TOPICS ON SYMBOLIC REES ALGEBRAS
FOR SPACE MONOMIAL CURVES

SHIRO GOTO, KOJI NISHIDA AND YASUHIRO SHIMODA[1]

§1. Introduction

Let A be a regular local ring of $\dim A = 3$ and $\mathfrak{p}$ a prime ideal in A of $\dim A/\mathfrak{p} = 1$. We put $R_s(\mathfrak{p}) = \sum_{n \geq 0} \mathfrak{p}^{(n)} t^n$ (here t denotes an indeterminate over A) and call it the symbolic Rees algebra of $\mathfrak{p}$. With this notation the authors [5, 6] investigated the condition under which the A-algebra $R_s(\mathfrak{p})$ is Cohen-Macaulay and gave a criterion for $R_s(\mathfrak{p})$ to be a Gorenstein ring in terms of the elements f and g in Huneke's condition [11, Theorem 3.1] of $R_s(\mathfrak{p})$ being Noetherian. They furthermore explored the prime ideals $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ in the formal power series ring $A = k[\![X, Y, Z]\!]$ over a field k defining space monomial curves $X = t^{n_1}$, $Y = t^{n_2}$ and $Z = t^{n_3}$ with $\mathrm{GCD}\,(n_1, n_2, n_3) = 1$ and proved that $R_s(\mathfrak{p})$ are Gorenstein rings for certain prime ideals $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$. In the present research, similarly as in [5, 6], we are interested in the ring-theoretic properties of '$R_s(\mathfrak{p})$ mainly for $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ and the results of [5, 6] will play key roles in this paper.

However, the problem whether $R_s(\mathfrak{p})$ is a finitely generated A-algebra is more fundamental and as far as the authors know, it remains open even for the general space monomial curves $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$. And the problem was one of the motivations of J. Herzog and B. Ulrich to develop a theory on self-linked curve singularities [9], where they gave a criterion for $R_s(\mathfrak{p})$ to be generated in degree two and proved that $\mathfrak{p}$ is self-linked and $R_s(\mathfrak{p})$ is a Gorenstein ring, once $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$. In our paper we shall succeed the research of Herzog and Ulrich [9]; of course our target is the next step, that is, the study of the A-subalgebra $A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$ of $R_s(\mathfrak{p})$. There are two purposes: one is to find the conditions under which $R_s(\mathfrak{p})$ is generated in degree three and the other one is to

see what kind of ring-theoretic properties $R_s(\mathfrak{p})$ possesses, when $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.

Now let us explain our results. Throughout the work [9] the information from [12] about $\mathfrak{p}^{(2)}$ played an important role and we will briefly summarize the results on $\mathfrak{p}^{(2)}$ in Sections 2 and 4, because they are still helpful for our purposes, too. But we furthermore need an explicit structure theorem of the third symbolic powers $\mathfrak{p}^{(3)}$, which we will discuss in Section 5. The ring-theoretic properties of $R_s(\mathfrak{p})$ heavily depend on whether $\mathfrak{p}$ is self-linked or not. In fact, the ring $R_s(\mathfrak{p})$ is necessarily Gorenstein, if $\mathfrak{p}$ is self-linked and if $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$. But $R_s(\mathfrak{p})$ cannot be Cohen-Macaulay and the characteristic of the ground field has to be 2, if $\mathfrak{p}$ is not self-linked although $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$. These assertions will be proved in Theorems (6.1) and (6.12), where the conditions under which the A-algebra $R_s(\mathfrak{p})$ is generated in degree three shall be summarized, too. As was proved in [9], $A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ is a Gorenstein ring if $\mathfrak{p}$ is self-linked. However, this assertion is no more true when $\mathfrak{p}$ is not self-linked. In fact, $A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ is not a Cohen-Macaulay ring for $\mathfrak{p} = \mathfrak{p}(13, 14, 17)$, while the rings $A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ are Cohen-Macaulay and of $r(A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]) = 3$ for a certain large class of prime ideals $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ (cf. Section 3 and [9]). Thus our main purposes will be attained to by the end of Section 6.

