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CONVERGING FACTORS FOR SOME ASYMPTOTIC MOMENT SERIES THAT ARISE IN NUMERICAL QUADRATURE

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Abstract

In this work the asymptotic behavior of the partial sums of the divergent asymptotic moment series $\sum_{i=1}^{\infty} \mu_i/z^i$, where μ_i are the moments of the weight functions $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, and $w(x) = x^\alpha E_m(x)$, $\alpha > -1$, $m + \alpha > 0$, on the interval $[0, \infty)$, is analyzed. Expressions for the converging factors are derived. These converging factors form the basis of some very accurate numerical quadrature formulas derived by the author for the infinite range integrals $\int_0^\infty w(x)f(x)\,dx$ with $w(x)$ as given above.

1. Introduction

Recently a new approach to numerical quadrature has been presented by the author, see [5]. In this approach numerical quadrature formulae $I_k[u]$ for the integral $I[u]$, where

$$I[u] = \int_a^b w(x)u(x)\,dx, \qquad I_k[u] = \sum_{i=1}^{k} A_{ki}u(x_{ki}), \tag{1.1}$$

are derived by forming a sequence of rational approximations $H_k(z)$, $k = 1, 2, \ldots$, to the function $H(z)$, where

$$H(z) = \int_a^b \frac{w(x)}{z - x}\,dx, \qquad H_k(z) = \sum_{i=1}^{k} \frac{A_{ki}}{z - x_{ki}}. \tag{1.2}$$

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The function $H(z)$ is analytic in the complex z-plane cut along the interval $[a, b]$.

We shall not deal with the motivation of this approach in this work, as that has been done in detail in [5]. We shall only state that from the motivation in the work above one could conclude heuristically that $I_k[u]$ would be a good approximation to $I[u]$ if $H_k(z)$ is a good approximation to $H(z)$ in the complex z-plane cut along $[a, b]$, and that $I_k[u] \to I[u]$ quickly as $k \to \infty$ if $H_k(z) \to H(z)$ quickly as $k \to \infty$ in the complex z-plane cut along $[a, b]$.

The rational approximations $H_k(z)$ in [5] are obtained by applying some modified version of the T-transformation of [1] to the moment series of $H(z)$, namely to the series

$$H(z) \sim \sum_{i=1}^{\infty} \frac{\mu_i}{z^i}, \quad \text{as } z \to \infty, z \notin [a, b], \tag{1.3}$$

where μ_i are the moments of $w(x)$,

$$\mu_i = \int_a^b w(x) x^{i-1} dx, \qquad i = 1, 2, \ldots. \tag{1.4}$$

It can easily be seen that if $[a, b]$ is finite, then the series (1.3) converges for $|z| > \max(|a|, |b|)$. If, however, $[a, b]$ is infinite, like $[0, \infty)$, then (1.3) is a divergent asymptotic series.

As explained in [5] (see also [3] and [4], in which convergence properties of the T-transformation are analyzed), the T-transformation, when applied to the sequence of the partial sums of the infinite series in (1.3), produces very good approximations to $H(z)$, provided that

$$H(z) = \sum_{i=1}^{n-1} \frac{\mu_i}{z^i} + R_n f(n), \tag{1.5}$$

where R_n are numbers related to the moments, and $f(y)$, as a function of the continuous variable y, has an asymptotic expansion of the form

$$f(y) \sim \sum_{i=0}^{\infty} \frac{\beta_i}{y^i} \quad \text{as } y \to \infty, \tag{1.6}$$

and is infinitely differentiable up to $y = \infty$. We notice that the term $R_n f(n)$ in (1.5) serves as a "converging factor" for the series in (1.3).

