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LOCALLY DEFINED FITTING CLASSES

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Fitting classes of finite solvable groups were first considered by Fischer, who with Gäschutz and Hartley (1967) showed in that in each finite solvable group there is a unique conjugacy class of "$\mathfrak{F}$-injectors", for $\mathfrak{F}$ a Fitting class. In general the behaviour of Fitting classes and injectors seems somewhat mysterious and hard to determine. This is in contrast to the situation for saturated formations and $\mathfrak{F}$-projectors of finite solvable groups which, because of the equivalence of saturated formations and locally defined formations, can be studied in a much more detailed way. However for those Fitting classes $\mathfrak{F}$ that are "locally defined" the theory of $\mathfrak{F}$-injectors can be made more explicit by considering various centralizers involving the local definition of $\mathfrak{F}$, giving results analogous to some of those concerning locally defined formations. Particular attention will be given to the subgroup $B(\mathfrak{S})$ defined by

$$B(\mathfrak{S}) = \prod C_{S_p}(G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}),$$

where the set $\{\mathfrak{F}(p)\}$ of Fitting classes locally defines $\mathfrak{F}$, and the S_p are the Sylow p-subgroups associated with a given Sylow system $\mathfrak{S} - B(\mathfrak{S})$ plays a role very much like that of Graddon's $\mathfrak{F}$-reducer in Graddon (1971). An $\mathfrak{F}$-injector of $B(\mathfrak{S})$ is an $\mathfrak{F}$-injector of G, and for certain simple $\mathfrak{F}$ $B(\mathfrak{S})$ is an $\mathfrak{F}$-injector of G.

All groups will be finite and solvable. Recall that a class $\mathfrak{F}$ of groups is a Fitting class if $\mathfrak{F}$ is closed under taking normal subgroups and if in each group G there is a unique normal subgroup that is maximal with respect to being in $\mathfrak{F}$; this subgroup, the $\mathfrak{F}$-radical, will be denoted by $G_{\mathfrak{F}}$. If $\mathfrak{F}$ and $\mathfrak{H}$ are Fitting classes then the extension class

$$\mathfrak{F}\mathfrak{H} = \{G: G/G_{\mathfrak{F}} \in \mathfrak{H}\}$$

is a Fitting class.

DEFINITION 1 (Fischer, Gäschutz and Hartley (1967)). *If $\mathfrak{F}$ is a Fitting class then $V \leqq G$ is an $\mathfrak{F}$-injector of G if $N \lhd\lhd G$ implies that $N \cap V$ is an $\mathfrak{F}$-maximal subgroup of N, where $\mathfrak{F}$-maximal means maximal with respect to being in $\mathfrak{F}$.*

By Fischer, Gäschutz and Hartley (1967) Satz 1 each group has a unique conjugacy class of $\mathfrak{F}$-injectors. It is easily seen that V an $\mathfrak{F}$-injector of G implies $V \geqq W_{\mathfrak{F}}$, for all W subnormal in G.

The following two propositinns hold for an arbitrary Fitting class, with the proof of Proposition 4 needing the lemma and first theorem of Fischer, Gäschutz and Hartley (1967). However when the Fitting class is locally defined the required results of Fischer, Gäschutz and Hartley (1967) can be replaced by Lemma 8 and Remark 14 below (which are independent of Proposition 4).

PROPOSITION 2. *If $\mathfrak{F}$ is a Fitting class and $G \in \mathfrak{F}\mathfrak{N}$, where $\mathfrak{N}$ is the Fitting class of nilpotent groups, then $G_{\mathfrak{F}}$ is an $\mathfrak{F}$-maximal subgroup of G.*

PROOF. Suppose $G_{\mathfrak{F}} \leqq F \leqq G$ with $F \in \mathfrak{F}$. If $F = G$ then $G_{\mathfrak{F}} = G$ so we may assume $F < G$. As $G/G_{\mathfrak{F}}$ is nilpotent there is a maximal subgroup M of G such that $F \leqq M \lhd G$. This means that

$$M_{\mathfrak{F}} = M \cap G_{\mathfrak{F}} = G_{\mathfrak{F}} \leqq F \leqq M$$

and so by induction $M_{\mathfrak{F}} = G_{\mathfrak{F}} = F$.