As a final topic we would like to include in Section 7 an answer to a question posed by Huckaba [10, Remark 2.6 (3)]. He asked the relationship between Huneke's condition [13, Theorem 1.4] of $R_s(\mathfrak{p})$ being Noetherian and his own condition for that. Our answer Theorem (7.1) will show that they are equivalent to each other.

In what follows let A denote a 3-dimensional regular local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{p}$ a prime ideal in A of $\dim A/\mathfrak{p} = 1$. For a given ideal I we denote by $\lambda(I)$ the analytic spread of I. Let $\ell_A(M)$ and $\mu_A(M)$ respectively denote, for each finitely generated A-module M, the length of M and the number of elements in a minimal system of generators for M.

§ 2. Preliminaries

Let X, Y, Z be a regular system of parameters for A and assume that $\mathfrak{p}$ is generated by the maximal minors of a matrix M of the form

$$M = \begin{bmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{bmatrix},$$

where α, β, γ, α', β', γ' are positive integers. Notice that if $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ and $\mu_A(\mathfrak{p}) = 3$, then the ideal $\mathfrak{p}$ satisfies the above assumption (cf. [7]).

The purpose of this section is to summarize certain results on the second symbolic powers $\mathfrak{p}^{(2)}$. To begin with we note the following

LEMMA (2.1). (1) $\alpha \neq \alpha'$, $\beta \neq \beta'$ or $\gamma \neq \gamma'$.

(2) $2\alpha \neq \alpha'$, $2\beta \neq \beta'$ or $\gamma \neq \gamma'$.

(3) If $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ for some positive integers n_1, n_2 and n_3, then $\alpha \neq 2\alpha'$, $2\beta \neq \beta'$ or $2\gamma \neq \gamma'$.

Proof. (1) See [6, Lemma (2.3)].

(2) Suppose that $2\alpha = \alpha'$, $2\beta = \beta'$ and $\gamma = \gamma'$. Then since $(X^{3\alpha} - Y^\beta Z^\gamma) - (Y^{3\beta} - X^\alpha Z^\gamma) = (X^\alpha - Y^\beta)(X^{2\alpha} + X^\alpha Y^\beta + Y^{2\beta} + Z^\gamma)$, we have $X^\alpha - Y^\beta \in \mathfrak{p}$ or $X^{2\alpha} + X^\alpha Y^\beta + Y^{2\beta} + Z^\gamma \in \mathfrak{p}$. Therefore at least one of Y^β and $Y^{2\beta} + Z^\gamma$ must be in the ideal $(X) + \mathfrak{p} = (X, Z^{2\gamma}, Y^\beta Z^\gamma, Y^{3\beta})$, which is impossible.

(3) Suppose that $\alpha = 2\alpha'$, $2\beta = \beta'$ and $2\gamma = \gamma'$. Then $Z^{3\gamma} - X^{\alpha'} Y^{2\beta} \in \mathfrak{p}$ and $X^{3\alpha'} - Y^\beta Z^{2\gamma} \in \mathfrak{p}$. Hence $3\gamma n_3 = \alpha' n_1 + 2\beta n_2$ and $3\alpha' n_1 = \beta n_2 + 2\gamma n_3$. These equations imply $\alpha' n_1 = \gamma n_3$ and so $X^{\alpha'} - Z^\gamma \in \mathfrak{p}$. Thus we get $Z^\gamma \in (X) + \mathfrak{p} = (X, Z^{3\gamma}, Y^\beta Z^{2\gamma}, Y^{3\beta})$, which is impossible.

After suitable permutations of the rows and columns of M, we may assume that M is one of the following types.

(I) $\alpha \leq \alpha'$, $\beta \leq \beta'$ and $\gamma \leq \gamma'$,

(II) $\alpha > \alpha'$, $\beta < \beta'$ and $\gamma < \gamma'$.