For $a = 0$, $b = 1$, and $w(x) = (1 - x)^\alpha x^\beta (-\log x)^\nu$, $\alpha + \nu > -1$, $\beta > -1$, it has been shown in [5] that (1.5) holds with $R_n = 1/(n^{\alpha+\nu+1} z^n)$ and

$$f(y) = y^{\alpha+\nu+1} \int_0^\infty e^{-yt} t^{\alpha+\nu} g(t)\, dt,$$

where

$$g(t) = \left[(1 - e^{-t})/t\right]^\alpha e^{-\beta t} / \left(1 - e^{-t}/z\right) = \sum_{i=0}^{\infty} g_i t^i;$$

hence (1.6) is also satisfied with $f(y) \sim \sum_{i=0}^{\infty} g_i(\alpha+\nu+i)!/y^i$ as $y \to \infty$, for all $z \notin [0,1]$. (Actually R_n, $f(n)$, $g(t)$, and g_i should be written as $R_n(z)$, $f(n,z)$, $g(t,z)$, and $g_i(z)$. However, z is fixed and we do not perform any operation with respect to z. Hence we omit z without fear of confusion.) This expansion can be obtained by applying Watson's lemma, see [2, page 71], to the integral representation of $f(y)$. Since (1.5) and (1.6) are satisfied, the numerical quadrature formulae for this $w(x)$ turn out to be very accurate. For more details and numerical results see [5].

We note that $f(y)/y$ is the Laplace transform of a function $\varphi(t)$ which is analytic at $t = 0$ and has a Maclaurin series with a *finite* radius of convergence (that of the Maclaurin series of $g(t)$). This follows from a more general result which is given in the appendix to the present work.

In a recent work, [6], very accurate numerical quadrature formulae for infinite range integrals with $a = 0$, $b = \infty$, and weight functions $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, and $w(x) = x^\alpha E_m(x)$, $\alpha > -1$, $m + \alpha > 0$, where $E_m(x)$ is the exponential integral, have been developed using the approach of [5]. These formulae are based on the results of Section 2 and Section 3 of the present work, which show that also in these cases the functions $H(z)$ satisfy (1.5) and (1.6). Furthermore, for $\operatorname{Re} z < 0$, it is shown that $f(y)/y$ for these cases are Laplace transforms of *entire* functions $\varphi(t)$, which, we believe, should be of some importance in the convergence analysis of $H_k(z)$ to $H(z)$.

2. The case $w(x) = x^\alpha e^{-x}$, $\alpha > -1$

Let $w(x) = x^\alpha e^{-x}$, $\alpha > -1$. Then from (1.4) we have

$$\mu_i = \int_0^\infty e^{-x} x^{\alpha+i-1}\, dx = (\alpha+i-1)!, \qquad i = 1, 2, \ldots. \tag{2.1}$$

By substituting the relation

$$\frac{1}{z-x} = \sum_{i=1}^{n-1} \frac{x^{i-1}}{z^i} + \frac{1}{z^n}\frac{x^{n-1}}{1-x/z} \tag{2.2}$$

and (2.1) in (1.2), we have

$$H(z) = \int_0^\infty \frac{x^\alpha e^{-x}}{z-x}\, dx = \sum_{i=1}^{n-1} \frac{\mu_i}{z^i} + \frac{J(\alpha+n-1, z)}{z^n}, \tag{2.3}$$

where

$$J(p,z) = \int_0^\infty \frac{e^{-x} x^p}{1-x/z}\, dx, \tag{2.4}$$

which is analytic for all z not on the positive real axis.

