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A formula for accelerating the convergence of a general series

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A weighted average of the partial sums of a series provides a quick and moderately powerful sum for any series in which the ratio of successive terms varies slowly along the series and this ratio is not close to +1 . Some properties of the sum are examined.

Let u_n be the n-th term of a series and S_n given by

$$S_n = u_1 + u_2 + \ldots + u_n$$

be a partial sum of n terms of an infinite series, S . Aitken's [1] two term formula for the sum may be written

$$T_{r,r+1} = \left(S_{r+1}/u_{r+1}-S_r/u_r\right)\Big/\left(1/u_{r+1}-1/u_r\right)$$

and Lubkin's [2] three term formula may be written

$$T_{r,r+2} = \left(S_{r+2}/u_{r+2}-2S_{r+1}/u_{r+1}+S_r/u_r\right)\Big/\left(1/u_{r+2}-2/u_{r+1}+1/u_r\right) .$$

We now consider a generalization of these formulae,

(1) $$T_{rs} = \sum_{i=0}^{s-r} W_i S_{r+i} \Big/ \sum_{i=0}^{s-r} W_i \quad \text{where} \quad W_i = (-)^i C_i^{s-r}/u_{r+i}$$

and C_i^k is the binomial coefficient $k!/i!(k-i)!$.

T_{rs} is a weighted average of the partial sums of the series and may

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be compared with the Euler sum of a slowly convergent alternating series,

$$S_r + \frac{1}{2} u_{r+1} + \frac{1}{4}(u_{r+1}-u_{r+2}) + \ldots + (\text{to } k \text{ terms}) = \sum_{i=0}^{k} C_i^k S_{r+i}/2^k .$$

This is a good approximation when the terms alternate and vary slowly in size.

An approximation to the residual $(S-T_{rs})$: Let $T_{rs} = S - E_{rs}$. If S_i is a convergent sequence, then the truncation error $(S-S_i)$ tends to zero as $i \to \infty$ and T_{rs} is a weighted mean of the S_i , so E_{rs} also tends to zero as r and $s \to \infty$ provided that the sum of the weights has a non-zero limit.

Let $u_{n+1}/u_n = R_n$; then

$$(2) \quad T_{rs} = S_{r-1} + u_r \frac{\sum_{i=0}^{s-r} (-)^i C_i^{s-r} (R_{s-i}R_{s-i+1}\ldots R_{s-1}) \sum_{j=0}^{s-i} (R_r R_{r+1}\ldots R_{s-1+j})}{\sum_{i=0}^{s-r} (-)^i C_i^{s-r} (R_{s-i}R_{s-i+1}\ldots R_{s-1})} .$$

For a rapidly convergent series (that is all the $|R_n| < 1$)

$$(3) \qquad T_{r+1,s+1} - T_{rs} = E_{rs} - E_{r+1,s+1} \doteqdot u_{s+1} \sum_{i=0}^{s-r} (-)^i C_i^{s-r} R_{s-i}/R_s ,$$

on picking out the largest term in the difference. We may conclude from this formula that if the ratio of terms, R_n , varies slowly with n then the $(s-r)$-th order finite difference of R_n will be small compared with R_n . Hence the first finite difference of E_{rs} will be small compared with u_{s+1} . If, furthermore, u_s decreases rapidly as s increases, so also will E_{rs} decrease as s increases with $(s-r)$ constant, so we can neglect $E_{r+1,s+1}$ in our approximation and equation (2) gives an estimate of E_{rs} . Similarly for a rapidly divergent series (that is all the $|R_n| > 1$) on picking out the largest term in the difference,

$$E_{rs} - E_{r+1,s+1} \doteqdot u_r R_r \sum_{i=0}^{s-r} (-)^i C_i^{s-r} / R_{r+i} \; . \tag{4}$$

In both cases we compare the residual in the formula with the smallest term of the original series used in the weighting. If the divergent series has a meaningful sum, then the residual in the formula will be small if the $(s-r)$-th finite difference of the reciprocal of the R_n is small compared with the reciprocal of R_n . If the transformation makes the series less divergent, $E_{r+1,s+1}$ will be larger than E_{rs} while E_{rs} will be the larger if the divergent series is converted to a convergent series, but in either case E_{rs} will be comparable with the expression (4).