For ease of reference we quote the following

LEMMA 3 (Fischer, Gäschutz and Hartley (1967)). *If $\mathfrak{F}$ is a Fitting class, $R \lhd G$, $G/R \in \mathfrak{N}$, U an $\mathfrak{F}$-maximal subgroup of R, V and W $\mathfrak{F}$-maximal subgroups of G with $U \leqq V \cap W$ then V and W are conjugate in G.* (cf. Lemma 8).

PROPOSITION 4. *If $\mathfrak{F}$ is a Fitting class, $G \in \mathfrak{F}\mathfrak{N}\mathfrak{N}$ and $V \leqq G$ then V is an $\mathfrak{F}$-injector of G if and only if $V \geqq G_{\mathfrak{F}}$ and V is an $\mathfrak{F}$-maximal subgroup of G.*

PROOF. Suppose that $V \geqq G_{\mathfrak{F}}$ and is an $\mathfrak{F}$-maximal subgroup of G. If the $\mathfrak{F}\mathfrak{N}$-radical of G is R then $G/R \in \mathfrak{N}$. We have $V \cap R \lhd V \in \mathfrak{F}$ so $V \cap R \in \mathfrak{F}$. Also $V \cap R \geqq G_{\mathfrak{F}} \cap R = G_{\mathfrak{F}} = R_{\mathfrak{F}}$. Thus by Proposition 2, as $R \in \mathfrak{F}\mathfrak{N}$, $V \cap R = R_{\mathfrak{F}}$ is an $\mathfrak{F}$-maximal subgroup of R. By Fischer, Gäschutz and Hartley (1967) there is an $\mathfrak{F}$-injector W of G so similarly $W \cap R = G_{\mathfrak{F}}$. Hence by Lemma 3 V and W are conjugate in G i.e. V is an $\mathfrak{F}$-injector of G.

DEFINITION 5. *A Fitting class $\mathfrak{F}$ is locally defined by the set $\{\mathfrak{F}(p)\}_{p \in \Sigma}$ of Fitting classes $\mathfrak{F}(p)$, for p in a set Σ of primes, if*

$$\mathfrak{F} = \mathfrak{C}_{\Sigma} \cap \bigcap_{p \in \Sigma} \mathfrak{F}(p)\mathfrak{C}_p\mathfrak{C}_{p'}$$

where $\mathfrak{C}_{\pi}$ is the class of π-groups (for a set π of primes).

REMARK 6. We may assume that $(\forall p)\mathfrak{F}(p) = \mathfrak{F}(p)\mathfrak{C}_p \leqq \mathfrak{F}$. Note that $\mathfrak{F}$ is also locally defined by the set $\{\mathfrak{F}_0(p)\mathfrak{C}_p\}$ where $\mathfrak{F}_0(p)$ is the Fitting class generated by the class

$$\{F \in \mathfrak{F} : F = \mathbf{O}^{p'}(F)\}.$$

If the theory were to parallel exactly that of locally defined formations one would expect to have $\mathfrak{F}(p)$ equal to $\mathfrak{F}_0(p)\mathfrak{C}_p$. However it is unclear if this happens, so for the rest of the paper we fix the local definition and do not consider the question of whether things are actually independent of that local definition.

From now on $\mathfrak{F}$ is locally defined with the local definition $\{\mathfrak{F}(p)\}_{p \in \Sigma}$ satisfying $\mathfrak{F}(p) = \mathfrak{F}(p)\mathfrak{C}_p \leqq \mathfrak{F}$.

If L/H is a section of G, and $K \leqq G$, then

$$\mathbf{C}_K(L/H) = \{g \in K : [g, L] \leqq H\}.$$

An $\mathfrak{F}$-maximal subgroup contains a number of these "centralizers".

LEMMA 7. *If $F \leqq G$, $F \in \mathfrak{F}$, and S is a p-subgroup of G then $F \cdot \mathbf{C}_S(F/F_{\mathfrak{F}(p)})$ is in $\mathfrak{F}$; while if also $F \lhd G$, $F \leqq H \in \mathfrak{F}$ then $\mathbf{C}_S(H/H_{\mathfrak{F}(p)}) \leqq \mathbf{C}_S(F/F_{\mathfrak{F}(p)})$.*

PROOF. Let $C = \mathbf{C}_S(F/F_{\mathfrak{F}(p)})$; then $[F, C] \leqq F_{\mathfrak{F}(p)}$. Thus $F \lhd FC$ and $F_{\mathfrak{F}(p)}C$ is in $\mathfrak{F}(p)\mathfrak{C}_p \subseteq \mathfrak{F}$. Therefore $FC = F \cdot F_{\mathfrak{F}(p)}C$ is in $\mathfrak{F}$.

