On isoptic families of curves

By H. W. Richmond.

1. Imagine that a number of straight lines, coplanar and concurrent, are united so as to form as it were a rigid frame. Imagine that this frame is moved (continuously) in the plane in such a way that two selected lines always touch two cycloids traced in the plane. Then it will be found that

Every line of the frame will move so as to envelope a cycloid.

Isoptic (and orthoptic) are names used by Charles Taylor for loci on which two tangents of curves intersect at a constant angle. The cycloids form a family of curves in which *each two members* have the same isoptic locus; they may therefore be described as forming an *isoptic family.* Isoptic loci are of no great importance or interest. Our aim here is to investigate this and other instances in which curves of a uniform type are enveloped by the various lines of a rigid frame.

2. In rectangular cartesian coordinates a curve is defined by the equation of its tangents

$$x \cos \psi + y \sin \psi = p \qquad \text{(i)}$$

when the relation between p and ψ is known. For a cycloid, however it may be placed, it is necessary and sufficient that

$$p = a \cos \psi + b \sin \psi + A\psi \cos \psi + B\psi \sin \psi \qquad \text{(ii)}$$

a, b, A, B being constants. A line inclined at an angle α to (i) and touching a second cycloid must have an equation

$$x \cos (\psi - \alpha) + y \sin (\psi - \alpha) = p' \qquad \text{(iii)}$$

where p' is (ii) with different constants a, b, A, B, and with $\psi - \alpha$ in place of ψ. But, on expanding the sines and cosines, we see that the addition of $-\alpha$ to ψ introduces no new type of term; p' is merely p with other constants.

A line concurrent with (i) and (iii) and inclined to (i) at an angle β has for its equation

$$x \cos (\psi - \beta) + y \sin (\psi - \beta) = p'' \qquad \text{(iv)}$$

where

$$p'' = kp + k'p' \qquad \text{(v)}$$

k and k' being constants whose values depend upon α and β. There-

https://doi.org/10.1017/S0950184300002743 Published online by Cambridge University Press

fore p'' is also of the form (ii), and the line (iv) envelopes another cycloid.

Since any value may be assigned to β in (iv) we have here the equation of a family of cycloids, simply infinite in number, which possess the property described in § 1. The family has been derived from two cycloids placed at random in a plane and two of their tangents. But these two cycloids are in no way exceptional members of the family; any two will serve as well. The equation is simplified if we choose two whose tangents meet at right angles, *i.e.* two for which α is $\pi/2$, for then k and k' take the values $\cos\beta$ and $\sin\beta$. The equation of the family of cycloids is

$$x \cos(\psi - \beta) + y \sin(\psi - \beta) = p \cos\beta + p' \sin\beta \qquad \text{(vi)}$$

p and p' being functions of ψ of the form (ii).

3. Formula (ii) which expresses p in terms of ψ in the case of a cycloid plainly implies that p must satisfy the differential equation

$$(D^2 + 1)^2 p = 0, \qquad \text{(vii)}$$

the operator D denoting differentiation with respect to ψ. We must consider why this should lead to the results we have obtained, and what other differential equations will lead to similar results.

It was seen to be essential in (v) that the sum of two solutions (of the differential equation for p) should also be a solution. The fact that (vii) is linear ensures this.

Again it is essential that the addition of $-\alpha$ to ψ should leave the form of p, and therefore the differential equation which gives p, unchanged. The coefficients in the linear differential equation must be independent of ψ, *i.e.* must be constants. Any linear differential equation with constant coefficients fulfils these two conditions.

But further the curves represented by (i) must be independent of the set of coordinates used. The form of p must be unaffected by a change of origin, which adds multiples of $\cos\psi$ and $\sin\psi$ to p, and by a rotation of the axes, which adds a constant to ψ. It has already been noted that the latter is without effect: as to the former, terms in $\cos\psi$ and $\sin\psi$ must already be contained in p; the differential operator in the equation must contain a factor $D^2 + 1$.

We may therefore enunciate a general theorem :—

If $F(D)$ is a polynomial in D, D denoting differentiation with respect to ψ, the curves given by (i)

$$x \cos\psi + y \sin\psi = p$$

https://doi.org/10.1017/S0950184300002743 Published online by Cambridge University Press

where p satisfies

$$(D^2+1)\,F(D)\,p=0 \qquad \text{(viii)}$$

are such that families having the properties described in § 1—*isoptic families—may be formed from them.*

4. Since F may be any polynomial the properties proved for cycloids in § 1 are shared by an infinity of other curves. For certain simple values of F they are well known curves, and in some of these the properties are also well known.

(a.) If $F(D)$ is a constant, $p=a\cos\psi+b\sin\psi$. The lines of the frame pass through points (a, b). The properties of lines through fixed points and inclined at constant angles were known in Euclid's day.

(b.) If $F(D)=D$, $p=c+a\cos\psi+b\sin\psi$. The curves are circles. From two circles (and two of their tangents) we derive as in § 1 a family of circles (vi). In one circle of the family we see that the radius c vanishes: one circle is a point, and lines through this point meet the tangents of a second circle perpendicularly. The isoptic locus is therefore the pedal of a circle, a Limaçon (or a Cardioid). In this case (which is well known) the equations of the family of circles may be expressed as

$$(x-A\sin^2\gamma)^2+(y-A\sin\gamma\cos\gamma)^2=C^2\sin^2\gamma.$$

Their centres lie on a circle $x^2+y^2=Ax$.

(c.) If $F(D)=D-k$, $p=a\cos\psi+b\sin\psi-c\,e^{k\psi}$. The curves are *Equiangular Spirals*, the angle between the curve and the radius vector being the same in all; its cotangent is k.

(d.) If $F(D)=D^2$, $p=a\cos\psi+b\sin\psi+c+C\psi$. The curves are *Involutes of circles.*

(e.) If $F(D)=D^2+n^2$ $\quad[n^2=1$ being excluded$]$,

$$p=a\cos\psi+b\sin\psi+A\cos n\psi+B\sin n\psi.$$

The curves are epicycloids or hypocycloids, the ratio of the radii of the fixed and the rolling circles being the same in all.

(f.) All other values of the polynomial $F(D)$ give equations of curves which satisfy the conditions imposed; but they are unfamiliar.

5. Up to this point it has been assumed that the lines forming the rigid frame are concurrent. A line coplanar but not concurrent with them obviously envelopes a curve parallel to one of the former

envelopes. Now the curves parallel to a circle are concentric circles. Hence the theorem of § 4 (b) may be extended; every line coplanar with the former lines envelopes a circle concentric with one of the former circles; and it was noticed that all these centres lie on a circle. (It is well known that if two sides of a triangle of given size and shape touch two given circles the third side must touch another circle.)

It may be pointed out that a similar extension from a frame of concurrent lines to the more general frame of all coplanar lines is valid also in § 4 (d); and whenever $F(D)$ is divisible by D.

Results such as these are remote from the trend of present day geometry and need not be developed further. I have relied for my facts upon Gino Loria's "Spezielle algebraische und transcendente ebene Kurven" (Teubner, 1902), an indispensable book of reference for curves which have a history. But see also "Curves, Special," in the *Encyclopaedia Britannica*, 14th Edition, Vol. 6, p. 897.

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