2.0486	0.06262
Stress Function Approach	12.3304	8.1468	5.4067	2.1028	0.06490
Jones & Mazumdar [14]	12.3268	8.1336	5.3930	2.0975	0.06481
Jones [13]	12.4177	8.1571	5.3794	2.0730	0.06367

TABLE 2. Central bending moment (M_x^*) of a shallow shell upon an elliptical base $(K^* = 1.3218,\ R_x/R_y = a^2/b^2)$

$M_x^* \times 10^2$					
a/b	1.1	1.3	1.5	2.0	5.0
Displacement Approach	6.6991	4.8620	3.5608	1.7993	0.2126
Stress Function Approach	6.7077	4.9018	3.6172	1.8496	0.2210
Jones & Mazumdar [14]	6.7057	4.8938	3.6082	1.8445	0.2203
Jones [13]	6.7701	4.9170	3.6042	1.8242	0.2165

TABLE 3. Deflection under load for a simply supported shallow shell elliptical in plan. $(K^* = 1.3218,\ R_x/R_y = a^2/b^2,\ \nu = 0.3)$

$WD_f/Pa^2 \times 10^2$							
a/b	1.0	1.1	1.2	1.3	1.4	1.5	2.0
Displacement Approach	4.3882	3.9348	3.4788	3.0486	2.6587	2.3139	1.1818
Stress Function Approach	4.3883	3.9531	3.5351	2.8342	2.7826	2.4559	1.3122

5.2. Bending of a shallow spherical shell on an equilateral triangular base

Consider the bending of a shallow spherical shell, in the form of an equilateral triangle in plan, by a uniformly distributed load (q_z) over its surface. The edges of the shell are assumed simply supported by shear diaphragm walls (Figure 6), while the spherical surface of the shell is described by the equation given in (1) with $R_x = R_y = R$ and $R_{xy} = 0$.

As a first approximation, the expression for the equation of the contour lines, in this case, has the form

$$u(x, y) = \left(x^3 - 3xy^2 - ax^2 - ay^2 + 4a^3/27\right)\left(4a^2/9 - x^2 - y^2\right), \quad (73)$$

which represents the exact expression for the equation of the constant deflection contours of the corresponding flat-plate problem $(R = 0)$ under similar loading and boundary conditions. Here, a denotes the perpendicular height of the equilateral triangular base, as shown in the figure.

Tables 4, 5 and 6 present centroidal results determined with the use of both the displacement and stress function approaches, and corresponding results computed from the approximate analytical solution to the problem [1].

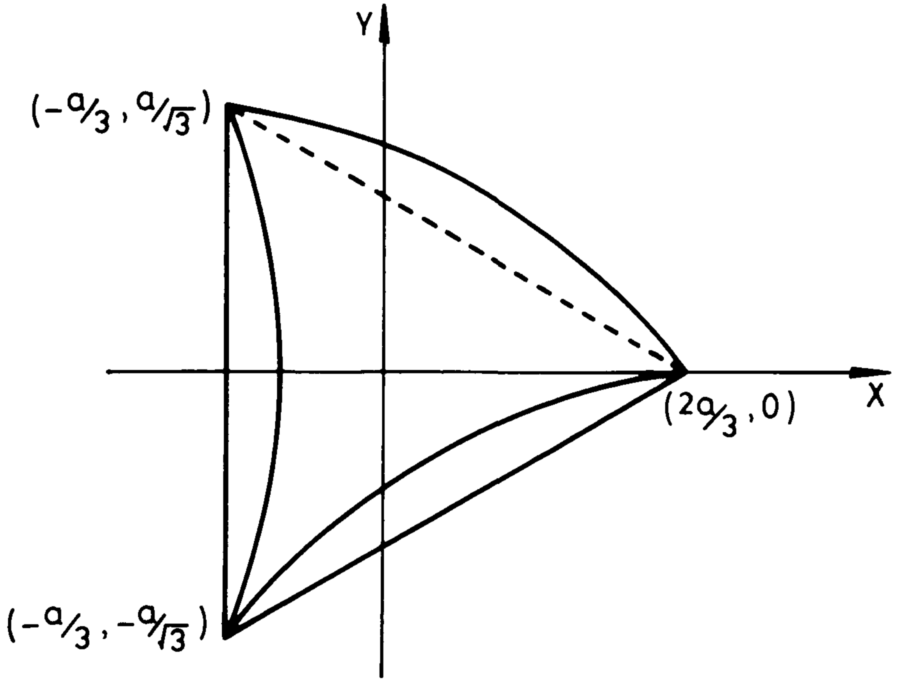

FIGURE 6. Shallow spherical shell, equilateral triangular in plan-form.

