On the Infinitesimal Rigidity of Homogeneous Varieties

J. M. LANDSBERG[*]
*Laboratoire de Mathématiques, Université Paul Sabatier, UFR-MIG 31062 Toulouse Cedex 4,
France; e-mail: jml@picard.ups-tlse.fr*

Abstract. Let $X \subset \mathbb{P}^N$ be a variety (respectively an open subset of an analytic submanifold) and let $x \in X$ be a point where all integer valued differential invariants are locally constant. We show that if the projective second fundamental form of X at x is isomorphic to the second fundamental form of a point of a Segre $\mathbb{P}^n \times \mathbb{P}^m$, $n, m \geqslant 2$, a Grassmaniann $G(2, n+2)$, $n \geqslant 4$, or the Cayley plane $\mathbb{OP}^2$, then X is the corresponding homogeneous variety (resp. an open subset of the corresponding homogeneous variety). The case of the Segre $\mathbb{P}^2 \times \mathbb{P}^2$ had been conjectured by Griffiths and Harris in [GH]. If the projective second fundamental form of X at x is isomorphic to the second fundamental form of a point of a Veronese $v_2(\mathbb{P}^n)$ and the Fubini cubic form of X at x is zero, then $X = v_2(\mathbb{P}^n)$ (resp. an open subset of $v_2(\mathbb{P}^n)$). All these results are valid in the real or complex analytic categories and locally in the C^∞ category if one assumes the hypotheses hold in a neighborhood of any point x. As a byproduct, we show that the systems of quadrics $I_2(\mathbb{P}^{m-1} \sqcup \mathbb{P}^{n-1}) \subset S^2 \mathbb{C}^{m+n}$, $I_2(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \subset S^2 \mathbb{C}^{2n}$ and $I_2(\mathbb{S}_5) \subset S^2 \mathbb{C}^{16}$ are stable in the sense that if $A_t \subset S^2 T^*$ is an analytic family such that for $t \neq 0$, $A_t \simeq A$, then $A_0 \simeq A$.

We also make some observations related to the Fulton–Hansen connectedness theorem.

Introduction

The intrinsic rigidity of homogenous spaces has been studied extensively (see [HM] and the references therein). In this paper we study the extrinsic and infinitesimal rigidity of homogeneous spaces.

Let $G/P \subset \mathbb{P}^N$ be an *n*-dimensional homogenous space embedded homogeneously, but not necessarily in its canonical embedding. Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be a point where all integer valued differential invariants are locally constant. (Such points are called *general* points by algebraic geometers and *generic* points by differential geometers. Since a generic point has a different meaning in algebraic geometry, we have avoided using either term.) This paper addresses the question: To what extent do the projective differential invariants of X at x need to

---

resemble those of a point of $G/P \subset \mathbb{P}^N$ in order to be able to conclude $X = G/P$ as a projective variety? Let $|II_{X,x}| \subset \mathbb{P}S^2 T_x^* X$ denote (the system of quadrics induced by) the projective second fundamental form of X at x. Our progress on this question is as follows (an open subset means open with respect to the manifold topology)

THEOREM 1. *Let $X^{n+m} \subset \mathbb{P}^{nm+n+m+z}$, $n, m \geq 2$, be an open subset of an analytic manifold not contained in a hyperplane and let $x \in X$ be a point where all integer valued differential invariants are locally constant. If the second fundamental form $|II_{X,x}|$ is isomorphic to $I_2(\mathbb{P}^{m-1} \sqcup \mathbb{P}^{n-1})$, the quadrics vanishing on the disjoint union of two projective spaces, then $z = 0$ and X is an open subset of the Segre $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{nm+n+m}$. The same result holds locally in the C^∞ category if the hypotheses hold in the neighborhood of any point x.*

The case $n = m = 2$ had been conjectured by Griffiths and Harris in [GH].

THEOREM 2. *Let $X^{2(m-2)} \subset \mathbb{P}^{\binom{m}{2}-1+z}$, $m \geq 6$, be an open subset of an analytic manifold not contained in a hyperplane and let $x \in X$ be a point where all integer valued differential invariants are locally constant. If the second fundamental form $|II_{X,x}|$ is isomorphic to $I_2(\mathbb{P}^1 \times \mathbb{P}^{n-1})$, the quadrics vanishing on the Segre variety, then $z = 0$ and X is an open subset of the Grassmanian $G(2, m)$. The same result holds locally in the C^∞ category if the hypotheses hold in the neighborhood of any point x.*

Note that the result is false for $m = 4$.

THEOREM 3. *Let $X^{16} \subset \mathbb{P}^{26+z}$ be an open subset of an analytic manifold not contained in a hyperplane and let $x \in X$ be a point where all integer valued differential invariants are locally constant. If the second fundamental form $|II_{X,x}|$ is isomorphic to $I_2(\mathbb{S}_5)$, the quadrics vanishing on the spinor variety, then $z = 0$ and X is an open subset of the Cayley plane in its canonical embedding. The same result holds locally in the C^∞ category if the hypotheses hold in the neighborhood of any point x.*

The *Fubini cubic form* of X at x, $F_{3X,x}$ is a relative differential invariant encoding the geometric information in the third derivatives of the embedding. It was first used by Fubini [F] to study hypersurfaces. See [L1] for a precise definition.