All we have to analyze then is $J(p, z)$. Now for $\operatorname{Re} z < 0$, we have $\operatorname{Re}[e^{i\arg(-z)}(1 - x/z)] > 0$ whenever $0 \leqslant x < \infty$. It then follows that

$$\frac{e^{-x}}{1 - x/z} = e^{-z} \int_{\substack{-z \\ (L)}}^{\infty} e^{-\tau(1-x/z)} \, d\tau, \tag{2.5}$$

where the integral is taken along the straight line path L in the τ-plane, which starts at $\tau = -z$ and extends to $\tau = \infty$, and along which $\arg \tau = \arg(-z)$, see Figure 1. Substituting (2.5) in (2.4), and changing the order of integration, we have

$$J(p, z) = e^{-z} \int_{\substack{-z \\ (L)}}^{\infty} d\tau \, e^{-\tau} \int_0^{\infty} dx \, e^{-(-\tau/z)x} x^p, \tag{2.6}$$

which, upon using the fact that

$$\int_0^{\infty} e^{-tx} x^q \, dx = \frac{q!}{t^{q+1}}, \qquad q > -1, \qquad \operatorname{Re} t > 0, \tag{2.7}$$

becomes

$$J(p, z) = e^{-z} p! \, (-z)^{p+1} \int_{\substack{-z \\ (L)}}^{\infty} d\tau \frac{e^{-\tau}}{\tau^{p+1}}, \tag{2.8}$$

where τ^s, for s real, is taken to be positive real for τ positive real, with its branch cut along the negative real axis, *i.e.*, $\tau^s = |\tau|^s e^{is\arg(\tau)}$, $|\arg(\tau)| < \pi$, similarly for $(-z)^s$. The change of order of integration in (2.6) is allowed since the integrand is absolutely integrable both at $x = \infty$ and $\tau = \infty e^{i\arg(-z)}$.

Now (2.8) can also be expressed as

$$J(p, z) = e^{-z} p! \, (-z)^{p+1} \int_{\substack{-z \\ (\Gamma)}}^{\infty} d\tau \frac{e^{-\tau}}{\tau^{p+1}}, \tag{2.9}$$

where the integral this time is taken along the contour Γ that starts at $\tau = -z$ and approaches the real τ-axis asymptotically, see Figure 1. Since the integral in (2.9)

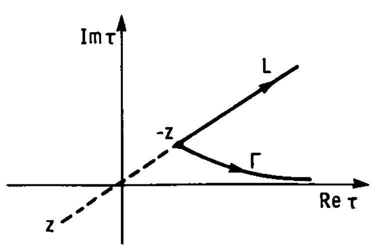

Figure 1. The contours Γ and L in the τ-plane for the case $\operatorname{Re} z < 0$.

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converges both when $\operatorname{Re} z < 0$ and $\operatorname{Re} z \geqslant 0$, by analytic continuation, $J(p, z)$ is given by (2.9) for all z, such that $z \notin [0, \infty)$.

LEMMA 1. *Let C be a contour in the complex plane that extends from ζ to infinity and define*

$$M[F; \zeta] = \int_{\zeta \atop (C)}^{\infty} \frac{F(\tau)}{\tau^{q+1}} d\tau. \tag{2.10}$$

Assume that along C the function $F(\tau)$ is infinitely differentiable and that $(S^iF)(\tau) = O(\tau^q)$ as $\tau \to \infty$ along C, where the operators S^i are defined as follows: $(SF)(\tau) = \tau F'(\tau)$, $S^0F = F$, $S^iF = S(S^{i-1}F)$, $i = 1, 2, \ldots$. Then for any positive integer N,

$$M[F; \zeta] = \frac{1}{\zeta^q} \sum_{i=0}^{N-1} \frac{(S^iF)(\zeta)}{q^{i+1}} + \frac{1}{q^N} M[S^NF; \zeta]. \tag{2.11}$$

PROOF. Integrating (2.10) by parts, we obtain

$$M[F; \zeta] = \frac{1}{\zeta^q} \frac{F(\zeta)}{q} + \frac{1}{q} M[SF; \zeta]. \tag{2.12}$$

Equation (2.11) now follows by repeated application of (2.12) to S^iF, $i = 1, 2, \ldots, N-1$.

