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ON THE NUMBER OF 2-HOOKS AND 3-HOOKS OF INTEGER PARTITIONS

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Abstract

Let $p_t(a, b; n)$ denote the number of partitions of n such that the number of t-hooks is congruent to a mod b. For $t \in \{2, 3\}$, arithmetic progressions r_1 mod m_1 and r_2 mod m_2 on which $p_t(r_1, m_1; m_2 n + r_2)$ vanishes were established in recent work by Bringmann, Craig, Males and Ono ['Distributions on partitions arising from Hilbert schemes and hook lengths', *Forum Math. Sigma* **10** (2022), Article no. e49] using the theory of modular forms. Here we offer a direct combinatorial proof of this result using abaci and the theory of t-cores and t-quotients.

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1. Introduction and statement of results

A partition λ, which is a nonincreasing sequence of positive integers summing to some integer n, can be represented visually by a collection of boxes arranged in left-justified rows. The row lengths are arranged in nonincreasing order and correspond to the size of each part of λ. Such a presentation is called the *Ferrers–Young diagram* for λ. In such a diagram, we may refer to the cells by their place in this array; we will denote the cell in row i and column j by (i, j).

We define the *hook* of (i, j) to be the collection of cells (a, b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The *length* of such a hook is the cardinality of this set, which we will denote by $h(i, j)$. We will call a hook a *t-hook* if its length is divisible by t. For example, the hook lengths $h(i, j)$ for the partition $3 + 2 + 1$ are labelled in their respective cells as

5	3	1
3	1	
1		

A study of hook numbers enriches many areas where partitions naturally arise. For instance, recall that the partitions of n parametrise the irreducible complex representations of the symmetric group on n letters. Moreover, the degree of such a representation $\rho_\lambda : S_n \to \mathrm{GL}(V_\lambda)$ is given by the Frame–Thrall–Robinson hook length formula: if λ is a partition of n and p_λ is an irreducible complex representation of S_n as above, then

$$\dim(V_\lambda) = \frac{n!}{\prod_{h(i,j)\in\mathcal{H}(\lambda)} h(i,j)},$$

where $\mathcal{H}(\lambda)$ is the multiset of hook lengths of cells in the Ferrers–Young diagram for λ [4].

Further, recall the famous q-series identities of Euler and Jacobi:

$$q\prod_{n=1}^{\infty}(1-q^{24n}) = q - q^{25} - q^{49} + q^{121} + q^{169} - \cdots$$
$$q\prod_{n=1}^{\infty}(1-q^{8n})^3 = q - 3q^{9} + 5q^{25} - 7q^{49} + 11q^{121} - \cdots.$$

These identities appear as specialisations of the Nekrasov–Okounkov hook length formula, which is derived by taking a z-deformation of the generating function for the partition function. In particular, for any complex number z,

$$\prod_{n=1}^{\infty}(1-q^n)^{z-1} = \sum_{\lambda} q^{|\lambda|} \prod_{h(i,j)\in\mathcal{H}(\lambda)} \Big(1 - \frac{z}{h(i,j)^2}\Big).$$

This result, in which hook lengths appear prominently, constitutes a significant generalisation of many new and classical number-theoretic results on q-series.

We are motivated to explore the number of t-hooks across all partitions of size n in an attempt to obtain an analogue of Dirichlet's theorem on primes in arithmetic progressions. There has been recent work on the distribution of partitions where the number of t-hooks lie in a fixed progression and it turns out that, unlike Dirichlet's theorem, equidistribution does not hold in general. Going beyond unequal distribution, there are situations where such counts are identically zero. For example, Table 1 shows results of [1] for congruence classes of the number of 2-hooks mod 3 for selected integers n.

This example highlights one of the more striking instances of the unequal distribution of t-hooks that has been observed in recent work (see [1, 3]). In particular, these results construct arithmetic progressions where these counts are identically zero. The existing proofs make use of the theory of theta functions to find exact formulae for the asymptotic behaviour of the number of t-hooks in a particular arithmetic progression. In this note, however, we argue the following result directly by appealing to the combinatorial study of hook lengths through abaci.