THEOREM 4. *Let*

$$X^n \subset \mathbb{P}^{\binom{n+2}{2}-1}, \quad n > 1,$$

be an open subset of an analytic manifold not contained in a hyperplane and let $x \in X$ be a point where all integer valued differential invariants are locally constant. If the third fundamental form at x, $III_{X,x}$, is zero and $F_{3X,x} = 0$, then X is an open subset of the Veronese $v_2(\mathbb{P}^n)$. The same result holds locally in the C^∞ category if the hypotheses hold in the neighborhood of any point x.

THEOREM 5. *The systems of quadrics,*

$$A = I_2(\mathbb{P}^{m-1} \sqcup \mathbb{P}^{n-1}) \subset S^2\mathbb{C}^{m+n}, \qquad I_2(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \subset S^2\mathbb{C}^{2n}$$

and $I_2(\mathbb{S}_5) \subset S^2\mathbb{C}^{16}$ are stable in the sense that if $A_t \subset S^2T^*$ is an analytic family such that for $t \neq 0$, $A_t \simeq A$, then $A_0 \simeq A$.

Note that in the analytic category, the results are also global, since an open subset determines an entire variety. One must be careful in the case of the real Cayley plane to insure that the normalization of the second fundamental form used in computations is possible over $\mathbb{R}$.

1. Previous Results

Monge showed that a curve in $\mathbb{P}^2$ is a conic if and only if a fifth order invariant is zero (see [L1, 3.6]). In higher dimensions, Fubini showed that to determine if a hypersurface is a quadric, all third order invariants must be zero [F]. In another direction, it is known that the Segre variety cannot be deformed as a submanifold of projective space (see, e.g. [HM] Section 3). Note that while $\mathbb{P}^1 \times \mathbb{P}^1$ is rigid as a submanifold of projective space in the sense of [HM], it fails to satisfy the analog of Theorem 1. In [L2] we showed that to determine if X is one of the four Severi varieties (that is, $\mathbb{AP}^2$ in its canonical embedding, where $\mathbb{A}$ is the complexification of one of the four real division algebras, i.e. $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, $\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$, $G(2, 6) \subset \mathbb{P}^{14}$, $E_6/P_1 \subset \mathbb{P}^{26}$), it is necessary to have agreement of second fundamental forms and a partial vanishing of the cubic form. (The case of $v_2(\mathbb{P}^2)$ had been proven earlier by Griffiths and Harris [GH].) Theorems 1–3 strengthen these results for the three Severi varieties with degenerate tangential varieties in the sense that they show it is only necessary to have agreement of second fundamental forms. This strengthening follows immediately from Proposition 6 below.

In the Euclidean geometry of submanifolds, if the Euclidean second fundamental form is surjective, then a submanifold is uniquely determined by second order data (sometimes even first, e.g. hypersurfaces of large dimension). In projective geometry, the order of data needed to obtain a complete set of functionally independent differential invariants is not known except in some special cases. For curves in $\mathbb{P}^2$ sixth order information is necessary and sufficient. For hypersurfaces of dimension greater than two, Jensen and Musso proved third order information is necessary and sufficient [JM].

2. Intrinsic vs Extrinsic Geometry

The intrinsic and extrinsic geometries of homogeous spaces are closely related. Given any G/P, the cone of minimal degree rational curves passing through each point is an essentially intrinsic object. The projectivization of this cone is contained in the base locus of $|II|$ in the standard embedding. (A line osculating to order two at a point of a homogeneous space G/P is contained in G/P since homogeneous varieties are cut out by quadrics.)

The intrinsic rigidity results in [HM] resemble ours and have a similar method of proof. Hwang and Mok prove Kähler rigidity of Hermitian symmetric spaces of the compact type under Kähler deformations by studing deformations of the cone of minimal degree rational curves. This cone naturally sits in a projective space, thus their study at the infinitesimal level is similar to our extrinsic problem. However, their results are different, which can most easily be seen by the fact that the quadric hypersurface is not rigid to order two, but is Kähler rigid, and even holomorphically rigid (see [H]). It would be desirable to rephrase the proofs here in terms of a geometric property of the base locus of the second fundamental form as in [HM] (see below).

3. Secant and Dual Varieties

Extremal degeneracies of auxilliary varieties often force homogeneity. Zak proved that if $X^n \subset \mathbb{CP}^{n+a}$ is a smooth variety not contained in a hyperplane, and $a < (n/2) + 2$ then the secant variety $\sigma(X)$ must be equal to $\mathbb{P}^{n+a}$ and if $a = (n/2) + 2$ and $\sigma(X) \neq \mathbb{P}^{n+a}$, then X must be a Severi variety (see [Z]). Zak also proved that if $X^n \subset \mathbb{P}^{n+a} = \mathbb{P}V$ is a smooth variety not contained in a hyperplane, then $\dim X^* \geqslant \dim X$, where $X^* \subset \mathbb{P}V^*$ denotes the dual variety of X. Ein showed that if $\dim X^* = \dim X$, and $a \geqslant (n/2)$, then X is either a hypersurface, $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^m)$, the Grassmanian $G(2, 5)$ or the ten dimensional spinor variety $\mathbb{S}_5$ (all in their canonical embeddings), see [E].

If $X^n \subset \mathbb{P}^{n+a}$ is a smooth variety with degenerate secant variety, then $a \leqslant \binom{n+1}{2}$ (see [Z], [L2]). A special case of Zak's Theorem on Scorza varieties states that if $a = \binom{n+1}{2}$ and $\sigma(X)$ is degenerate, then X must be a Veronese $v_2(\mathbb{P}^n)$.