COROLLARY 2. *Let the contour C be such that*

$$\int_{\zeta \atop (C)}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}} = O(|\zeta|^{-q}) \quad \text{as } q \to \infty, \tag{2.13}$$

and let $(S^iF)(\tau) = o(1)$ as $\tau \to \infty$ along C, $i = 0, 1, 2, \ldots$. Then $M[F; \zeta]$, as $q \to \infty$, has an asymptotic expansion given by

$$M[F; \zeta] \sim \frac{1}{\zeta^q} \sum_{i=0}^{\infty} \frac{(S^iF)(\zeta)}{q^{i+1}}. \tag{2.14}$$

PROOF. It is enough to show that in (2.11)

$$\zeta^q M[S^NF; \zeta] = O(q^{-1}) \quad \text{as } q \to \infty. \tag{2.15}$$

Now from (2.12)

$$\zeta^q M[S^NF; \zeta] = \frac{(S^NF)(\zeta)}{q} + \frac{1}{q} \zeta^q M[S^{N+1}F; \zeta]. \tag{2.16}$$

Since $(S^iF)(\tau) = o(1)$ as $\tau \to \infty$ along C for any i, there exist finite positive constants λ_i such that $\lambda_i = \max_{\tau \in C} |(S^iF)(\tau)|$, $i = 0, 1, \ldots$. Substituting in

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(2.16) the integral representation of $M[S^{N+1}F;\zeta]$ and taking the modulus of both sides, we obtain

$$|\zeta^q M[S^N F;\zeta]| \leqslant \frac{\lambda_N}{q} + \frac{\lambda_{N+1}}{q}|\zeta|^q \int_{\substack{\zeta \\ (C)}}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}}. \tag{2.17}$$

Using now (2.13) in (2.17), the result follows.

REMARK 3. *Parametric representations for two types of contours* C, *for which* (2.13) *is valid, are given below*:
(*Type* 1) $\tau = \zeta r$, *r real*, $1 \leqslant r < \infty$,
(*Type* 2) $C = C_1 \cup C_2$, *where* C_1 *is defined parametrically by* $\tau = \zeta e^{i\theta}$, $\min(0,\theta_0) \leqslant \theta \leqslant \max(0,\theta_0)$ *for some fixed* θ_0 *such that* $0 < |\theta_0| < \pi$, *and* C_2 *is defined parametrically by* $\tau = \zeta r e^{i\theta_0}$, *r real*, $1 \leqslant r < \infty$. *For contours of Type* 1

$$\int_{\substack{\zeta \\ (C)}}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}} = \frac{1}{|\zeta|^q},$$

whereas for those of Type 2

$$\int_{\substack{\zeta \\ (C)}}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}} = \frac{|\theta_0|}{|\zeta|^q} + \frac{1}{q|\zeta|^q}.$$

Note that the contour L *in Figure* 1 *is a contour of Type* 1 *with* $\zeta = -z$.

Going back to (2.9), we can see that the above lemma and its corollary can be applied quite easily to the integral on the right hand side by letting $F(\tau) = e^{-\tau}$, $q = p$, and $\zeta = -z$. Since the integrand is analytic everywhere except on the branch cut along the negative real axis, the contour Γ can be deformed to a contour of Type 2 as in Figure 2.

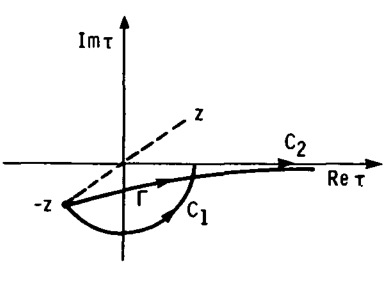

Figure 2. Deformation of Γ to a contour of Type 2.