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TABLE 4. Centroidal displacement of a shallow spherical shell, equilateral triangle in plan, for various values of the curvature parameter K^*.

$W^* \times 10^4$					
K^*	5	10	20	50	100
Displacement Approach	10.1361	9.6610	8.1289	3.7888	1.2159
Stress Function Approach	10.1400	9.6752	8.1696	3.8467	1.2405
Bhattacharya [1]	10.1209	9.6495	8.1259	3.7812	1.2014

TABLE 5. Centroidal bending moment of a shallow spherical shell, equilateral triangle in plan, for various values of the curvature parameter K^*

$M_x^* \times 10^2$					
K^*	5	10	20	50	100
Displacement Approach	2.3679	2.2476	1.8602	0.76998	0.15232
Stress Function Approach	2.3689	2.2511	1.8701	0.78265	0.15378
Bhattacharya [1]	2.3647	2.2445	1.8565	0.76064	0.14787

TABLE 6. Centroidal membrane force of a shallow spherical shell, equilateral triangle in plan, for various values of the curvature parameter K^*.

$N_x^*/K^* \times 10^4$					
K^*	5	10	20	50	100
Displacement Approach	5.4609	5.2047	4.3788	2.0394	0.6535
Stress Function Approach	5.0045	4.7749	4.0327	1.9015	0.6161
Bhattacharya [1]	5.0604	4.8248	4.0629	1.8906	0.6007

The nondimensional quantities appearing in the tables are defined as follows;

$$W^* = WD_f/(q_z a^4), \tag{74}$$

$$M_x^* = M_x/(q_z a^2), \tag{75}$$

$$N_x^* = \left(\frac{D_f}{Eh}\right)^{\frac{1}{2}} \frac{N_x}{q_z a^2}, \tag{76}$$

$$K^* = \left(\frac{Eh}{D_f}\right)^{\frac{1}{2}} \frac{a^2}{R}, \tag{77}$$

5.3. Bending of a shallow spherical shell upon a square base

A technically important problem which appears as a test example in many papers concerned with the numerical analysis of shallow shells [9, 15] is the bending of a shallow spherical shell on a square base. The surface of the shell is under uniformly distributed load (q_z), while the boundary of the shell is simply supported on shear diaphragm walls, as shown in Figure 7.

The exact solution to the above problem exists in the form of a double infinite Fourier series for both W and Φ [23].

In order to apply the present method to the above problem, an expression for the equation describing the contours of the deflected surface of the shell is required. Since the function u vanishes along the boundary, and because of the symmetrical nature of the problem, the expression for u may be taken