4. The Refined Third Fundamental Form and Connectedness

Let $X^n \subset \mathbb{P}^{n+a}$ be an open subset of a complex manifold. Let $x \in X$ be a point where all integer valued differential invariants are locally constant, and let $v \in T_x X$ be a generic tangent vector. If the mapping $II_v : T_x X \to N_x X$, defined by $w \mapsto II(v, w)$, is not surjective, there is a well defined third order invariant, called *the third fundamental form refined with respect to v*, III^v (see [L2] for details). Given a system of quadrics $A \subset S^2 T^*$ and $v \in T$, let $\mathrm{Ann}(v) = \{q \in A \mid [v] \in q_{\mathrm{sing}}\}$,

and let $\mathrm{Singloc}(A) = \{v \in T \mid [v] \in q_{\mathrm{sing}} \, \forall q \in A\}$. Note that $|II_{X,x}|/\mathrm{Ann}(v)$ is a well defined system of quadrics on $\mathrm{Singloc}(\mathrm{Ann}(v))$. With these notations

$$III^v \in S^3(\mathrm{Singloc}(\mathrm{Ann}(v)))^* \otimes N_x X / II_v(T).$$

A special case of the Fulton–Hansen connectedness theorem [FH] states that if $\dim \sigma(X) \neq 2n+1$ or $\dim \tau(X) \neq 2n$, then $\sigma(X) = \tau(X)$ for any projective variety X.

A consequence of the Fulton–Hansen theorem is that if X is a smooth variety with degenerate secant variety, then the refined third fundamental form is zero at points where all integer valued differential invariants are locally constant. In fact, the refined third fundamental form being zero implies $\sigma(X) = \tau(X)$ in the case X is smooth, see [L2]. In our original proof of Zak's theorem, we used the consequence of the connectedness theorem that $III^v \equiv 0$ to prove the rigidity of varieties that infinitesimally looked like Severi varieties.

If $A \subset S^2 T^*$ is a system of quadrics, its *prolongation*, $A^{(1)}$ is defined by $A^{(1)} = S^3 T^* \cap (A \otimes T^*)$.

PROPOSITION 6. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Let $x \in X$ be a point where all integer valued differential invariants are locally constant, and let $v \in T_x X$ be a generic tangent vector. With the notations of the paragraphs above, consider $|II_{X,x}|/\mathrm{Ann}(v)$ as a system of quadrics on $\mathrm{singloc}(\mathrm{Ann}(v))$. Then $|III^v| \subseteq (|II_{X,x}|/\mathrm{Ann}(v))^{(1)}$.*

Proposition 6 follows from the formula [L2, 13.1]. The first line of [L2, 13.1] shows that $|III^v| \subseteq (|II_{X,x}|/\mathrm{Ann}(v)) \otimes T^*$, and the second line shows that it is symmetric.

Zak's Theorem on Scorza varieties indicates that perhaps Theorem 4 is not the optimal result. A positive answer to the following question would provide a local version of Zak's Theorem.

QUESTION 7. *Let $X^n \subset \mathbb{P}^{\binom{n+2}{2}-1}$ be an open subset of a complex manifold not contained in a hyperplane. Let $x \in X$ be a point where all integer valued differential invariants are locally constant, and let $v \in T_x X$ be a generic tangent vector. If $III^v = 0$, must X be an open subset of the Veronese $v_2(\mathbb{P}^n)$?*

The difference between knowing that the cubic form is zero and knowing that the refined third fundamental form with respect to all tangent directions is zero is a difference of

$$\binom{n+1}{2}\binom{n+2}{3} - (n+1)\binom{n+1}{2} \quad \text{vs} \quad \binom{n}{2}\binom{n}{3} - \binom{n+1}{2}$$

equations. From the proof of Theorem 3, one sees how to construct the germ of a negative answer, but there is no reason to believe any germ will extend to a smooth variety.

Using Proposition 6, we obtain a stronger infinitesimal rigidity result than in [L2] by observing that if $|II|$ is the second fundamental form of a Severi variety other than $v_2(\mathbb{P}^2)$, then $|II|^{(1)} = 0$ and thus $(II/\text{Ann}(v))^{(1)} = 0$. Theorems 1, 2, 3 in the case of Severi varieties follow from Proposition 4 and this observation. The proofs given here of Theorems 1 and 2 are better than those in [L2] because here the basis vectors used for $T_x X$ are in the baselocus of $|II|$. In contrast, in [L2] a basis consisting of essentially generic vectors (although not a generic basis) was used. One could write out a corresponding better proof for the $\mathbb{OP}^2$ case as well.

QUESTION 8. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with degenerate tangential variety. Let $x \in X$ be a point where all integer valued differential invariants are locally constant, and let $v \in T_x X$ be a generic tangent vector. With the notation of the paragraphs above consider $|II_{X,x}|/\text{Ann}(v)$ as a system of quadrics on* $\text{singloc}(\text{Ann}(v))$. *Must $(|II_{X,x}|/\text{Ann}(v))^{(1)} = 0$?*

An affirmative answer to Question 8 would provide a new proof of the Fulton–Hansen Theorem relating the dimensions of $\sigma(X)$ and $\tau(X)$ that is differential-geometric and elementary in nature in the case X is smooth. (In particular, one would not need Deligne's Bertini Theorem.)