We note that $\mathfrak{p}$ is self-linked (resp. not self-linked) if and only if M has type (I) (resp. type (II)) (cf. [9, Corollary 1.10]). We put

$$a = Z^{\gamma+\gamma'} - X^{\alpha'} Y^{\beta'}, \quad b = X^{\alpha+\alpha'} - Y^\beta Z^{\gamma'} \quad \text{and} \quad c = Y^{\beta+\beta'} - X^\alpha Z^\gamma.$$

PROPOSITION (2.2) (cf. [6, 12, 19]). Suppose that M has type (I). Then there exists $d_2 \in \mathfrak{p}^{(2)}$ such that $\mathfrak{p}^{(2)} = (d_2) + \mathfrak{p}^2$ and

$$X^{\alpha'} d_2 = Z^{\gamma'-\gamma} ac - Y^{\beta'-\beta} b^2,$$
$$Y^\beta d_2 = ab - X^{\alpha'-\alpha} Z^{\gamma'-\gamma} c^2,$$
$$Z^\gamma d_2 = X^{\alpha'-\alpha} Y^{\beta'-\beta} bc - a^2.$$

We furthermore have

$$d_2 \equiv -Z^{\gamma+2\gamma'} \bmod (X) \qquad \text{if } \alpha < \alpha',$$
$$\equiv -Z^{\gamma+2\gamma'} \bmod (Y) \qquad \text{if } \beta < \beta',$$
$$\equiv -X^{\alpha+2\alpha'} Y^{\beta'-\beta} \bmod (Z) \qquad \text{if } \gamma < \gamma'.$$

PROPOSITION (2.3) (cf. [15]).   *Suppose that M has type* (II).   *Then there exists $d_2 \in \mathfrak{p}^{(2)}$ such that $\mathfrak{p}^{(2)} = (d_2) + \mathfrak{p}^2$ and*

$$X^{\alpha'}d_2 = Z^{\gamma'-\gamma}ac - Y^{\beta'-\beta}b^2,$$
$$Y^{\beta}d_2 = X^{\alpha-\alpha'}ab - Z^{\gamma'-\gamma}c^2,$$
$$Z^{\gamma}d_2 = Y^{\beta'-\beta}bc - X^{\alpha-\alpha'}a^2.$$

Hence we have $d_2 \equiv -Y^{\beta+2\beta'}Z^{\gamma'-\gamma} \bmod (X)$.

COROLLARY (2.4).   (1)   *Suppose that M has type* (I).   *Then we have*

$$\mu_A(\mathfrak{p}^{2)}) = 4 \qquad \text{if } \beta = \beta' \text{ or } \gamma = \gamma',$$
$$= 5 \qquad \text{if } \beta < \beta' \text{ and } \gamma < \gamma'.$$

(2)   *If M has type* (II)*, then $\mu_A(\mathfrak{p}^{(2)}) = 7$.*

Proof.   See [15, (2.3)] for the proof of the assertion (2).

(1)   First we assume that $\beta = \beta'$. Then by (2.1) $\alpha < \alpha'$ or $\gamma < \gamma'$. If $\alpha < \alpha'$, by (2.2) we have $(X) + \mathfrak{p}^{(2)} = (X, Z^{\gamma+2\gamma'}, Y^{2\beta}Z^{\gamma+\gamma'}, Y^{3\beta}Z^{\gamma'}, Y^{4\beta})$ so that $\mu_A((X) + \mathfrak{p}^{(2)}/(X)) = 4$.   Hence $\mu_A(\mathfrak{p}^{(2)}) = 4$, because

$$(X) + \mathfrak{p}^{(2)}/(X) \cong \mathfrak{p}^{(2)}/(X) \cap \mathfrak{p}^{(2)}$$
$$= \mathfrak{p}^{(2)}/X\mathfrak{p}^{(2)}.$$

Similarly we get $\mu_A(\mathfrak{p}^{(2)}) = 4$ also for the case $\gamma < \gamma'$, because $(Z) + \mathfrak{p}^{(2)} = (Z, Y^{4\beta}, X^{\alpha'}Y^{3\beta}, X^{\alpha+\alpha'}Y^{2\beta}, X^{\alpha+2\alpha'})$.   A similar argument works to prove the assertion in the case where $\gamma = \gamma'$, too.

