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CHARACTERS AND POINT EVALUATIONS

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ABSTRACT. We give a simple proof that, if X is a Lindelöf topological space, and A is an algebra of continuous real-valued functions on X which is inverse-closed, local and z-regular, then every character on A is a point evaluation. We also give a number of examples to illustrate both the applications of this theorem and its limitations.

Let X be a topological space, and let A be an algebra of continuous real-valued functions on X which contains the constants. A *character* on A is an algebra homomorphism $\phi: A \longrightarrow \mathbb{R}$ such that $\phi(1) = 1$. For example, each point evaluation $f \longmapsto f(x)$ $(x \in X)$ is a character on A. This note concerns the converse: under what conditions on X and A is every character a point evaluation?

Of course, much work has been done on this question, but mostly in cases where X is compact or locally compact, or where A has some topological structure. The following result, which assumes neither, may therefore be of interest, especially in view of its very simple proof.

THEOREM. *Let X and A be as above, and assume further that:*
(a) X is Lindelöf *(every open cover of X has a countable subcover);*
(b) A is inverse-closed *(if $f \in A$ and $f(x) \neq 0$ for all $x \in X$, then $1/f \in A$);*
(c) A is local *(if $f|U \in A|U$ for each U in some open cover of X, then $f \in A$);*
(d) A is z-regular *(given a closed set $F \subset X$ and $x \in X \setminus F$, there exists $f \in A$ such that $f \neq 0$ on F and $f = 0$ on a neighbourhood of x).*
Then every character on A is a point evaluation.

PROOF. Before embarking on the proof proper, we make two elementary observations about characters ϕ on A. The first is that if f is an invertible element of A, then $\phi(f) \neq 0$. This is clear because

$$\phi(f)\phi(f^{-1}) = \phi(ff^{-1}) = \phi(1) = 1.$$

The second, which follows from the first, is that if $f, g \in A$ and $f \geq g$ then $\phi(f) \geq \phi(g)$. For if not, say $\phi(f) = \phi(g) - \lambda$ where $\lambda > 0$, then $f - g + \lambda$ is non-zero on X, and hence invertible by (b), while at the same time $\phi(f - g + \lambda) = 0$, contradicting our first remark.

To prove the theorem, we argue by contradiction. Suppose that ϕ is a character on A which is not a point evaluation. Then for each $x \in X$, there exists $g_x \in A$ with $g_x(x) \neq \phi(g_x) = 0$. Applying (d), we can find $f_x \in A$ such that $f_x \neq 0$ on $g_x^{-1}(0)$ and $f_x = 0$ on an open neighbourhood U_x of x. Since $f_x^2 + g_x^2 > 0$ everywhere on X, it is invertible

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by (b), and so by our earlier observation $\phi(f_x^2 + g_x^2) \neq 0$. As $\phi(g_x) = 0$, this implies that $\phi(f_x) \neq 0$, and multiplying f_x by a constant, we can in fact suppose that $\phi(f_x) = 1$.

Now the sets $(U_x)_{x\in X}$ form an open cover of X, so by (a) there is a countable subcover $(U_{x_n})_{n\geq 1}$. If this subcover is actually finite, then for some n we have $f_{x_1}\cdots f_{x_n} = 0$, giving

$$0 = \phi(f_{x_1}\cdots f_{x_n}) = \phi(f_{x_1})\cdots\phi(f_{x_n}) = 1,$$

which is obviously absurd. On the other hand, if the subcover is infinite, then consider the function

$$f = \sum_{n=1}^{\infty} f_{x_1}^2 \cdots f_{x_n}^2.$$

Each $x \in X$ has a neighbourhood on which this sum is finite, so by (c) it follows that $f \in A$. But then by our observation at the beginning,

$$\phi(f) \geq \phi\Big(\sum_{n=1}^{N} f_{x_1}^2 \cdots f_{x_n}^2\Big) = \sum_{n=1}^{N} \phi(f_{x_1})^2 \cdots \phi(f_{x_n})^2 = N$$

for every $N \geq 1$, which is also absurd. We have thus arrived at a contradiction, and the theorem is proved. ■

REMARK. The same result also holds for complex-valued functions, under the further assumption that A is *self-adjoint* ($f \in A \Rightarrow \bar{f} \in A$). To see this, either repeat the proof above, replacing f_x^2 by $|f_x|^2$ where appropriate, or else apply the theorem directly to the real-valued functions in A.