A variant of Question 8 is as follows: Let $A \subset S^2 \mathbb{C}^n$ be a system of quadrics with a tangential defect (i.e. the quadrics in A satisfy a polynomial relation). What additional conditions can one impose on A to imply that if $A = |II_{X,x}|$ where $x \in X$ is a point where all integer valued differential invariants are locally constant, then any tensor corresponding to $|III_{X,x}^v|$ must be zero?

5. Ideas Towards More Geometric Proofs

While the proofs here are rather short, it would be desirable to have more geometric arguments. The rigidity statements in [HM] are proven by exploiting that $\Lambda^2 T_x X$ is generated by elements of the form $v \wedge v'$ where $[v] \in \text{Base}|II|$ and $v' \in T_{[v]} \text{Base}|II|$. In [LM] we show that if X is homogeneous, if $\sigma(\text{Base}|II|) = \mathbb{P}T_x X$, then $|II|^{(1)} = 0$, so in particular $III^v \equiv 0$. Thus, part of the results here follow from geometric arguments, but it is not in general true that all third-order information can be recovered from III^v. Accordingly, some further geometric properties of $\text{Base}|II|$ are needed.

6. Other Open Problems

The program of LeBrun and Salamon to classify the quaternionic-Kähler manifolds with positive scalar curvature (see [Le], [LS]) has reduced (via the twistor

transform) the classification problem to classifying the contact Fano manifolds. (It is generally conjectured that the only quaternionic-Kähler manifolds with positive scalar curvature are homogenous.) The only known contact Fano manifolds are the adjoint varieties. Given a contact Fano manifold projectively embedded via its contact line bundle, the base locus of its second fundamental form must be a Legendrian variety. So it is of particular interest to determine the extent a variety must resemble a homogeneous Legendrian variety (resp. adjoint variety) before one can conclude that it is a homogeneous Legendrian variety (resp. adjoint variety).

Another problem is to determine rigidity for the cases of $\mathbb{P}^1 \times \mathbb{P}^n$ and $G(2, 5)$, which are not covered by the theorems above. By Ein's results on dual varieties, one might conjecture that these varieties are rigid to second order as well.

PROOFS

We will use formulas for projective differential invariants derived in [L1].

The idea of the proofs is as follows: given any variety $X \subset \mathbb{P}V$, one has the first order adapted frame bundle $\pi \colon \mathcal{F}^1_X \to X$. The elements $f \in \mathcal{F}^1 = \mathcal{F}^1_X$ are bases of V that respect the flag $\hat{x} \subset \hat{T}_x X \subset V$, where $\hat{x}$ is the line in V corresponding to x and $\hat{T}_x X$ is the cone over the embedded tangent space. In particular, each $f \in \mathcal{F}^1$ determines a splitting of the flag which we denote $\hat{x} + T + N$. Although it is not in general a Lie group, $\mathcal{F}^1 \subset \mathrm{GL}(V)$.

Write the pullback of the Maurer–Cartan form of $\mathrm{Gl}(V)$ to $\mathcal{F}^1$ as

$$\Omega = \begin{pmatrix} \omega^0_0 & \omega^0_\beta & \omega^0_\nu \\ \omega^\alpha_0 & \omega^\alpha_\beta & \omega^\alpha_\nu \\ 0 & \omega^\mu_\beta & \omega^\mu_\nu \end{pmatrix}$$

with index ranges $1 \leqslant \alpha, \beta \leqslant \dim X$, $\dim X + 1 \leqslant \mu, \nu \leqslant \dim \mathbb{P}V$.

If $X = G/P$, one can reduce $\mathcal{F}^1$ until it is isomorphic to G (with fiber isomorphic to P). In that case one obtains the Maurer–Cartan form symbolically as

$$\Omega_G = \begin{pmatrix} \omega^0_0 & \omega^0_\beta & 0 \\ \omega^\alpha_0 & \omega^\alpha_\beta = \rho_T(\mathfrak{h}) & \omega^\alpha_\nu = A_2(\omega^0_\beta) \\ 0 & \omega^\mu_\beta = A_1(\omega^\alpha_0) & \omega^\mu_\nu = \rho_N(\mathfrak{h}) \end{pmatrix},$$

where H is the semi-simple part of P, $T = T_x X$, $N = N_x X$ are H-modules, and A_1, A_2 are H-equivariant maps. The zero in the upper right hand block indicates that any infinitesimal change in the splitting satisfies the 'transversality' condition that $\mathrm{d}N \subseteq \{T + N\}$. The dependence of the ω^α_ν block on the forms ω^0_β indicates that if one changes the choice of T, there is a corresponding change in choice of N mandated.

If X is a variety with the same second fundamental form as G/P, by restricting bases in $T_x X$ and $N_x X$ to be we can reduce $\mathcal{F}_X^1$ to a bundle $\mathcal{F}_X^2$ where the pullback of the the Maurer–Cartan form looks like

$$\Omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 \\ \omega_0^\alpha & \omega_\beta^\alpha = \rho_T(\mathfrak{h}) + w_1 & \omega_\nu^\alpha \\ 0 & \omega_\beta^\mu = A_1(\omega_0^\alpha) & \omega_\nu^\mu = \rho_N(\mathfrak{h}) + w_2 \end{pmatrix}$$

where w_1, w_2 are linear combinations of the other forms appearing in the Maurer–Cartan form. The proofs proceed by showing that there are reductions of $\mathcal{F}_X^2$ to G by restricting the admissible splittings that reduce to Ω_G.