Next assume that $\beta < \beta'$ and $\gamma < \gamma'$. Then by (2.2) we have $(Y) + \mathfrak{p}^{(2)} = (Y, Z^{\gamma+2\gamma'}, X^{\alpha}Z^{2\gamma+\gamma'}, X^{2\alpha}Z^{2\gamma}, X^{2\alpha+\alpha'}Z^{\gamma}, X^{2\alpha+2\alpha'})$ and so $\mu_A((Y) + \mathfrak{p}^{(2)}/(Y)) = 5$. Therefore $\mu_A(\mathfrak{p}^{(2)}) = 5$.

§ 3.   The Cohen-Macaulay property of the A-algebra $A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$

Let us maintain the same notation as in Section 2.   The purpose is to prove the following

THEOREM (3.1).   *Suppose that M has type* (II) *and $(\alpha - 2\alpha')(\beta' - 2\beta)$ $(\gamma' - 2\gamma) \geq 0$.   Then the A-algebra $R = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ is a Cohen-Macaulay ring with $r(R) = 3$, where $r(R)$ denotes the Cohen-Macaulay type of R (cf. [8, p. 4]).*

We divide the proof of Theorem (3.1) into several steps.   First assume that M has type (II). Let T_1, T_2, T_3 and T_4 be indeterminates over A and

put $S = A[T_1, T_2, T_3, T_4]$. Let $\psi \colon S \to R$ be the homomorphism of A-algebras such that $\psi(T_1) = at$, $\psi(T_2) = bt$, $\psi(T_3) = ct$ and $\psi(T_4) = d_2 t^2$, where d_2 is the element of $\mathfrak{p}^{(2)}$ given in (2.3). Let $P = \operatorname{Ker} \psi$. Then P is a prime ideal in S of $\operatorname{ht}_S P = 3$ containing the following five elements

$$
\begin{aligned}
F_1 &= X^\alpha T_1 + Y^{\beta'} T_2 + Z^\gamma T_3, \\
F_2 &= Y^\beta T_1 + Z^\gamma T_2 + X^{\alpha'} T_3, \\
G_1 &= X^{\alpha'} T_4 + Y^{\beta' - \beta} T_2^2 - Z^{\gamma' - \gamma} T_1 T_3, \\
G_2 &= Y^\beta T_4 + Z^{\gamma' - \gamma} T_3^2 - X^{\alpha - \alpha'} T_1 T_2, \\
G_3 &= Z^\gamma T_4 + X^{\alpha - \alpha'} T_1^2 - Y^{\beta' - \beta} T_2 T_3.
\end{aligned}
$$

We put $J = (F_1, F_2, G_1, G_2, G_3)S$. Then $J \subseteq P$ and we have

LEMMA (3.2). S/J *is a Gorenstein ring of* $\dim S/J = 4$.

Proof. Because J is generated by the Pfaffians of order 4 in the following 5 by 5 antisymmetric matrix

$$
\begin{bmatrix}
0 & T_4 & T_3 & -T_1 & T_2 \\
-T_4 & 0 & -X^{\alpha - \alpha'} T_1 & Y^{\beta' - \beta} T_2 & -Z^{\gamma' - \gamma} T_3 \\
-T_3 & X^{\alpha - \alpha'} T_1 & 0 & Z^\gamma & Y^\beta \\
T_1 & -Y^{\beta' - \beta} T_2 & -Z^\gamma & 0 & X^{\alpha'} \\
-T_2 & Z^{\gamma' - \gamma} T_3 & -Y^\beta & -X^{\alpha'} & 0
\end{bmatrix},
$$

to see that S/J is a Gorenstein ring of $\dim S/J = 4$ it is enough by [2, Theorem 2.1] to show that $\operatorname{grade}_S J = 3$. We put $B = A[T_1, T_2, T_3]$. Then $F_1, F_2 \in B$ and $R(\mathfrak{p}) := \sum_{n \geq 0} \mathfrak{p}^n t^n \cong B/(F_1, F_2)B$ (cf. [21, Theorem 3.6]). Hence $(F_1, F_2)B \in \operatorname{Spec} B$ with $\operatorname{ht}_B(F_1, F_2)B = 2$ and so $(F_1, F_2)S \in \operatorname{Spec} S$ with $\operatorname{ht}_S(F_1, F_2)S = 2$. Obviously $G_1 \notin (F_1, F_2)S$. Thus we get that $\operatorname{grade}_S J \geq 3$, while $\operatorname{ht}_S J \leq 3$ as $J \subseteq \mathfrak{m} S$. Hence $\operatorname{grade}_S J = 3$ as required.