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Applying now Lemma 1 and Corollary 2 to (2.9), we obtain

$$J(p, z) \sim -ze^{-z}p! \sum_{i=0}^{\infty} \frac{\gamma_i(z)}{p^{i+1}} \quad \text{as } p \to \infty, \tag{2.18}$$

where

$$\gamma_i(z) = (S^iF)(\tau)|_{\tau=-z} = \left(z\frac{d}{dz}\right)^i e^z, \qquad i = 0, 1, 2, \ldots. \tag{2.19}$$

Using the Maclaurin series expansion of e^z, $\gamma_i(z)$ can be expressed by the infinite series

$$\gamma_i(z) = \sum_{k=0}^{\infty} \frac{k^i}{k!} z^k, \qquad i = 0, 1, 2, \ldots. \tag{2.20}$$

As explained in the Introduction, what goes into the T-transformation is a converging factor in terms of an infinite series in inverse powers of y. Such an expansion can also be found quite easily. Letting $p = \alpha + y - 1$, we can express (2.9) in the form

$$J(\alpha + y - 1, z) = e^{-z}(\alpha + y - 1)!\,(-z)^{\alpha+y-1} \int_{-z\ (\Gamma)}^{\infty} d\tau \frac{(e^{-\tau}/\tau^{\alpha-1})}{\tau^{y+1}}. \tag{2.21}$$

Applying now Lemma 1 and Corollary 2 with $F(\tau) = e^{-\tau}/\tau^{\alpha-1}$, $q = y$ and $\zeta = -z$, with the contour Γ deformed as above, we obtain

$$J(\alpha + y - 1, z) \sim e^{-z}(\alpha + y - 1)!\,(-z)^{\alpha-1} \sum_{i=0}^{\infty} \frac{\varepsilon_i(z)}{y^{i+1}} \quad \text{as } y \to \infty, \tag{2.22}$$

where

$$\varepsilon_i(z) = \left(\tau\frac{d}{d\tau}\right)^i \frac{e^{-\tau}}{\tau^{\alpha-1}}\bigg|_{\tau=-z}, \qquad i = 0, 1, 2, \ldots. \tag{2.23}$$

We can therefore conclude that (1.5) and (1.6) are satisfied with

$$R_n = \frac{(\alpha + n - 1)!}{nz^n} = \frac{\mu_n/z^n}{n} \tag{2.24}$$

and

$$f(y) \sim (-z)^{\alpha-1}e^{-z} \sum_{i=0}^{\infty} \frac{\varepsilon_i(z)}{y^i} \quad \text{as } y \to \infty. \tag{2.25}$$

REMARK 4. *When Levin's t-transformation is applied to the sequence of the partial sums of the series* (1.3), *one takes* $R_n = \mu_n/z^n$. *From* (2.18) *we see that* R_n *should actually be* $(\mu_n/z^n)n^{-1}$. *However, the results of using either* R_n *in the T-transformation are about the same, see* [5].

We now wish to show that the function $f(y)$ can be expressed in terms of a Laplace transform of an *entire* function whenever $\operatorname{Re} z < 0$. Making the change of variable $\tau = -ze^{\xi}$ in the integral in (2.8), we obtain

$$J(p, z) = -ze^{-z}p! \int_0^{\infty} e^{-p\xi} \exp(ze^{\xi})\, d\xi, \tag{2.26}$$

or

$$J(\alpha + y - 1, z) = -ze^{-z}(\alpha + y - 1)! \int_0^{\infty} e^{-y\xi} h(\xi, z)\, d\xi, \tag{2.27}$$

where $h(\xi, z) = e^{(1-\alpha)\xi} \exp(ze^{\xi})$ is an entire function of ξ. Hence by (1.5) and (2.24)

$$f(y) = -ze^{-z}y \int_0^{\infty} e^{-y\xi} h(\xi, z) d\xi,$$

which is a Laplace transform. Using Watson's lemma in (2.26) (or (2.27)), the existence of the asymptotic expansions in (2.18) (or (2.22)) and the expressions given in (2.20) and (2.23) can be re-established.