In practice, we prove this by showing the invariants $F_k \in \pi^*(S^k T^* X \otimes N X)$ that contain the geometric information of the kth derivative of the embedding $X \to \mathbb{P}^N$, are zero for $k > 2$. In frames one writes $F_k = r_{\alpha_1 \ldots \alpha_k}^\mu \omega_0^{\alpha_1} \ldots \omega_0^{\alpha_k} \otimes e_\mu$, where ω_0^β is a basis of the semi-basic forms and e_μ is a basis of $N_x X(1)$, and the $r_{\alpha_1 \ldots \alpha_k}^\mu$ are functions defined on $\mathcal{F}^1$. F_k measures the infinitesimal motion of X away from its embedded tangent space to $(k-1)$-st order.

We recall the following formulae from [L1]

$$r_{\alpha\beta\gamma}^\mu \omega_0^\gamma = -\mathrm{d}q_{\alpha\beta}^\mu - q_{\alpha\beta}^\mu \omega_0^0 - q_{\alpha\beta}^\nu \omega_\nu^\mu + q_{\alpha\delta}^\mu \omega_\beta^\delta + q_{\beta\delta}^\mu \omega_\alpha^\delta, \tag{L1 2.15}$$

$$r_{\alpha\beta\gamma\delta}^\mu \omega_0^\delta = -\mathrm{d}r_{\alpha\beta\gamma}^\mu - 2r_{\alpha\beta\gamma}^\mu \omega_0^0 - r_{\alpha\beta\gamma}^\nu \omega_\nu^\mu + \mathfrak{S}_{\alpha\beta\gamma} r_{\alpha\beta\delta}^\mu \omega_\gamma^\delta - $$
$$- \mathfrak{S}_{\alpha\beta\gamma} q_{\alpha\delta}^\mu q_{\beta\gamma}^\nu \omega_\nu^\delta + \mathfrak{S}_{\alpha\beta\gamma} q_{\alpha\beta}^\mu \omega_\gamma^0, \tag{L1 2.17}$$

$$r_{\alpha\beta\gamma\delta\varepsilon}^\mu \omega_0^\varepsilon = -\mathrm{d}r_{\alpha\beta\gamma\delta}^\mu - 3r_{\alpha\beta\gamma\delta}^\mu \omega_0^0 - r_{\alpha\beta\gamma\delta}^\nu \omega_\nu^\mu + $$
$$+ \mathfrak{S}_{\alpha\beta\gamma\delta}(r_{\alpha\beta\gamma\varepsilon}^\mu \omega_\delta^\varepsilon + 2r_{\alpha\beta\gamma}^\mu \omega_\delta^0 - $$
$$- (r_{\alpha\beta\varepsilon}^\mu q_{\gamma\delta}^\nu + r_{\alpha\beta\gamma}^\nu q_{\delta\varepsilon}^\mu) \omega_\nu^\varepsilon - q_{\alpha\beta}^\mu q_{\gamma\delta}^\nu \omega_\nu^0). \tag{L1 2.20}$$

The functions $r_{\alpha_1 \ldots \alpha_k}^\mu$ vary in the fiber as follows: Under a motion

$$e_\alpha \mapsto e_\alpha + g_\alpha^0 e_0,$$

$$e_\mu \mapsto e_\mu + g_\mu^\alpha e_\alpha + g_\mu^0 e_0,$$

the corresponding changes in the coefficients of F_3, F_4 are as follows

$$\Delta r_{\alpha\beta\gamma}^\mu = \mathfrak{S}_{\alpha\beta\gamma} g_\alpha^0 q_{\beta\gamma}^\mu + \mathfrak{S}_{\alpha\beta\gamma} g_\nu^\delta q_{\alpha\beta}^\nu q_{\gamma\delta}^\mu,$$
$$\Delta r_{\alpha\beta\gamma\delta}^\mu = \mathfrak{S}_{\alpha\beta\gamma\delta} g_\alpha^0 r_{\beta\gamma\delta}^\mu + \mathfrak{S}_{\alpha\beta\gamma\delta} g_\nu^\varepsilon (r_{\alpha\beta\gamma}^\nu q_{\delta\varepsilon}^\mu + q_{\alpha\beta}^\nu r_{\gamma\delta\varepsilon}^\mu) + g_\nu^0 q_{\alpha\beta}^\mu q_{\gamma\delta}^\nu. \tag{L1 2.24}$$

Proof of Theorem 1. $z = 0$ because the prolongation of $|II|$ is zero. Let V have basis $\{e_0, e_i, e_s, e_{si}\}$, $1 \leqslant i, j, k, l \leqslant n$, $n + 1 \leqslant s, t, u, v \leqslant n + m$ adapted such that $x = [e_0]$, $\hat{T}_x X = \{e_0, e_i, e_s\}$ and $II_{X,x} = \omega_0^i \omega_0^s \otimes e_{is}$ (see, e.g., [GH] or [L1]). Note that the forms ω_j^i, ω_t^s are all independent and independent of the semi-basic forms because they represent infinitesimal motions that preserve our normalization of II.