TABLE 1. 2-Hooks modulo 3.

n	$p_2(0,3;n)$	$p_2(1,3;n)$	$p_2(2,3;n)$
300	≈ 0.7347	≈ 0.2653	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
600	≈ 0.6977	≈ 0.3022	0
900	≈ 0.6837	≈ 0.3163	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4500	≈ 0.6669	≈ 0.3330	0
4800	≈ 0.6669	≈ 0.3330	0
5100	≈ 0.6668	≈ 0.3331	0

THEOREM 1.1 [1, Theorem 1.3]. *Let $(\cdot/\ell)$ denote the Legendre symbol and* $\mathrm{ord}_\ell(n)$ *the ℓ-adic valuation of n. Then the following statements hold.*

(1) *If ℓ is an odd prime and a_1, a_2 are integers for which $(-16a_1 + 8a_2 + 1/\ell) = -1$, then $p_2(a_1, \ell; \ell k + a_2) = 0$.*
(2) *If $\ell \equiv 2 \bmod 3$ is prime and a_1, a_2 are integers for which $\mathrm{ord}_\ell(-9a_1 + 3a_2 + 1) = 1$, then $p_3(a_1, \ell^2; \ell^2 k + a_2) = 0$.*

EXAMPLE 1.2. Let $\ell = 5$, $a_1 = 1$ and $a_2 = 1$. Observe that $(-16a_1 + 8a_2 + 1/5) = -1$ and $\mathrm{ord}_5(-9a_1 + 3a_2 + 1) = 1$. Thus, if $n \equiv 1 \bmod 5$, we have $p_2(1, 5, n) = 0$; that is, there does not exist a partition λ of n where the number of 2-hooks of λ is equivalent to 1 mod 5. Likewise, if $n \equiv 1 \bmod 25$, then $p_3(1, 25; n) = 0$.

In the following sections, we make use of a well-known bijection between integer partitions and their decompositions into t-cores and t-quotients. We are then able to approach the theorem directly using the theory of abaci and a naturally appearing positive definite binary quadratic form intimately related to the problem.

2. Nuts and bolts

2.1. Cores and quotients. Fix an integer t. A partition which does not contain any t-hooks is called a *t-core*. We may construct a t-core $\tilde{\lambda}$ from an arbitrary partition λ by removing rim t-hooks from the Ferrers–Young diagram of λ. Moreover, this t-core is unique. To this end, [5] describes an algorithm for finding both the core and quotient of a partition by systematically removing rim-hooks from λ to build a t-quotient and arrive at the t-core of a partition.

The following is a well-known bijection which identifies a partition with its t-core and a t-tuple of partitions called its t-quotient. In particular, let P denote the space of all partitions, $c_t(P)$ all t-core partitions and $q_t(P)$ the space of t-quotients, which is isomorphic to the direct product of t copies of P. With this notation, there is a bijection $\varphi : P \to c_t(P) \times q_t(P)$ given by

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$$\varphi(\lambda) = (\tilde{\lambda}, \lambda_0, \ldots, \lambda_{t-1}).$$

In particular,

$$|\lambda| = |\tilde{\lambda}| + t \cdot \sum_{i=0}^{t-1} |\lambda_i|. \tag{2.1}$$

This decomposition is classical and gives rise to the fact that for a given partition λ, the size of its corresponding t-quotient given by $\sum_{i=0}^{t-1} |\lambda_i|$ is a count of the number of t-hooks contained in λ (for example, see [8, Theorem 3.3]). To make full use of the content of this bijection, we will revisit this result in the context of abaci theory.

2.2. Abaci. We may encode the data of a partition λ of n in an abacus. To do this, let $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \lambda_s > 0\}$. For $1 \leq i \leq s$, define the ith *structure number* B_i by $B_i = \lambda_i - i + s$; that is, the structure number B_i is the hook number of the entry in the ith row and first column of the Young diagram.

Now, create an abacus with t vertical runners labelled 0 through $t-1$, each infinitely long. We will place beads on the runners in accordance with the structure numbers. In particular, the B_i are positive integers and thus there exist (by Euclidean division) unique integers r_i and c_i such that

$$B_i = t(r_i - 1) + c_i, \quad 0 \leq c_i \leq t-1, \quad r_i \geq 1.$$

We place a bead representing B_i in the row and column position (r_i, c_i) on the abacus.

For example, consider the partition $\lambda = (5, 3, 2, 1)$. We find $B_1 = \lambda_1 - 1 + 4 = 5 - 1 + 4 = 8$. Likewise, $B_2 = 5$, $B_3 = 3$ and $B_4 = 1$. The corresponding abacus with $t = 3$ is

$$\begin{array}{c|ccc} 1 & \cdot & \circ & \cdot \\ 2 & \circ & \cdot & \circ \\ 3 & \cdot & \cdot & \circ \end{array}$$

Conversely, given an abacus with beads in positions $\{(r_i, c_i)\}$, we can construct a decreasing sequence $B_1 \geq \cdots \geq B_k$ by defining $B_i = t(r_i - 1) + c_i$. Then we can find a corresponding partition by setting $\lambda_i = B_i + i - s$.