We now give some simple applications of this theorem.

COROLLARY 1. *If X is a completely regular Lindelöf space, then every character on $C(X)$ is a point evaluation.*

PROOF. It suffices to verify that conditions (a)–(d) hold. Certainly (a), (b) and (c) are clear. To check (d), take a closed set $F \subset X$ and $x \in X \setminus F$. As X is completely regular, there exists a continuous function g on X such that $g = 0$ on F and $g(x) = 1$. Then setting $f = \max(\frac{1}{2} - g, 0)$, we have $f \in C(X)$, with $f \neq 0$ on F and $f = 0$ on a neighbourhood of x. ■

In fact it is well known (see *e.g.* [2, §10.5(c)]) that the precise condition for every character on $C(X)$ to be a point evaluation is that X be *realcompact* (homeomorphic to a closed subset of a product of copies of $\mathbb{R}$). Thus Corollary 1 says that every completely regular Lindelöf space is realcompact, a fact which is also well known (see *e.g.* [2, §8.2]). Nonetheless, the proof of Corollary 1 seems simpler than most, even for the special case $X = \mathbb{R}$.

COROLLARY 2. *If X is a separable normed space, then every character on $C^k(X)$ ($k \geq 1$) and on $C^\infty(X)$ is a point evaluation.*

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PROOF. Again conditions (a), (b) and (c) are clear (note that for a metric space, Lindelöf is equivalent to separable). To prove (d), it suffices to construct $f \in C^\infty(X)$ such that $f = 0$ on $\{x : \|x\| \leq 1\}$ and $f > 0$ on $\{x : \|x\| > 1\}$; the general case then follows by considering $f\big((x-a)/r\big)$ for various a, r. Moreover, since X is separable, it can be isometrically embedded in ℓ_∞, so in fact it is enough to construct such an f on ℓ_∞.

To do this, take $\psi \in C^\infty(\mathbb{R})$ such that

$$\psi = 0 \text{ on } [-1,1], \quad \psi > 0 \text{ on } \mathbb{R} \setminus [-1,1] \quad \text{and} \quad \psi = 1 \text{ on } \mathbb{R} \setminus [-2,2],$$

and define $f \colon \ell_\infty \longrightarrow \mathbb{R}$ by

$$f(\xi_1, \xi_2, \ldots) = \sum_{k \geq 1} a_k \psi(\xi_k), \quad \text{where } a_k = \Big(2^k \sum_{j=0}^{k} \sup_{\mathbb{R}} |\psi^{(j)}|\Big)^{-1}.$$

Then $\sum_k a_k \psi^{(j)}(\xi_k)$ is uniformly convergent for each j, so $f \in C^\infty(X)$. Also

$$\|(\xi_1, \xi_2, \ldots)\|_\infty \leq 1 \Rightarrow f(\xi_1, \xi_2, \ldots) = \sum_{k \geq 1} 0 = 0,$$

$$\|(\xi_1, \xi_2, \ldots)\|_\infty > 1 \Rightarrow f(\xi_1, \xi_2, \ldots) \geq \sup_{k \geq 1} a_k \psi(\xi_k) > 0,$$

so f has the desired properties. ∎

Corollary 2 was proved in [1] under the additional assumption that X is complete. That paper also contains the idea for the construction of the function f above, but the rest of the proof is more complicated. The demonstration given above makes it clear that the linear structure of X plays a relatively minor rôle, and that the same result is true, for example, if X is any smooth manifold based on a separable normed space.

COROLLARY 3. *Let (X, d) be a separable metric space, and let A be any one of the following algebras of functions on X:*

(i) $\{f : f$ *is locally uniformly continuous on* $X\}$,

(ii) $\{f : f$ *is locally Lipschitz on* $X\}$,

(iii) $\big\{f : |f(y) - f(x)| = O\big(d(x,y)^\alpha\big)$ *as* $y \to x$, *all* $x \in X\big\}$ $(0 < \alpha \leq 1)$,

(iv) $\big\{f : |f(y) - f(x)| = o\big(d(x,y)^\alpha\big)$ *as* $y \to x$, *all* $x \in X\big\}$ $(0 < \alpha < 1)$.

Then every character on A is a point evaluation.