Assuming $F \lhd G$, $F \leqq H \in F$ then

$$[F, \mathbf{C}_S(H/H_{\mathfrak{F}(p)})] \leqq F \cap H_{\mathfrak{F}(p)} = F_{\mathfrak{F}(p)}.$$

The conjugacy statement of the next lemma is a special case of Lemma 3 but the proof is different and does not involve the Carter subgroup.

LEMMA 8. *If $M \lhd G$, $G/M \in \mathfrak{N}$, F is an $\mathfrak{F}$-maximal subgroup of M and $\mathfrak{S} = \{S_p\}$ is a Sylow system of G reducing into $\mathbf{N}_G(F)$ then*

$$C = \prod_{p \in \Sigma} \mathbf{C}_{S_p}(F/F_{\mathfrak{F}(p)})$$

is an $\mathfrak{F}$-maximal subgroup of G with $F = C \cap M$. Any $\mathfrak{F}$-maximal subgroup H of G with $F \leqq H$ is a conjugate of C in G.

PROOF. We show first that C is a subgroup of G. Let

$$S_\Sigma = \prod_{p \in \Sigma} S_p \text{ and } S^p = S_{p'} \text{ then}$$

$$C_0 = S_\Sigma \cap \bigcap_{p \in \Sigma} (S^p \cap \mathbf{N}_G(F)) \cdot \mathbf{C}_G(F/F_{\mathfrak{F}(p)})$$

is a well-defined group. By the Dedekind identity $C = C_0$, so C is also a group.

Next we show that C is in $\mathfrak{F}$ and that $F = C \cap M$. Let

$$(\forall p) A_p = \mathbf{C}_{S_p}(F/F_{\mathfrak{F}(p)});$$

then for p not equal to q

$$[A_p, A_q] \leqq X = M \cap \mathbf{C}_{S_pS}\,(F/(F_{\mathfrak{F}(p)} \cap F_{\mathfrak{F}(q)}))$$

using the fact that G/M is nilpotent, and also the three subgroup lemma, which

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is applicable since $A_p \leqq \mathbf{N}_G(F)$ and $F_{\mathfrak{F}(p)} \lhd \mathbf{N}_G(F)$. Because the Sylow system $\mathfrak{S}$ reduces into $\mathbf{N}_G(F)$, $\mathfrak{S}$ reduces as well into $\mathbf{C}_M(F/(F_{\mathfrak{F}(p)} \cap F_{\mathfrak{F}(q)}))$. Thus X has as a Sylow p-subgroup, $M \cap \mathbf{C}_{S_p}(F/(F_{\mathfrak{F}(p)} \cap F_{\mathfrak{F}(q)}))$, which is a subgroup of F by Lemma 7, since F is $\mathfrak{F}$-maximal in M. Hence $[A_p, A_q] \leqq X \leqq F$. Therefore as $S_p \cap F \leqq A_p$ (because $S_p \cap F \leqq F_{\mathfrak{F}(p)}$) we have proved that $F \lhd C$ and $C/F \in \mathfrak{N}$. In particular $F \cdot A_p \lhd C$; but by Lemma 7 $F \cdot A_p$ is in $\mathfrak{F}$ so we obtain $C = \Pi\,(F \cdot A_p) \in \mathfrak{F}$. Finally F is an $\mathfrak{F}$-maximal subgroup of $M \lhd G$ and $F \leqq C \cap M \in \mathfrak{F}$ so in fact $F = C \cap M$.

Now suppose H is any $\mathfrak{F}$-maximal subgroup of G which contains F. Then $F \leqq H \cap M \in \mathfrak{F}$. Using again the fact that F is $\mathfrak{F}$-maximal in M we have $F = H \cap M \lhd H$. Let $\mathfrak{T} = \{T_p\}$ be a Sylow system of G reducing into H and into $\mathbf{N}_G(F)$. We shall show that H has the same form as C. Applying Lemma 7 we see that

$$(\forall p \in \Sigma)\mathbf{C}_{T_p}(H/H_{\mathfrak{F}(p)}) = T_p \cap H \leqq H$$