3. The case $w(x) = x^{\alpha}E_m(x), \alpha > 1, m + \alpha > 0$

Let $w(x) = x^{\alpha}E_m(x)$, where $E_m(x) = \int_1^{\infty} (e^{-xt}/t^m)dt$. Then for $\alpha > -1$ and $m + \alpha > 0$

$$\mu_i = \int_0^{\infty} dx\, x^{\alpha+i-1} \int_1^{\infty} dt \frac{e^{-xt}}{t^m}, \tag{3.1}$$

which, upon changing the order of integration and using (2.7), becomes

$$\mu_i = \frac{(\alpha + i - 1)!}{m + \alpha + i - 1}, \qquad i = 1, 2, \ldots. \tag{3.2}$$

Substituting again (2.2) in (1.2), and using (3.2), we obtain

$$H(z) = \int_0^{\infty} \frac{x^{\alpha}E_m(x)}{z - x} dx = \sum_{i=1}^{n-1} \frac{\mu_i}{z^i} + \frac{K(\alpha + n - 1, z)}{z^n}, \tag{3.3}$$

where

$$K(p, z) = \int_0^{\infty} \frac{x^p E_m(x)}{1 - x/z} dx = \int_0^{\infty} dx \frac{x^p}{1 - x/z} \int_1^{\infty} dt \frac{e^{-xt}}{t^m}, \tag{3.4}$$

which, upon changing the order of integration and then making the change of variable $xt = u$ in the integral with respect to x (t fixed), is seen to be

$$K(p, z) = \int_1^{\infty} \frac{dt}{t^{m+p+1}} J(p, tz), \tag{3.5}$$

where $J(p, z)$ is as defined in the previous section.

We now consider the case $\operatorname{Re} z < 0$. Since in (3.5) $1 \leqslant t < \infty$, for this case $\operatorname{Re}(tz) < 0$ too. Hence we can make use of the integral representation for $J(p, z)$ given in (2.26), with z replaced by tz. We then obtain

$$K(p, z) = -zp!\int_1^\infty \frac{dt}{t^{m+p}} e^{-tz} \int_0^\infty d\xi\, e^{-p\xi} \exp(tze^{\xi}). \tag{3.6}$$

Making the change of variable $t = e^\sigma$ in (3.6), we have

$$K(p, z) = -zp!\int_0^\infty d\sigma e^{-(m+p-1)\sigma} \exp(-ze^\sigma) \int_0^\infty d\xi e^{-p\xi} \exp(ze^{\xi+\sigma}). \tag{3.7}$$

Making the further change of variables $(\xi, \sigma) \to (\omega, \sigma')$, where $\omega = \xi + \sigma$, and $\sigma' = \sigma$, we can express (3.7) as

$$K(p, z) = -zp!\int_0^\infty d\omega e^{-p\omega} \exp(ze^\omega) B(\omega, z), \tag{3.8}$$

where

$$B(\omega, z) = \int_0^\omega d\sigma e^{(1-m)\sigma} \exp(-ze^\sigma). \tag{3.9}$$

Note that $B(\omega, z)$ is an entire function of ω. Letting $p = \alpha + y - 1$, and writing the integrand in equation (3.8) in the form $e^{-y\omega}A(\omega, z)$, it is very easy to see that $A(\omega, z)$ is an entire function of ω, hence

$$K(\alpha + y - 1, z) \sim \frac{(\alpha + y - 1)!}{y} \sum_{i=0}^\infty \frac{-z\varepsilon_i(z)}{y^i} \quad \text{as } y \to \infty, \tag{3.10}$$

where

$$\varepsilon_i(z) = \partial^i A(\omega, z)/\partial \omega^i |_{\omega=0}, \qquad i = 0, 1, \ldots. \tag{3.11}$$

A similar result for the case $\operatorname{Re} z \geqslant 0$ could probably be obtained, however this seems to be rather complicated and shall not be pursued further in this work.