To show X is an open subset of the Segre, we will show all higher differential invariants are zero. We see immediately that

$$r_{tu\beta}^{si} = 0 \ \forall \beta \quad \text{and} \quad t, u \neq s,$$

$$r_{jk\beta}^{si} = 0 \ \forall \beta \quad \text{and} \quad j, k \neq i,$$

(these equations include the equations for the refined third fundamental form being zero). The nonzero coefficients of F_3 satisfy the following equations. (Here and in what follows, we use the convention that if an index appears more than twice it is *not* to be summed over. E.g. there is no sum over s in (s1).) From now on, if latin indicies are distinct, we assume they are not equal.

$$r_{sts}^{si}\omega_0^s + r_{stk}^{si}\omega_0^k + r_{sti}^{si}\omega_0^i = \omega_t^i, \tag{s1}$$

$$r_{ijs}^{si}\omega_0^s + r_{ijt}^{si}\omega_0^t + r_{iji}^{si}\omega_0^i = \omega_j^s, \tag{s2}$$

$$r_{ss\beta}^{si}\omega_0^\beta = 2\omega_s^i, \tag{s3}$$

$$r_{ii\beta}^{si}\omega_0^\beta = 2\omega_i^s. \tag{s4}$$

Since the right-hand side of (s1), resp. (s2), is independent of s, resp. i, we conclude (assuming $n, m \geqslant 2$) $r_{sts}^{si} = 0$, $r_{iji}^{si} = 0$ and

$$r_{stk}^{si} = r_{utk}^{ui}, \tag{s5}$$

$$r_{ijt}^{si} = r_{kjt}^{sk}. \tag{s6}$$

Now

$$\Delta r_{stj}^{si} = g_{(tj)}^i, \qquad \Delta r_{ijt}^{si} = g_{(jt)}^s,$$

$$\Delta r_{sti}^{si} = g_{(ti)}^i + g_t^0, \qquad \Delta r_{ijs}^{si} = g_{(js)}^s + g_j^0.$$

Fixing a particular (i, s), use $g_{tj}^i, g_{tj}^s, g_{ti}^i, g_{js}^s$ to normalize all these terms to zero. By (s5,s6) the normalizations send the terms to zero for all i, s. Now (s1,s2) imply $\omega_t^i = 0$, $\omega_i^t = 0$ for all i, t so (s3,s4) imply $r_{ss\beta}^{si} = 0$, $r_{ii\beta}^{si} = 0$.

We have now reduced to frames where $F_3 = 0$. (At this point one has a projective connection on TX isomorphic to that on the Segre.) The coefficients of F_4,

$r^{si}_{\alpha\beta\gamma\delta}$ are zero unless, among $\alpha\beta\gamma\delta$, two are in the s, t, u range, two are in the i, j, k range, at least one is equal to s, and one is equal to i. We use $\Delta r^{si}_{ssii} = g^0_{si}$ to normalize $r^{si}_{ssii} = 0$ for all s, i. This uses up all the freedom to normalize differential invariants. (The g^0_α terms were useless as they always appeared with a corresponding g^ε_μ term.)

The remaining coefficients of F_4 that are potentially nonzero are $r^{si}_{stji}, r^{si}_{ssij}, r^{si}_{iist}$.

We now examine the coefficients of F_5. Fortunately, most of these are immediately seen to be zero. $r^{si}_{\alpha\beta\gamma\delta\varepsilon}$ is zero if four or all of the lower indices are all in the same range. Moreover, there must be at least two indicies that are either i or s. Consider

$$r^{si}_{sujks}\omega^s_0 + r^{si}_{sujki}\omega^i_0 = r^{si}_{suji}\omega^i_k + r^{si}_{suki}\omega^i_j,$$

$$r^{si}_{tuiks}\omega^s_0 + r^{si}_{tuiki}\omega^i_0 = r^{si}_{suik}\omega^s_t + r^{si}_{stik}\omega^s_u,$$

$$r^{si}_{ssjki}\omega^i_0 = r^{si}_{suji}\omega^i_k + r^{si}_{suki}\omega^i_j,$$

$$r^{si}_{tuiks}\omega^s_0 + r^{si}_{tuiki}\omega^i_0 = r^{si}_{suik}\omega^s_t + r^{si}_{stik}\omega^s_u.$$

In all equations the forms on the right hand side are all independent and independent of the forms on the left-hand side (which are independent as well). Thus all coefficients appearing are zero, in particular, all coefficients of F_4 are zero. Now consider $r^{si}_{stjii}\omega^i_0 + r^{si}_{stjis}\omega^s_0 = \omega^0_{tj}$.

Since the right-hand side is independent of i, s, we conclude both sides are zero. Using that $\omega^0_{tj} = 0$ for all t, j, the equations $r^{si}_{ssii\beta}\omega^\beta_0 = 2\omega^0_{si}$ imply $F_5 = 0$. We easily see the coefficients of F_6 are all zero and thus all higher differential invariants are zero. $\qquad\square$

Proof of Theorem 2. Again $z = 0$ because $|II|^{(1)} = 0$.