LEMMA (3.3). $\operatorname{Ass}_S S/J = \{P, \mathfrak{m} S\}$.

Proof (cf. Proof of [9, Theorem 2.1]). Obviously $\operatorname{Ass}_S S/J \supseteq \{P, \mathfrak{m} S\}$, so we shall prove the reverse inclusion. Since $\operatorname{Ass}_S S/J = \operatorname{Min}_S S/J$ by (3.2), we see that $\mathfrak{m} S$ is the unique associated prime ideal of S/J including $(X^{\alpha'}, Y^\beta, Z^\gamma)A$. Hence there exists $\xi \in (X^{\alpha'}, Y^\beta, Z^\gamma)A$ such that $\xi \notin Q$ for any $Q \in \operatorname{Ass}_S S/J \setminus \{\mathfrak{m} S\}$. We write $\xi = c_1 X^{\alpha'} + c_2 Y^\beta + c_3 Z^\gamma$, where $c_i \in A$ for $1 \leq i \leq 3$, and we put $G = c_1 G_1 + c_2 G_2 + c_3 G_3$. Then $G = \xi T_4 - \eta$ with $\eta \in A[T_1, T_2, T_3]$. Let $S' := S[1/\xi] = A[1/\xi][T_1, T_2, T_3, T_4]$ and let $\rho := \eta/\xi$. Then since $S'/(F_1, F_2, T_4 - \rho) \cong A[1/\xi][T_1, T_2, T_3]/(F_1, F_2)$, we get that $(F_1,$

$F_2, T_4 - \rho)S' \in \operatorname{Spec} S'$ and $\operatorname{ht}_{S'}(F_1, F_2, T_4 - \rho)S' = 3$ (cf. the proof of (3.2)). Let $Q \in \operatorname{Ass}_S S/J \setminus \{\mathfrak{m}S\}$. Then we get $JS' = QS'$ because $(F_1, F_2, T_4 - \rho)S'$ $\subseteq JS' \subseteq QS'$. This means that $PS' = QS'$ so that $P = Q$, which completes the proof.

As $JS_P = PS_P$ by the proof of (3.3), we know that P is the P-primary component of J. Let $C = S_{\mathfrak{m}S}$ and $\mathfrak{a} = JC \cap S$. Then we get

$$J = P \cap \mathfrak{a},$$

which implies $P = [J \colon \mathfrak{a}]_S$ and $\mathfrak{a} = [J \colon P]_S$ so that we have the following

LEMMA (3.4). $P/J = [(0) \colon \mathfrak{a}/J]_{S/J}$ and $\mathfrak{a}/J = [(0) \colon P/J]_{S/J}$.

Let $\alpha_1 = \min \{\alpha', \alpha - \alpha'\}$, $\beta_1 = \min \{\beta, \beta' - \beta\}$ and $\gamma_1 = \{\gamma, \gamma' - \gamma\}$ and put $K = (X^{\alpha_1}, Y^{\beta_1}, Z^{\gamma_1})S$.

PROPOSITION (3.5). R is a Cohen-Macaulay ring of $r(R) = 3$, if $JC = KC$.

Proof.   As K is $\mathfrak{m}S$-primary, we have $\mathfrak{a} = JC \cap S = KC \cap S = K$. Hence as S/K is Cohen-Macaulay, we see by (3.4) and [16, Proposition (1.3)] that $R = S/P$ is Cohen-Macaulay, too. Let K_R be the canonical module of R. Then we get

$$\begin{aligned} K_R &\cong \operatorname{Hom}_{S/J}(S/P, S/J) && \text{(by (3.2))} \\ &\cong [(0) \colon P/J]_{S/J} \\ &= K/J && \text{(by (3.4))}. \end{aligned}$$

Therefore we have

$$\begin{aligned} r(R) &= \mu_R(K/J) \\ &= \ell_R(K/(NK + J)) \\ &= \ell_R(K/NK) \\ &= 3, \end{aligned}$$

where $N = \mathfrak{m}R + (T_1, T_2, T_3, T_4)R$.