LEMMA 2.1. *Removing a t-hook from a partition λ is equivalent to sliding a bead up one row on the abacus representing λ.*

PROOF. Let $\{B_1, B_2, \ldots, B_k\}$ be a set of structure numbers for the partition λ. Removing a rim t-hook T from λ is equivalent to subtracting t from some structure number B_ℓ, resulting in the set of structure numbers $\{B_1, \ldots, B_{\ell-1}, B_\ell - t, B_{\ell+1}, \ldots, B_k\}$ for $\lambda \setminus T$ [7, Lemma 2.7.13].

By construction, the bead position on the abacus for B_ℓ is given by (r_ℓ, c_ℓ), where $B_\ell = t(r_\ell - 1) + c_\ell$. Then

$$B_\ell - t = t(r_\ell - 1) + c_\ell - t = t(r_\ell - 2) + c_\ell = t((r_\ell - 1) - 1) + c_\ell,$$

and so removing a t-hook has the effect of sliding a bead up one row on the abacus. □

Since rows in the abacus representing the partition λ are labelled with nonnegative integers, and sliding a bead upward on the abacus fixes the column while subtracting one from the row number, the process of removing t-hooks can only be done finitely many times before arriving at an abacus representing the t-core of λ.

Furthermore, since removing a t-hook corresponds to sliding a bead upward on the abacus, an abacus representing a t-core partition has no gaps between beads in any column, and any nonempty column has a bead in the first row [9, Theorem 4]. Following this observation, we may restate (2.1) in terms of the t-hooks contained in λ.

COROLLARY 2.2. *Let $h_t(\lambda)$ denote the number of t-hooks contained in λ and let $\tilde{\lambda}$ denote its t-core. Then $|\lambda| = |\tilde{\lambda}| + t \cdot h_t(\lambda)$.*

We may thus denote the abaci of t-cores by t-tuples of nonnegative integers, indicating the number of beads in each column. However, there are multiple abaci which represent a single t-core partition.

LEMMA 2.3 [9, Lemma 1]. *The two abaci*

$$\mathfrak{A}_1 = (a_0, a_1, \ldots, a_{t-1}) \quad \text{and} \quad \mathfrak{A}_2 = (a_{t-1} + 1, a_0, a_1, \ldots, a_{t-2})$$

represent the same t-core partition.

By repeatedly applying the above lemma, we may find a unique abacus representation for a given t-core containing zero beads in the first column. Thus, a tuple $(0, a_1, \ldots, a_{t-1})$ uniquely represents a t-core.

2.3. Structure theorem for 2- and 3-cores. We now specialise to the cases where t is two or three. Using the theory of abaci, we are able to completely classify 2- and 3-core partitions by looking at the divisors of $8n + 1$ and $3n + 1$, respectively.

THEOREM 2.4. *Let $c_t(n)$ denote the number of t-core partitions of n.*

(1) *We have $c_2(n) = 1$ if $8n + 1$ is an odd square, and 0 otherwise.*
(2) *We have $c_3(n) = \sum_{d|3n+1} (d/3)$. In particular, $c_3(n) \neq 0$ if and only if for all primes $p \equiv 2 \bmod 3$, $\mathrm{ord}_p(3n + 1)$ is even.*

REMARK 2.5. The equality in (1) is classical by considering the self-conjugate partitions arising from triangular numbers. The equality in (2) was first proven in [6] by comparing coefficients of closely related modular forms.

SKETCH OF PROOF. The case $t = 2$. First, we show that n is a triangular number if and only if $8n + 1$ is an odd square. A triangular number has the form $n = k(k + 1)/2$ for some integer k. Then

$$8n + 1 = 8 \cdot \frac{k(k+1)}{2} + 1 = 4k(k + 1) + 1 = 4k^2 + 4k + 1 = (2k + 1)^2.$$

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Certainly, if n is a triangular number, we have a partition λ where the the parts are given by $\lambda_i = k - i + 1$ for $1 \leq i \leq k$, where $n = k(k+1)/2$. We can check that this partition is indeed a 2-core by simply noting every hook is symmetric, so the hook length is of the form $2k+1$ for suitable k.

Now suppose λ is a 2-core partition of n. Then some abacus $\mathfrak{A} = (0, a)$ uniquely represents λ and has the shape

$$\begin{array}{c|cc} 1 & \cdot & \circ \\ \vdots & \vdots & \vdots \\ a & \cdot & \circ \end{array}$$