We have shown that, at least for $\operatorname{Re} z < 0$, $H(z)$ satisfies (1.5) with (1.6), where $R_n = [(\alpha + n - 1)!/z^n]n^{-1}$, such that the expansion $f(y)/y$ is the Laplace transform of an entire function. Note that this R_n is *independent* of m and is the *same* as that obtained for $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, whereas in the t-transformation of Levin $R_n = \mu_n/z^n$, hence this R_n depends on m through $\mu_n = (\alpha + n - 1)!/(m + \alpha + n - 1)$. These observations have some important implications in the development of the new numerical quadrature formulae as explained in [6].

Appendix

The Laplace transform $\bar{u}(p)$ of a function $u(t)$ is defined by

$$\bar{u}(p) = \mathfrak{L}[u(t); p] = \int_0^\infty e^{-pt} u(t)\, dt. \tag{A.1}$$

THEOREM. *Let $g(t)$ be an analytic function at $t = 0$ and let its Maclaurin series*

$$g(t) = \sum_{i=0}^{\infty} g_i t^i \tag{A.2}$$

have radius of convergence ρ. Define

$$G(p) = \mathfrak{L}[t^{\sigma} g(t); p], \qquad \sigma > -1. \tag{A.3}$$

Then there exists a function $h(t)$, which is analytic at $t = 0$ and has a Maclaurin series with radius of convergence ρ, such that

$$\bar{h}(p) = p^{\sigma} G(p). \tag{A.4}$$

Actually for $|t| < \rho$

$$h(t) = \sum_{i=0}^{\infty} \frac{(\sigma + i)!}{i!} g_i t^i. \tag{A.5}$$

PROOF. We shall deal with the case $-1 < \sigma < 0$ first. Since $-\sigma - 1 > -1$, $p^{\sigma} = \mathfrak{L}[t^{-\sigma-1}/(-\sigma - 1)!;\ p]$. Hence $p^{\sigma} G(p)$ is the Laplace transform of the convolution

$$h(t) = \int_0^t \frac{(t - \tau)^{-\sigma-1}}{(-\sigma - 1)!} \tau^{\sigma} g(\tau) d\tau. \tag{A.6}$$

Let $t \leqslant \bar{\rho} < \rho$. Then substituting (A.2) in (A.6), and changing the order of summation and integration, we obtain

$$h(t) = \sum_{i=0}^{\infty} g_i \int_0^t \frac{(t - \tau)^{-\sigma-1}}{(-\sigma - 1)!} \tau^{\sigma+i} d\tau. \tag{A.7}$$

This is allowed since the Maclaurin series of $g(t)$ converges absolutely and uniformly for $|t| \leqslant \bar{\rho} < \rho$. Using the fact that the integral in the summation is a convolution, (A.5) is easily obtained. Using the ratio test, the infinite power series in (A.5) can be shown to have the same radius of convergence as that of (A.2), namely ρ. This completes the proof for the case $-1 < \sigma < 0$.

If we let $\sigma = 0$, then (A.4) and (A.5) reduce to $h(t) = g(t)$, which is trivially true. If σ is a positive integer, (A.4) and (A.5) are again true, for in this case $h(t) = (d^{\sigma}/dt^{\sigma})[t^{\sigma} g(t)]$. The case $\sigma > 0$, σ not an integer, can be dealt with as follows: Define the function $\tilde{g}(t)$ by $t^{\sigma} g(t) = t^{\tilde{\sigma}} \tilde{g}(t)$, where $\tilde{\sigma} = \sigma - [\sigma] - 1$. Then $-1 < \tilde{\sigma} < 0$ and $\tilde{g}(t) = t^{[\sigma]+1} g(t)$. Then as we have shown above, (A.4) and (A.5) hold with σ and $g(t)$ replaced by $\tilde{\sigma}$ and $\tilde{g}(t)$, respectively. Using the definitions of $\tilde{\sigma}$ and $\tilde{g}(t)$, (A.4) and (A.5) can now be easily obtained. This completes the proof of the theorem.

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