Now we are ready to prove Theorem (3.1).

Proof of Theorem (3.1).

It is enough to show that $JC = KC$ if $(\alpha - 2\alpha')(\beta' - 2\beta)(\gamma' - 2\gamma) \geq 0$. Let us respectively denote by x, y, z and t_i the images of X, Y, Z and T_i in C/JC. As G_1, G_2 and G_3 are in JC, we have

(*1) $$x^{\alpha'}t_4 + y^{\beta'-\beta}t_2^2 - z^{\gamma'-\gamma}t_1t_3 = 0,$$

(*2) $$y^{\beta}t_4 + z^{\gamma'-\gamma}t_3^2 - x^{\alpha-\alpha'}t_1t_2 = 0,$$

(*3) $$z^{\gamma}t_4 + x^{\alpha-\alpha'}t_1^2 - y^{\beta'-\beta}t_2t_3 = 0.$$

By the equations (*1) and (*2) we get

$$\begin{aligned}0 &= x^{\alpha'}t_3t_4 + y^{\beta'-\beta}t_2^2t_3 - t_1(z^{\gamma'-\gamma}t_3^2)\\&= x^{\alpha'}t_3t_4 + y^{\beta'-\beta}t_2^2t_3 - t_1(x^{\alpha-\alpha'}t_1t_2 - y^{\beta}t_4),\end{aligned}$$

so that

$$x^{\alpha_1}(x^{\alpha'-\alpha_1}t_3t_4 - x^{\alpha-\alpha'-\alpha_1}t_1^2t_2) + y^{\beta_1}(y^{\beta-\beta_1}t_1t_4 + y^{\beta'-\beta-\beta_1}t_2^2t_3) = 0.$$

Suppose that $\gamma' \geq 2\gamma$. Then by (*1) and (*3) we have

$$\begin{aligned}0 &= x^{\alpha'}t_4^2 + y^{\beta'-\beta}t_2^2t_4 - z^{\gamma'-2\gamma}t_1t_3(z^{\gamma}t_4)\\&= x^{\alpha'}t_4^2 + y^{\beta'-\beta}t_2^2t_4 - z^{\gamma'-2\gamma}t_1t_3(y^{\beta'-\beta}t_2t_3 - x^{\alpha-\alpha'}t_1^2),\end{aligned}$$

so that

$$x^{\alpha_1}(x^{\alpha'-\alpha_1}t_4^2 + x^{\alpha-\alpha'-\alpha_1}z^{\gamma'-2\gamma}t_1^3t_3) + y^{\beta_1}(y^{\beta'-\beta-\beta_1}t_2^2t_4 - y^{\beta'-\beta-\beta_1}z^{\gamma'-2\gamma}t_1t_2t_3^2) = 0.$$

Thus we get

$$U\begin{bmatrix} x^{\alpha_1} \\ y^{\beta_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$U = \begin{bmatrix} x^{\alpha'-\alpha_1}t_3t_4 - x^{\alpha-\alpha'-\alpha_1}t_1^2t_2 & y^{\beta-\beta_1}t_1t_4 + y^{\beta'-\beta-\beta_1}t_2^2t_3 \\ x^{\alpha'-\alpha_1}t_4^2 + x^{\alpha-\alpha'-\alpha_1}z^{\gamma'-2\gamma}t_1^3t_3 & y^{\beta'-\beta-\beta_1}t_2^2t_4 - y^{\beta'-\beta-\beta_1}z^{\gamma'-2\gamma}t_1t_2t_3^2 \end{bmatrix}.$$