since H is $\mathfrak{F}$-maximal in G and thus

$$H = \prod_{p \in \Sigma} \mathbf{C}_{T_p}(H/H_{\mathfrak{F}(p)}).$$

Again by Lemma 7 (with G replaced by $\mathbf{N}_G(F)$) we have $(\forall p \in \Sigma)$

$$\mathbf{C}_{T_p}(H/H_{\mathfrak{F}(p)}) = \mathbf{C}_{T_p \cap H}(H/H_{\mathfrak{F}(p)}) \leqq \mathbf{C}_{T_p}(F/F_{\mathfrak{F}(p)})$$

from which we obtain

$$H \leqq L = \prod_{p \in \Sigma} \mathbf{C}_{T_p}(F/F_{\mathfrak{F}(p)}).$$

L is a subgroup of G in F and so $H = L$ by the $\mathfrak{F}$-maximality of H. This completes the proof for now H is conjugate in $\mathbf{N}_G(F)$ to C (Sylow systems in $\mathbf{N}_G(F)$ being conjugate) and hence C is itself $\mathfrak{F}$-maximal in G.

DEFINITION 9. *If $\mathfrak{S} = \{S_p\}$ is a Sylow system of G then the subgroup $B(\mathfrak{S})$ of G is defined by*

$$B(\mathfrak{S}) = S_\Sigma \cap \bigcap_{p \in \Sigma} S^p \cdot \mathbf{C}_G(G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}) = \prod_{p \in \Sigma} \mathbf{C}_{S_p}(G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}),$$

where $S_\Sigma = \Pi_{p \in \Sigma} S_p$ and $S^p = S_{p'}$, and where the second equality follows from the Dedekind identity.

The subgroup $B(\mathfrak{S})$ will turn out to contain an $\mathfrak{F}$-injector of G and in some cases to be an $\mathfrak{F}$-injector of G.

LEMMA 10. *If $M \lhd \lhd G$ then $M \cap B(\mathfrak{S}) = B(\mathfrak{S} \cap M)$, where $\mathfrak{S} \cap M$ is the Sylow system of M to which $\mathfrak{S}$ reduces.*

PROOF. We may assume that $M \lhd G$ with $\mathbf{O}^p(G) \leqq M$ for some p in Σ. Intersecting $B(\mathfrak{S})$ with M gives

$$M \cap B(\mathfrak{S}) = \mathbf{C}_{M \cap S_p}(G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}) \cdot \prod_{q \in \Sigma - p} \mathbf{C}_S\,(G_{\mathfrak{F}}/G_{\mathfrak{F}(q)}).$$

Because $G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}$ and $M_{\mathfrak{F}}/M_{\mathfrak{F}(p)}$ are p'-groups while $G_{\mathfrak{F}}/M \cap G_{\mathfrak{F}}$ is a p-group, the factors $G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}$ and $M_{\mathfrak{F}}/M_{\mathfrak{F}(p)}$ are G-isomorphic, and thus

$$\mathbf{C}_{M \cap Sp}(G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}) = \mathbf{C}_{M \cap S_p}(M_{\mathfrak{F}}/M_{\mathfrak{F}(p)}).$$

Now fix q unequal to p. We show that $Q = \mathbf{C}_{S_q}(M_{\mathfrak{F}}/M_{\mathfrak{F}(q)})$ and $\mathbf{C}_{S_q}(G_{\mathfrak{F}}/G_{\mathfrak{F}(q)})$ are equal. Let $A = G_{\mathfrak{F}}/G_{\mathfrak{F}(q)}$, which is a q'-group on which the q-group Q acts. By a corollary to the Schur-Zassenhaus theorem $A = \mathbf{C}_A(A) \cdot [A, Q]$ and $[A, Q] = [A, Q, Q]$. We have $[Q, G_{\mathfrak{F}}] \leqq M \cap G_{\mathfrak{F}} = M_{\mathfrak{F}}$ as $Q \leqq M$, and so

$$[G_{\mathfrak{F}}, Q, Q] \leqq [M_{\mathfrak{F}}, Q] \leqq M_{\mathfrak{F}(q)} \leqq G_{\mathfrak{F}(q)}.$$

Hence $[A, Q, Q] = [G_{\mathfrak{F}}, Q, Q]G_{\mathfrak{F}(q)}/G_{\mathfrak{F}(q)} = 1$ and $A = \mathbf{C}_A(Q)$. This just means that $Q \leqq \mathbf{C}_{S_q}(G_{\mathfrak{F}}/G_{\mathfrak{F}(q)})$. The reverse inclusion follows trivially from $S_q \leqq M$ and the lemma is proved.