PROOF. Again it suffices to check that condition (d) holds. Given a closed set $F \subset X$ and $x \in X \setminus F$, let

$$f(z) = \operatorname{dist}(z, E), \quad \text{where } E = \{y \in X : d(x,y) \leq \operatorname{dist}(x,F)/2\}.$$

Then f belongs to all the algebras listed above, $f = 0$ on the neighbourhood E of x, and $f > 0$ on F. ∎

We now give some examples to show that the Theorem breaks down if any one of the hypotheses (a)–(d) is omitted.

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EXAMPLE 1. Let $A = C(\Omega)$, where Ω is the space of all countable ordinals with the order topology, *i.e.* the topology with base given by sets of the form

$$\{\omega \in \Omega : \alpha \leq \omega < \beta\} \quad (\alpha, \beta \in \Omega).$$

Then X is completely regular, so as in Corollary 1, A satisfies properties (b), (c) and (d). However, Ω also has the property that every continuous function $f: \Omega \longrightarrow \mathbb{R}$ is constant on all sufficiently large ω (see [2, §5.12(c)]), and so, denoting this constant by $\phi(f)$, we obtain a character $\phi: A \longrightarrow \mathbb{R}$ which is not a point evaluation at any $\omega \in \Omega$. Thus the Theorem breaks down if condition (a) is omitted (this example is classical).

EXAMPLE 2. Let $X = [-1, 1]$, and let A be the algebra of all continuous functions $f: [-1, 1] \longrightarrow \mathbb{R}$ of the form $f = p + g$, where p is a polynomial and g vanishes on a neighbourhood of 0. Clearly X is Lindelöf, so (a) holds, and it is easily checked that A satisfies (c) and (d). However, if we define $\phi: A \longrightarrow \mathbb{R}$ by

$$\phi(p + g) = p(2),$$

then ϕ is a character on A which is not a point evaluation at any $x \in [-1, 1]$. Hence the Theorem breaks down if condition (b) is omitted.

EXAMPLE 3. Let $X = \mathbb{R}$, and A be the algebra of continuous functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ of the form $f = \lambda + g$, where λ is a constant and g has compact support. Then (a), (b) and (d) clearly hold. However, if we define $\phi: A \longrightarrow \mathbb{R}$ by

$$\phi(\lambda + g) = \lambda,$$

then ϕ is a character on A which is not a point evaluation at any $x \in \mathbb{R}$. Therefore the Theorem also breaks down if condition (c) is omitted.

EXAMPLE 4. Let X be the (separable) Hilbert space ℓ_2, and let A be the algebra of functions $f: \ell_2 \longrightarrow \mathbb{R}$ of the form $f = h \circ \pi_n$, where $\pi_n: \ell_2 \longrightarrow \mathbb{R}^n$ denotes the projection onto the first n coordinates, $h: \mathbb{R}^n \longrightarrow \mathbb{R}$ is real-analytic, and n may be any positive integer. This algebra is evidently inverse-closed, and it is also local, because two real-analytic functions on $\mathbb{R}^n$ which agree on a non-empty open set must agree on the whole of $\mathbb{R}^n$. Thus conditions (a), (b) and (c) all hold. However, if we define $\phi: A \longrightarrow \mathbb{R}$ by

$$\phi(h \circ \pi_n) = h(1, 1, \ldots, 1),$$

then ϕ is a character on A which is not a point evaluation at any $\boldsymbol{\xi} \in \ell_2$, since such a $\boldsymbol{\xi}$ would have to satisfy $\xi_j = 1$ for all j. Thus the Theorem also breaks down if condition (d) is omitted.

We end with two questions.

(1) Does the Theorem remain valid if, in condition (a), we replace 'Lindelöf' by 'realcompact'? As mentioned earlier, this will true if $A = C(X)$, but the classical proof does not appear to extend to more general algebras.

(2) Does the Theorem remain valid if, in condition (d), we replace 'z-regular' by the more usual 'regular'? (A is *regular* if, given a closed set $F \subset X$ and $x \in X \setminus F$, there exists $f \in A$ such that $f = 0$ on F and $f(x) = 1$). Z-regular is so called because it amounts to saying that the interiors of the zero-sets of elements of A form a base for the topology on X. Regular just means that the topology on X equals the weak topology generated by A. Thus regularity appears the more natural, but z-regularity is sometimes easier to verify—witness Corollary 2 above.

REFERENCES

1. J. Arias-de-Reyna, *A real-valued homomorphism on algebras of differentiable functions*, Proc. Amer. Math. Soc. **104**(1988), 1054–1058.
2. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer–Verlag, 1976.

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