In operating on a slowly convergent or divergent series the expression for the residual is more complicated than (3) or (4) but for a slowly convergent alternating series, formula (1) approximates to the Euler sum which is known to be a good approximation for such a series. However, if u_r is the reciprocal of a polynomial in n , then in equation (1), $\sum W_i$ is the finite difference of a polynomial, so will be zero when $(s-r)$ is sufficiently large. In this case T_{rs} may be unbounded. Lubkin's formula which is $T_{r,r+2}$ fails when u_n is the reciprocal of a linear function of n , but this is no loss because $\sum n^{-1}$ is itself a divergent series. For a slowly convergent series of positive terms formula (1) is found to be unsatisfactory because the sum of the weights is small, but it works well on many other series.

As an example we sum the series $\sum (-)^n n!$ to show the simplicity and power of the sum. We tabulate $(10)!/u_n$ and $(10)!S_n/u_n$; then we may introduce the binomial coefficients and calculate any T_{rs} as desired. The set T_{0n} is listed in the third column of the table. Any one of these may be computed as desired and provides a relatively simple and moderately powerful method for summing a series. If greater accuracy is desired, it may be observed that the T_{0n} column appears to be a convergent sequence, so formula (1) may be applied again to this sequence. In this case the labour involved becomes more comparable with that of the other more powerful methods such as repeated use of Aitken's transform [1], Shanks' e_n processes [4], Wynn's E and ρ algorithms [5, 6] and Rutishauser's Q-D algorithm [3]. Euler's transform is useless on this series.

TABLE. ESTIMATES OF $\sum (-)^n n!$

$(10)!/n!$	$(10)!S_n/n!$	T_{0n}
3628800	3628800	
3628800	0	.50000
1814400	3628800	.57143
604800	-2419200	.58824
151200	3024000	.59330
30240	-3024000	.59508
5040	3124800	.59578
720	-3182400	.59608
90	3231000	.59621
10	-3269800	.59628
1	331820	.59631

As a second example we sum the series

$$\pi = 4 - 4/3 + 4/5 + \ldots .$$

In this example $R_n = (2n-1)/(2n+1)$. R_n changes rapidly with n when n is small, so any finite difference of R_n or R_n^{-1} which includes terms with n near zero is not small. In this case we can do better by taking a lower order finite difference further along the series. If we use 16 terms it is best to calculate $T_{6,16}$; then we get an error of 1 in the 10-th digit. This series is also particularly good for Euler's formula since R_n is close to 1 . However we lose strength if we include the first terms where R_n is not close to 1 . On applying Euler's formula to the latter 10 or 11 of 16 terms, we get an error of 2 in the 9-th digit. Aitken's formula develops the same weakness when R_n is changing

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rapidly. With somewhat more labour and six repetitions of Aitken's transform we use up 12 terms of the series before Aitken's transform becomes unstable at the start of the series. At this stage the error is in the 14-th digit. Aitken's transform may be repeated but each repetition after the sixth uses up three terms because of the initial instability so the convergence is less rapid. Aitken's method however remains the best of these three methods per number of terms used if the amount of effort used is not an important restriction.

If we tabulate T_{1s} as s runs from 2 to 16 we obtain a convergent sequence of estimates of π using equation 1 and if desired we may repeat equation 1 on T_{1s} to get a better estimate of the sum. Alternatively, we may tabulate $T_{r,16}$ as r runs from 1 to 15 and this time we obtain an asymptotic sequence with a minimum at $r = 6$ mentioned above. Equation 1 may be repeated on this sequence to get a better estimate.

In conclusion, equation (1) is a weighted Euler sum which may be used when a suitable Euler weight is unknown.

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