LEMMA 11. *If $G_{\mathfrak{F}} \leqq H \leqq G$, $H \in \mathfrak{F}$ and $\mathfrak{S}$ reduces into H then $H \leqq B(\mathfrak{S})$.*

PROOF. Since $G_{\mathfrak{F}}$ and H are both in $\mathfrak{F}$ they have Sylow p-subgroups respectively $S_p \cap G_{\mathfrak{F}} \leqq G_{\mathfrak{F}(p)}$ and $S_p \cap H \leqq H_{\mathfrak{F}(p)}$. We have

$$[H \cap S_p, G_{\mathfrak{F}} \cap S_p] \leqq G_{\mathfrak{F}(p)}$$

and

$$[H \cap S_p, G_{\mathfrak{F}} \cap S^p] \leqq H_{\mathfrak{F}(p)} \cap G_{\mathfrak{F}} = G_{\mathfrak{F}(p)}$$

so that

$$[H \cap S_p, G_{\mathfrak{F}}] \leqq G_{\mathfrak{F}(p)}.$$

Hence

$$(\forall p \in \Sigma) H \cap S_p \leqq \mathbf{C}_{S_p}(G_{\mathfrak{F}}/G_{\mathfrak{F}(p)})$$

and thus

$$H = \prod_{p \in \Sigma} H \cap S_p \leqq B(\mathfrak{S}).$$

THEOREM 12. *If $V \leqq G$ is such that $\mathfrak{S}$ reduces into V then V is an $\mathfrak{F}$-injector of G if and only if V is an $\mathfrak{F}$-injector of $B(\mathfrak{S})$.*

PROOF. One method of proof would be to use Lemma 11 and the results of Fischer, Gäschutz and Hartley (1967), including the fact that an $\mathfrak{F}$-injector of a group G is also an $\mathfrak{F}$-injector of any subgroup containing it. To keep things within the context of locally defined Fitting classes we give a second proof, based on Lemmas 8 and 10, of the sufficiency part of the theorem, noting that necessity follows from Lemma 11.

Suppose that V is an $\mathfrak{F}$-injector of $B(\mathfrak{S})$. If M is a subnormal proper subgroup of G then $V \cap M$ is an $\mathfrak{F}$-injector of $M \cap B(\mathfrak{S})$, which equals $B(\mathfrak{S} \cap M)$ by Lemma 10. By induction on the group order $V \cap M$ is an $\mathfrak{F}$-injector of M i.e.

$V \cap M$ is $\mathfrak{F}$-maximal in M. Now take $M \lhd G$ such that G/M is nilpotent but non-trivial. According to Lemma 8,

$$W = \prod_{p \in \Sigma} \mathbf{C}_{S_p}((V \cap M)/(V \cap M)_{\mathfrak{F}(p)})$$

is $\mathfrak{F}$-maximal in G and $W \geqq V \cap M$. We have

$$V = \Pi\, (V \cap S_p) \leqq \Pi\, \mathbf{C}_{S_p}((V \cap M)/(V \cap M)_{\mathfrak{F}(p)}) = W$$

since $V \cap S_p \leqq V_{\mathfrak{F}(p)}$ and $(V \cap M)_{\mathfrak{F}(p)} = V_{\mathfrak{F}(p)} \cap M$. Clearly $\mathfrak{S}$ reduces into W so by Lemma 11 $V \leqq W \leqq B(\mathfrak{S})$. By assumption V is $\mathfrak{F}$-maximal in $B(\mathfrak{S})$ and therefore $V = W$ and is $\mathfrak{F}$-maximal in G. Thus V is an $\mathfrak{F}$-injector of G.