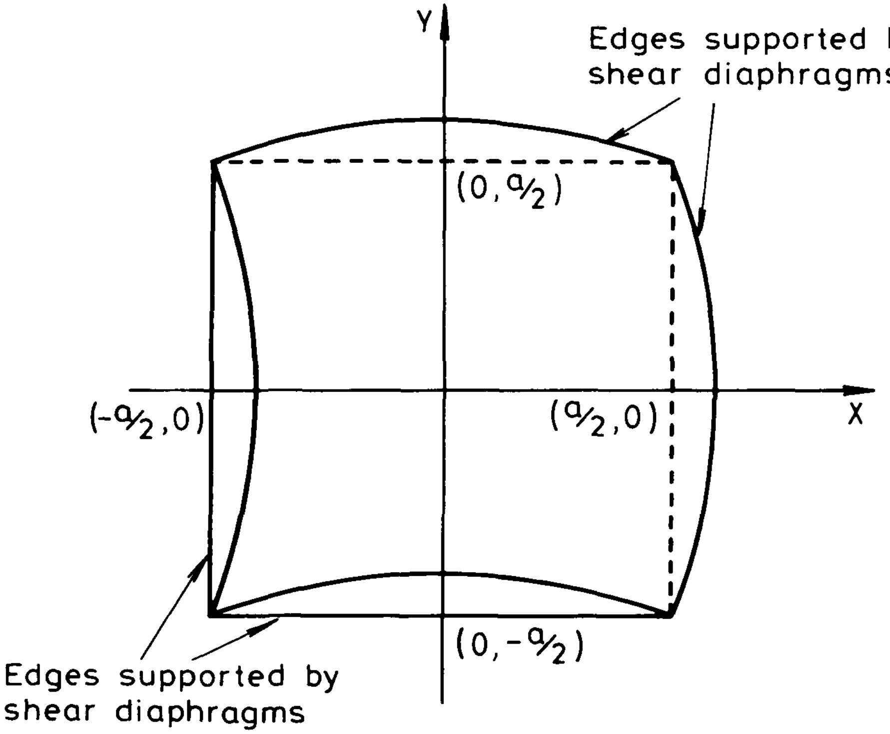

FIGURE 7. Shallow spherical shell upon a square base.

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in the boundary form

$$u(x, y) = \left(x^2 - a^2/4\right)\left(y^2 - a^2/4\right), \qquad (78)$$

where a is the dimension of the square base.

The central displacement, bending moment and membrane stress, computed independently with the aid of both the stress function and displacement approach are presented in Table 7, together with the corresponding results, obtained by a finite element method [8], and the exact values for these quantities which are also quoted in the above-mentioned reference. Only four contour elements are used in the present analysis, and two effective terms are retained in the displacement approach [4].

TABLE 7. Shallow spherical shell on a square base, under uniform load q_z.

$Rh/a^2 = 0.02$				
	Stress Function Approach	Displacement Approach	Finite Element Method [8]	Exact [8]
EhW/q_zR^2	1.0023	0.99895	1.009819	1.009785
$M_x/q_zRh \times 10^3$	−8.8623	−8.3982	−8.873	−8.487
$N_x/q_zR \times 1.0$	5.0260	5.0161	5.111	5.049
$Rh/a^2 = 0.005$				
	Stress Function Approach	Displacement Approach	Finite Element Method [8]	Exact [8]
EhW/q_zR^2	0.98303	0.99800	1.000241	1.000429
$M_x/q_zRh \times 10^5$	−3.2371	−3.0238	–	−3.1
$N_x/q_zR \times 10$	5.0059	5.0172	5.013	5.002

It is apparent from Table 7 that, with only four elements and an approximate expression for u, the present results are sufficiently accurate for most practical purposes.

Next, in order to explore the numerical convergence of the two approaches, the value of the strain energy Π_u, computed for increasing values of m, where m denotes the number of elements used to discretise the shell region, is summarised in Table 8, together with the total number of degrees of freedom required for solution.

In the table, Π_u^* represents the non-dimensional value of the strain energy defined by

$$\Pi_u^* = 10Eh\Pi_u/(q_z^2a^2R^2). \qquad (79)$$

TABLE 8. Strain energy convergence for a simply supported shallow spherical shell on a square base.

$\Pi_u^*(Rh/a^2 = 0.02)$				
Number of Elements	Stress Function Approach	Net DOF	Displacement Approach	Net DOF
2	3.837439	10	3.800501	26
4	3.84301	18	3.808480	54
6	3.843432	26	3.807086	82
Exact		3.89958		
$\Pi_u^*(Rh/a^2 = 0.005)$				
Number of Elements	Stress Function Approach	Net DOF	Displacement Approach	Net DOF
2	4.397121	10	4.377570	26
4	4.402422	18	4.397054	54
6	4.406677	26	4.400775	82
Exact		4.44990		

Although the numerical convergence of the present method, towards the exact value of the strain energy for this problem, is relatively slow, it is clear from the table that the accuracy of the results is adequate for practical purposes, even if the shell is modelled by only two elements. Furthermore, the stress function approach requires fewer degrees of freedom in the solution process.

6. Conclusion

Two generalised shallow-shell bending elements are developed for the analysis of doubly-curved shallow shells having arbitrary shaped plan forms. The elements are defined using the concept of iso-deflection contour lines of the shell in combination with standard finite element techniques. One element employs displacements as the element unknowns, while the other element is hybrid, in that both a displacement component and a stress function are utilised as the basic element variables. Several illustrative examples are included to demonstrate the accuracy and relative convergence of the two elements when used for static analysis purposes.

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