Because

$$\begin{aligned}\det U &= -x^{\alpha'-\alpha_1}y^{\beta'-\beta-\beta_1}z^{\gamma'-2\gamma}t_1t_2t_3^3t_4 - x^{\alpha-\alpha'-\alpha_1}y^{\beta'-\beta-\beta_1}t_1^2t_2^3t_4\\&\quad - x^{\alpha'-\alpha_1}y^{\beta-\beta_1}t_1t_4^3 - x^{\alpha-\alpha'-\alpha_1}y^{\beta-\beta_1}z^{\gamma'-2\gamma}t_1^4t_3t_4,\end{aligned}$$

if $(\alpha - 2\alpha')(\beta' - 2\beta) \geq 0$ or $\gamma' = 2\gamma$, we see that $\det U$ is a unit of C/JC so that $z^{\gamma_1} = 0$ as well as $x^{\alpha_1} = y^{\beta_1} = 0$, which implies $JC = KC$.

We consider the case where $\gamma' \leq 2\gamma$. Then by (*1) and (*3) we get

$$\begin{aligned}0 &= z^{2\gamma-\gamma'}t_4(z^{\gamma'-\gamma}t_1t_3) + x^{\alpha-\alpha'}t_1^3t_3 - y^{\beta'-\beta}t_1t_2t_3^2\\&= z^{2\gamma-\gamma'}t_4(x^{\alpha'}t_4 + y^{\beta'-\beta}t_2^2) + x^{\alpha-\alpha'}t_1^3t_3 - y^{\beta'-\beta}t_1t_2t_3^2,\end{aligned}$$

so that

$$x^{\alpha_1}(x^{\alpha'-\alpha_1}z^{2\gamma-\gamma'}t_4^2 + x^{\alpha-\alpha'-\alpha_1}t_1^3t_3) + y^{\beta_1}(y^{\beta'-\beta-\beta_1}z^{2\gamma-\gamma'}t_2^2t_4 - y^{\beta'-\beta-\beta_1}t_1t_2t_3^2) = 0.$$

Hence we have

$$U\begin{bmatrix} x^{\alpha_1} \\ y^{\beta_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$U = \begin{bmatrix} x^{\alpha'-\alpha_1}t_3t_4 - x^{\alpha-\alpha'-\alpha_1}t_1^2t_2 & y^{\beta-\beta_1}t_1t_4 + y^{\beta'-\beta-\beta_1}t_2^2t_3 \\ x^{\alpha'-\alpha_1}z^{2\gamma-\gamma'}t_4^2 + x^{\alpha-\alpha'-\alpha_1}t_1^3t_3 & y^{\beta'-\beta-\beta_1}z^{2\gamma-\gamma'}t_2^2t_4 - y^{\beta'-\beta-\beta_1}t_1t_2t_3^2 \end{bmatrix}.$$

If $(\alpha - 2\alpha')(\beta' - 2\beta) \leq 0$ or $\gamma' = 2\gamma$, we see that $\det U$ is a unit in $C(JC)$ whence $x^{\alpha_1} = y^{\beta_1} = z^{\gamma_1} = 0$. Thus we have $JC = KC$ in this case, too. This together with the result in the case where $\gamma' \geq 2\gamma$ completes the proof of Theorem (3.1).

COROLLARY (3.6). *Suppose that M has type* (II) *and satisfies one of the following conditions*
    (1) $\alpha' = \beta = \gamma = 1$.
    (2) $\alpha = 2\alpha'$, $\beta' = 2\beta$ or $\gamma' = 2\gamma$.
Then $R = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ is a Cohen-Macaulay ring of $r(R) = 3$.

EXAMPLE (3.7). $R = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ is a Cohen-Macaulay ring of $r(R) = 3$ for the following prime ideals $\mathfrak{p}$:
    (1) $\mathfrak{p} = \mathfrak{p}(n^2 + 2n + 2, n^2 + 2n + 1, n^2 + n + 1)$, where $2 \leq n \in \mathbf{Z}$.
    (2) $\mathfrak{p} = \mathfrak{p}(n^2, n^2 + 1, n^2 + n + 1)$, where $3 \leq n \in \mathbf{Z}$.
    (3) $\mathfrak{p} = \mathfrak{p}(17, 10, 19)$.