Remark 13. For a locally defined Fitting class $\mathfrak{F}$ Theorem 12 provides another proof of the existence of $\mathfrak{F}$-injectors since once it is shown that G not in $\mathfrak{F}$ implies $B(\mathfrak{S}) < G$ then by induction we may assume that $B(\mathfrak{S})$ has an $\mathfrak{F}$-injector which by Theorem 12 is an $\mathfrak{F}$-injector of G. The statement (G not in $\mathfrak{F}$ implies $B(\mathfrak{S}) < G$) can be proved as follows (cf. Corollary 15): if $G \in \mathfrak{F}\mathfrak{N}$ then by Lemma 7 and Proposition 2 $B(\mathfrak{S}) = G_{\mathfrak{F}} \in \mathfrak{F}$ and so using induction and Lemma 10 we have that $G \in \mathfrak{F}\mathfrak{N}^{n+1}$ implies $B(\mathfrak{S}) \in \mathfrak{F}\mathfrak{N}^{n}$ for $n \geqq 0$, where $\mathfrak{N}^0 = \{1\}$ and $\mathfrak{N}^{n+1} = \mathfrak{N} \cdot \mathfrak{N}^n$.

For certain Fitting classes — including locally defined Fitting classes (as is shown in Hartley (1969)) — an $\mathfrak{F}$-injector V is characterized by the condition:

$$V \in \mathfrak{F} \text{ and } (\forall H \geqq V) V \geqq H_{\mathfrak{F}}.$$

For the classes considered here this characterization comes easily from Theorem 12, for suppose $V \leqq G$ satisfies the condition and G is not in $\mathfrak{F}$. Then V is contained, by Lemma 11, in $B(\mathfrak{S}) < G$ for some $\mathfrak{S}$, is an $\mathfrak{F}$-injector of $B(\mathfrak{S})$ by induction and hence of G by Theorem 12.

If $B_1 = B(\mathfrak{S})$, $B_2 = B(\mathfrak{S} \cap B_1), \cdots B_n = B(\mathfrak{S} \cap B_{n-1}), \cdots$ (noting that $\mathfrak{S}$ reduces into $B(\mathfrak{S})$) then by Theorem 12 and the decreasing nature of the $B(\mathfrak{S})$ there is an n such that B_n is an $\mathfrak{F}$-injector of G. This process of obtaining the $\mathfrak{F}$-injectors is analogous to that used by Graddon (1971) to obtain the $\mathfrak{F}$-projectors — the $B(\mathfrak{S})$ corresponding to the $\mathfrak{F}$-reducers, and the $\mathfrak{F}$-radical to the $\mathfrak{F}$-normalizers.

We now examine some of the instances when $B(\mathfrak{S})$ is actually in $\mathfrak{F}$.

Theorem 14. *$B(\mathfrak{S})$ is in $\mathfrak{F}$ if and only if all $\mathfrak{F}$-maximal subgroups containing $G_{\mathfrak{F}}$ are $\mathfrak{F}$-injectors.*

Proof. Suppose all $\mathfrak{F}$-maximal subgroups containing $G_{\mathfrak{F}}$ are $\mathfrak{F}$-injectors. By Lemma 7 $(\forall p \in \Sigma)$

$$A_p = G_{\mathfrak{F}} \cdot \mathbf{C}_{S_p}(G_{\mathfrak{F}}/G_{\mathfrak{F}(p)}) \in \mathfrak{F},$$

and thus by hypothesis there is an $\mathfrak{F}$-injector V of G with $A_p \leqq V$. Now V is contained in a conjugate of $B(\mathfrak{S})$ so by the definition of $B(\mathfrak{S})$ $(\forall p \in \Sigma)$ $|A_p|_p = |V|_p = |B(\mathfrak{S})|_p$ (p-parts). Hence $|V| = |B(\mathfrak{S})|$ as $B(\mathfrak{S})$ is a Σ-group, and $B(\mathfrak{S})$ is an $\mathfrak{F}$-injector of G.

The converse is immediate from Lemma 11 and Theorem 12.

Corollary 15. *For $n \geqq 2$, $G \in \mathfrak{F}\mathfrak{N}^n$ implies that $B(\mathfrak{S}) \in \mathfrak{F}\mathfrak{N}^{n-2}$.*

Proof. If $G \in \mathfrak{F}\mathfrak{N}^2$ then $B(\mathfrak{S}) \in \mathfrak{F}$ by Proposition 4 and Theorem 14. Now if $G \in \mathfrak{F}\mathfrak{N}^n$, $n \geqq 3$, let M be the $\mathfrak{F}\mathfrak{N}^{n-1}$-radical of G. Then by induction and Lemma 10

$$M \cap B(\mathfrak{S}) = B(\mathfrak{S} \cap M) \in \mathfrak{F}\mathfrak{N}^{n-3}.$$

Thus, as G/M is nilpotent, we have $B(\mathfrak{S}) \in \mathfrak{F}\mathfrak{N}^{n-2}$.