Note that $B_i = 2(a-i)+1$, $1 \leq i \leq a$, and that a is the number of parts of the partition. Then recalling $\lambda_i = B_i + i - a$, we have $\lambda_i = 2(a-i)+1+i-a = (a-i)+1$, and

$$n = \sum_{i=1}^{a} \lambda_i = \sum_{k=1}^{a} k = \frac{k(k+1)}{2}.$$

The case $t = 3$. Every 3-core partition can be uniquely represented by an abacus of the form $\mathfrak{A} = (0, a, b)$ for some nonnegative integers a and b. Working backwards, we obtain an expression for n in terms of the structure numbers determined by this abacus:

$$\begin{aligned} n &= \sum_{i=1}^{a+b} \lambda_i \\ &= \sum_{i=1}^{a+b} B_i + \sum_{i=1}^{a+b} i - \sum_{i=1}^{a+b} (a+b) \\ &= \sum_{i=1}^{a+b} B_i + \frac{(a+b)(a+b+1)}{2} - (a+b)^2. \end{aligned}$$

Now, we need only compute the structure numbers. We may do this by considering the beads in column one and column two separately. We have

$$\begin{aligned} \sum_{i=1}^{a+b} B_i &= \sum_{i=1}^{a} (3(i-1)+1) + \sum_{j=1}^{b} (3(j-1)+2) \\ &= 3 \cdot \frac{a(a+1)}{2} - 2a + 3 \cdot \frac{b(b+1)}{2} - b. \end{aligned}$$

Combining with the above and simplifying, we ultimately arrive at

$$n = a^2 - ab + b^2 + b.$$

Define $x := -a + 2b + 1$, $y := a + b + 1$. Then

$$3n + 1 = x^2 - xy + y^2.$$

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We now have an expression for $3n+1$ in terms of a positive definite binary quadratic form with discriminant $D=-3$. The ring of integers of the imaginary quadratic number field $K=\mathbb{Q}(\sqrt{-3})$ is given by $\mathbb{Z}[\omega]$, where $\omega=(1+\sqrt{-3})/2$. The norm on $\mathbb{Z}[\omega]$ simplifies to $N(\alpha+\omega\beta)=\alpha^2-\alpha\beta+\beta^2$. The desired equality follows from constructing a correspondence between ideals in $\mathcal{O}_K$ with norm $3n+1$ as in the proof of Theorem 4.1 in [2].

In particular, every ideal of $\mathcal{O}_K$ is principal and factors uniquely into a product of finitely many (not necessarily distinct) prime ideals. Then given an integer prime p that divides $3n+1$, it follows that p is either congruent to 1 or 2 mod 3. In the latter case, p is inert and the principal ideal (p) in $\mathcal{O}_K$ is prime with norm p^2. Such p must have an even exponent in the prime factorisation of $3n+1$ in $\mathbb{Z}$. To determine whether n admits a 3-core partition, we may compute the prime factorisation of $3n+1$ in $\mathbb{Z}$ and check for even exponents on all primes p where $p\equiv 2 \bmod 3$. $\square$

3. Proof of Theorem 1.1

Suppose ℓ is an odd prime. Write $n=\ell m+a_2$ and suppose $\lambda\vdash n$ such that $h_2(\lambda)=\ell k+a_1$, that is, $h_2(\lambda)\equiv a_1 \bmod \ell$. Denote $|\tilde{\lambda}|$ by $\tilde{n}$. Using Corollary 2.2, we may write

$$n=\tilde{n}+2(\ell k+a_1)=\ell m+a_2$$

so $\tilde{n}=-2a_1+a_2+\ell(m-2k)$. Then,

$$8\tilde{n}+1=-16a_1+8a_2+1+\ell(8m-16k).$$

Now, $8\tilde{n}+1\equiv -16a_1+8a_2+1 \bmod \ell$. If $(-16a_1+8a_2+1/\ell)=-1$, then $8\tilde{n}+1$ cannot be an odd square. Since $\tilde{\lambda}$ was assumed to be a 2-core, we have reached a contradiction. Thus, no such λ can exist.

Next, suppose ℓ is a prime which is 2 mod 3. Write $n=\ell^2 m+a_2$ and suppose $\lambda\vdash n$ such that $h_3(\lambda)=\ell^2 k+a_1$ or $h_2(\lambda)\equiv a_1 \bmod \ell^2$. We may write

$$n=\tilde{n}+3(\ell^2 k+a_1)=\ell^2 m+a_2$$

so $\tilde{n}=-3a_1+a_2+\ell^2(m-3k)$. Then

$$3\tilde{n}+1=-9a_1+3a_2+1+\ell^2(3m-9k).$$

Now if $\mathrm{ord}_\ell(-9a_1+3a_2+1)=1$, then $\ell\mid -9a_1+3a_2+1$ but $\ell^2\nmid -9a_1+3a_2+1$. Since $\ell^2\mid \ell^2(3n-9k)$, we conclude that ℓ divides $3\tilde{N}+1$ but ℓ^2 does not. Since $\tilde{\lambda}$ was assumed to be a 3-core, we have reached a contradiction; therefore, no such λ can exist. $\square$

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