Let V have basis $\{e_0, e_{1j}, e_{2j}, e_{jk}\}$, where $3 \leqslant j, k, l \leqslant n + 2$, $\{\alpha\} = \{1j, 2j\}$. Normalize such that $II = (\omega^{1j}_0\omega^{2k}_0 - \omega^{1k}_0\omega^{2j}_0) \otimes e_{jk}$, $j < k$. Note that the forms $\omega^{1i}_{1j}, \omega^{2i}_{2j}, \omega^{1i}_{1i}, \omega^{2i}_{2i}, \omega^{2i}_{1i}, \omega^{1i}_{2i}$ are all independent and independent of the semi-basic forms because they represent infinitesimal motions that preserve our normalization of II. We have

$$r^{ij}_{(1k)(1l)\beta} = 0 \; \forall i, j, k, l \text{ distinct and } \forall \beta, \tag{g1}$$

$$r^{ij}_{(2k)(2l)\beta} = 0 \; \forall i, j, k, l \text{ distinct and } \forall \beta, \tag{g2}$$

(these equations include that the refined third fundamental form is zero). From now on, assume all indices are distinct. Using (g1), (g2), we have

$$r^{(ij)}_{(1i)(1k)(1i)}\omega^{1i}_0 + r^{(ij)}_{(1i)(1k)(1j)}\omega^{1j}_0 + r^{(ij)}_{(1i)(1k)(2i)}\omega^{2i}_0 + r^{(ij)}_{(1i)(1k)(2j)}\omega^{2j}_0 + r^{(ij)}_{(1i)(1k)(2l)}\omega^{2l}_0$$

$$= \omega^{(2j)}_{(1k)}. \tag{g3}$$

The right-hand side of (g3) is independent of i, so comparing with the same expression using m instead of i, (here we use that $n \geqslant 4$) we obtain

$$r_{(1i)(1k)(1i)}^{(ij)} = 0, \tag{g4}$$

$$r_{(1i)(1k)(1j)}^{(ij)} = r_{(1m)(1k)(1j)}^{(mj)}, \tag{g5}$$

$$r_{(1i)(1k)(2i)}^{(ij)} = r_{(1m)(1k)(2i)}^{(mj)}, \tag{g6}$$

$$r_{(1i)(1k)(2j)}^{(ij)} = r_{(1m)(1k)(2j)}^{(mj)}, \tag{g7}$$

$$r_{(1i)(1k)(2l)}^{(ij)} = r_{(1m)(1k)(2l)}^{(mj)}. \tag{g8}$$

Now

$$\Delta r_{(1i)(1k)(2l)}^{(ij)} = g_{(kl)}^{(2j)},$$
$$\Delta r_{(1i)(1k)(2j)}^{(ij)} = g_{(jk)}^{(2j)} + g_{(1k)}^{0}.$$

Using these equations and the corresponding equations with the role of 1 and 2 reversed, we reduce to frames where

$$r_{(1i)(1k)(2l)}^{(ij)} = 0, \qquad r_{(1i)(1k)(2j)}^{(ij)} = 0, \qquad r_{(2i)(2k)(1l)}^{(ij)} = 0, \qquad r_{(2i)(2k)(1j)}^{(ij)} = 0.$$

In these frames, $\omega_{2j}^{1k}, \omega_{1j}^{2k} = 0$, hence

$$0 = r_{(1i)(1i)\beta}^{(ij)}\omega_0\beta = -2\omega_{1i}^{2j} \tag{g9}$$

and similarly with the role of 1 and 2 reversed. Thus the only nonzero terms left in F_3 are $r_{(1i)(1j)(1k)}^{(ij)}, r_{(2i)(2j)(2k)}^{(ij)}$. Consider $r_{(1i)(1j)(1i)\beta}^{(ij)}\omega_0^{\beta} = 2r_{(1i)(1j)(1k)}^{(ij)}\omega_{1i}^{1k}$. Both sides of (g10) must be zero because the forms ω_{1i}^{1k} are all independent and independent of the semi-basic forms. The analogous equation holds with 2's. Hence we see $F_3 = 0$.

To have a nonzero coefficient of F_4, $r_{\alpha\beta\gamma\delta}^{(ij)}$, in the lower indicies there must be two 1's and two 2's, and at least two of the k-indices must be i or j. Consider

$$r_{(1i)(1k)(2l)(2j)}^{(ij)}\omega_0^{2j} = \omega_{kl}^{2j}, \tag{g11}$$

$$r_{(1i)(1i)(2l)(2j)}^{(ij)}\omega_0^{2j} = 2\omega_{il}^{2j}. \tag{g12}$$

Since the right-hand side of (g11) is independent of i, we conclude (after switching the roles of i and j) that $r_{(1i)(1k)(2l)(2j)}^{(ij)}$ is independent of i, j (with neither k, l equal to i or j, but $k = l$ is possible). Using $\Delta r_{(1i)(1k)(2l)(2j)}^{(ij)} = g_{kl}^{0}$ we normalize all these

terms to zero. This implies $\omega_{il}^{2j} = 0$ and thus $r_{(1i)(1i)(2l)(2j)}^{(ij)} = 0$ for all i, j, l distinct as well, and similarly with the role of 1 and 2 reversed. Thus the remaining nonzero terms in F_4 are $r_{(1i)(1i)(2j)(2j)}^{(ij)}, r_{(1i)(2i)(1j)(2j)}^{(ij)}$. Consider

$$r_{(1i)(2j)(1k)(2l)(1i)}^{(ij)}\omega_0^{1i} + r_{(1i)(2j)(1k)(2l)(1j)}^{(ij)}\omega_0^{1j} + r_{(1i)(2j)(1k)(2l)(2i)}^{(ij)}\omega_0^{2i} +$$
$$+ r_{(1i)(2j)(1k)(2l)(2j)}^{(ij)}\omega_0^{2j} = -\omega_{kl}^0. \tag{g13}$$

Since the right-hand side of (g13) is independent of i, j, we conclude $\omega_{kl}^0 = 0$ and, hence, the left-hand side is zero as well. Now it is easy to see the rest of the terms in F_5 are zero and all higher forms are zero. $\quad\square$

Remark. While the last step used $m \geqslant 4$ a second time, there is an alternate argument that avoids it here. One first observes the ω_0^{kl} are semi-basic and then uses the equation for $r_{(1i)(1i)(2j)(2k)\beta}^{(ij)}\omega_0^\beta$.