Theorem 16. *$B(\mathfrak{S})$ is in $\mathfrak{F}$ if $\mathfrak{F}$ has one of the forms*

(a) *$\mathfrak{F} = \mathfrak{K} \cdot \cap_i \mathfrak{C}_i\mathfrak{C}_{i'}$ where $\mathfrak{K}$ is a Fitting class, and $\mathfrak{C}_i = \mathfrak{C}_{\pi_i}$, $\mathfrak{C}_{i'} = \mathfrak{C}_{\pi_i'}$, for mutually disjoint sets π_i of primes such that the union of the π_i is all primes. Such $\mathfrak{F}$ include classes of the type $\mathfrak{K}\mathfrak{N}$,*

(b) *$\mathfrak{F} = \mathfrak{K} \cdot \mathfrak{C}_\pi\mathfrak{C}_{\pi'}$, for a set π of primes.*

Proof. (a) $\mathfrak{F}$ has the local definition $\mathfrak{F}(p) = \mathfrak{K}\mathfrak{C}_i$ for p in π_i. Let

$$Q_i = S_{\pi_i} \cap \mathfrak{C}_G(G_{\mathfrak{F}}/G_{\mathfrak{K}\mathfrak{C}_i})$$

where S_{π_i} is the Hall π_i-subgroup associated with a Sylow system $\mathfrak{S}$. Now $B(\mathfrak{S}) = \prod Q_i$. Since each $C_iC_{i'}$ contains the nilpotent groups,

$$[Q_i, Q_j] \leqq \mathbf{C}_G(G_{\mathfrak{F}}/G_{\mathfrak{K}}) \leqq G_{\mathfrak{F}}, \text{ for } i \neq j.$$

Therefore $Q_iG_{\mathfrak{F}} \lhd B(\mathfrak{S})$; but as $Q_iG_{\mathfrak{K}\mathfrak{C}_i}$ is normal in $Q_iG_{\mathfrak{F}}$ and is in $\mathfrak{K}\mathfrak{C}_i \leqq \mathfrak{F}$ so also is $Q_iG_{\mathfrak{F}}$ in $\mathfrak{F}$. Hence

$$B(\mathfrak{S}) = \prod_i Q_iG_{\mathfrak{F}} \in \mathfrak{F}.$$

(b) $\mathfrak{F}$ now has the local definition $\mathfrak{F}(p) = \mathfrak{K}\mathfrak{C}_\pi$, for p in π, and $\mathfrak{F}(p) = \mathfrak{F}$ otherwise. Thus

$$B(\mathfrak{S}) = S_{\pi'} \cdot \mathbf{C}_{S_\pi}(G_{\mathfrak{F}}/G_{\mathfrak{K}\mathfrak{C}_\pi}) = S_{\pi'} \cdot \mathbf{C}_G(G_{\mathfrak{F}}/G_{\mathfrak{K}\mathfrak{C}_\pi});$$

but $G_{\mathfrak{K}\mathfrak{C}_\pi} \leqq \mathbf{C}_G(G_{\mathfrak{F}}/G_{\mathfrak{K}\mathfrak{C}_\pi}) \leqq G_{\mathfrak{F}}$ since $\mathbf{C}_G(L_{\pi'}(G)) \leqq \mathbf{O}_{\pi\pi'}(G)$. Therefore

$$B(\mathfrak{S}) = S_{\pi'} \cdot G_{\mathfrak{K}\mathfrak{C}_\pi} \in \mathfrak{F} = \mathfrak{K}\mathfrak{C}_\pi\mathfrak{C}_{\pi'}.$$

Corollary 17. *If $\mathfrak{F}$ has either of the forms $\mathfrak{K} \cdot \cap_i \mathfrak{C}_i\mathfrak{C}_{i'}$ or $\mathfrak{K} \cdot \mathfrak{C}_\pi\mathfrak{C}_{\pi'}$ then the $\mathfrak{F}$-injectors of G are just those $\mathfrak{F}$-maximal subgroups of G that contain $G_{\mathfrak{F}}$.*

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Fischer has already pointed out this characterization for the special case $\mathfrak{F} = \mathfrak{N}$. The proof of Theorem 16 depends on the special nature of the classes considered so presumably $B(\mathfrak{S})$ need not in general be in $\mathfrak{F}$.

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