Proof of Theorem 4. $III = 0$ and X not contained in a hyperplane implies that $|II| = \mathbb{P}S^2T^*$. Take a basis (e_0, e_j, e_{ij}) of V, $1 \leqslant i, j, k \leqslant n$ such that $II = \omega_0^i\omega_0^j \otimes e_{ij}$. With this normalization the forms ω_j^i are all independent and independent of the semi-basic forms. We cannot use the g_{jk}^i, g_j^0 to make normalizations because we assumed $F_3 = 0$. The coefficients of F_4, r_{klmp}^{ij}, must be zero if three or four of the lower coefficients are different from i, j. Regarding the other coefficients

$$r_{ikli}^{ij}\omega_0^i + r_{iklj}^{ij}\omega_0^j = \omega_{kl}^j, \tag{v1}$$

$$r_{jklj}^{jj}\omega_0^j = 2\omega_{kl}^j \tag{v2}$$

and one has the corresponding equations with $k = l$. Combining (v1) and (v2) we conclude $r_{ikli}^{ij}, r_{ikki}^{ij} = 0$, (here we use $n \geqslant 3$, see [GH] or [L2] for a proof when $n = 2$) and if $n \geqslant 4$ we also have, since the right-hand side of (v1) is independent of i, $r_{iklj}^{ij} = r_{mklj}^{mj}$. The variability $\Delta r_{iklj}^{ij} = g_{kl}^0$, $\Delta r_{ikkj}^{ij} = g_{kk}^0$ allows us to normalize $r_{iklj}^{ij}, r_{ikkj}^{ij} = 0$ which implies $\omega_{kl}^i, \omega_{kk}^i = 0$, in turn implying $r_{ikli}^{ii}, r_{ikli}^{ii} = 0$. Since $\omega_{ik}^j, \omega_{ii}^j = 0$ we have $r_{iik\beta}^{ij}\omega_0^\beta = 0$, $r_{iik\beta}^{ii}\omega_0^\beta = 0$ so $r_{ikii}^{ij}, r_{iikj}^{ij}, r_{iiii}^{ij}, r_{iiij}^{ij} = 0$. The only potentially nonzero terms in F_4 are $r_{ijij}^{ij}, r_{iiii}^{ii}$. Consider $r_{ijkl\beta}^{ij}\omega_0^\beta = \omega_{kl}^0$ which implies ω_{kl}^0 is semi-basic. Using $r_{iijk\beta}^{ij}\omega_0^\beta = r_{iijj}^{ij}\omega_k^j + \omega_{ik}^0$, we conclude $r_{iijj}^{ij} = 0$, since ω_k^j is independent of the semi-basic forms. Using the corresponding equation with i replacing j we see $F_4 = 0$. Now it is easy to see that F_5 and all higher invariants are zero. $\quad\square$

Acknowledgement

It is a pleasure to thank Jun-Muk Hwang for useful conversations.

References

[E]   Ein, L.: Varieties with small dual varieties, *I, Invent. Math.* **86** (1986), 63–74.
[F]   Fubini, G.: Il problema della deformazione proiettiva delle ipersuperficie, *Rend. Acad. Naz. dei Lincei* **27** (1918), 147–155.
[FH]   Fulton, W. and Hansen, J.: A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings, *Ann. Math.* **110** (1979), 159–166.
[GH]   Griffiths, P. and Harris, J.: Algebraic geometry and local differential geometry, *Ann. Sci. École Norm. Sup.* **12** (1979), 355–432.
[H]   Hwang, J-M.: Nondeformability of the complex hyperquadric, *Invent. Math.* **120** (1995), 317–338.
[HM]   Hwang, J-M. and Mok, N.: Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation, preprint.
[JM]   Jensen, G. and Musso, E.: Rigidity of hypersurfaces in complex projective space, *Ann. Sci. École Norm. Sup.* **27** (1994), 227–248.
[L1]   Landsberg, J. M.: Differential-geometric characterizations of complete intersections, *J. Differential Geom.* **44** (1996), 32–73.
[L2]   Landsberg, J. M.: On degenerate secant and tangential varieties and local differential geometry, *Duke Math. J.* **85** (1996), 1–30.
[LM]   Landsberg, J. M. and Manivel, L.: On the geometry of homogeneous varieties, in preparation.
[Le]   LeBrun, C.: Fano manifolds, contact structures and quaternionic geometry, *Internat. J. Math.* **6** (1995), 419–437.
[LS]   LeBrun, C. and Salamon, S.: Strong rigidity of quaternion-Kähler manifolds, *Invent. Math.* **118** (1994), 109–132.
[Z]   Zak, F.: *Tangents and Secants of Algebraic Varieties*, Transl. Math. Monogr. 127, Amer. Math. Soc., Providence, 1993.