Proof. The prime ideals $\mathfrak{p}$ of (1), (2) and (3) are respectively generated by the maximal minors of the matrix

$$\begin{bmatrix} X^n & Y^n & Z^{n+1} \\ Y & Z & X \end{bmatrix}, \quad \begin{bmatrix} X^n & Y^n & Z^{n-1} \\ Y & Z & X \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X^3 & Y^4 & Z^2 \\ Y^3 & Z & X \end{bmatrix}.$$

Hence we see by (3.6) that R is a Cohen-Macaulay ring of $r(R) = 3$.

EXAMPLE (3.8). Let $\mathfrak{p} = \mathfrak{p}(13, 14, 17)$. Then the A-algebra $R = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ is not a Cohen-Macaulay ring.

Proof. Notice that $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} X^3 & Y^3 & Z^3 \\ Y & Z & X^2 \end{bmatrix}$$

of type (II). We put $a = Z^4 - X^2Y^3$, $b = X^5 - YZ^3$ and $c = Y^4 - X^3Z$. Then by (2.3) there exists $d_2 \in \mathfrak{p}^{(2)}$ such that

$$X^2 d_2 = Y^2 b^2 - Z^2 ac,$$
$$Yd_2 = Z^2 c^2 - Xab \quad \text{and}$$
$$Zd_2 = Xa^2 - Y^2 bc.$$

Let $S = A[T_1, T_2, T_3, T_4]$ and P be as in the proof of (3.1). Then the following elements

$$F_1 = X^3 T_1 + Y^3 T_2 + Z^3 T_3,$$
$$F_2 = YT_1 + ZT_2 + X^2 T_3,$$
$$G_1 = X^2 T_4 - Y^2 T_2^2 + Z^2 T_1 T_3,$$
$$G_2 = YT_4 - Z^2 T_3^2 + XT_1 T_2 \quad \text{and}$$
$$G_3 = ZT_4 - XT_1^2 + Y^2 T_2 T_3$$

are in P. We put $J = (F_1, F_2, G_1, G_2, G_3)S$, $Q = \mathfrak{m}S$, $C = S_Q$ and $\mathfrak{a} = JC \cap S$. As is shown in the proof of (3.1), we have $J = P \cap \mathfrak{a}$. Therefore by (3.4) and [16, Proposition 1.3] it is enough to show that $S/\mathfrak{a}$ is not Cohen-Macaulay. First we prove

CLAIM 1. $Q^2 C \subseteq JC$.

Proof of Claim 1. We respectively denote by x, y, z and t_i the images of X, Y, Z and T_i in C/JC. It suffices to show that $(x, y, z)^2 = (0)$. Since G_1, G_2 and G_3 are in J, we have

(*1) $x^2 t_4 - y^2 t_2^2 + z^2 t_1 t_3 = 0,$

(*2) $yt_4 - z^2 t_3^2 + xt_1 t_2 = 0,$

(*3) $zt_4 - xt_1^2 + y^2 t_2 t_3 = 0.$

By the equations (*1) and (*2) we get $x^2 t_3 t_4 - y^2 t_2^2 t_3 + yt_1 t_4 + xt_1^2 t_2 = 0$ and so $y = ux$, where $u = (t_1^2 t_2 + xt_3 t_4)/(yt_2^2 t_3 - t_1 t_4)$. Then by (*3) we have $zt_4 - xt_1^2 + u^2 x^2 t_2 t_3 = 0$ so that $z = vx$ with $v = (t_1^2 - u^2 xt_2 t_3)/t_4$. Thus the equation (*1) yields $x^2(t_4 - u^2 t_2^2 + v^2 t_1 t_3) = 0$. Let $Q' = QC/JC$. Then as $u \equiv -t_1 t_2/t_4 \bmod Q'$ and as $v \equiv t_1^2/t_4 \bmod Q'$, we get $t_4 - u^2 t_2^2 + v^2 t_1 t_3 \equiv (t_4^3 - t_1^2 t_2^2 + t_1^5 t_3)/t_4^2 \bmod Q'$. Hence $t_4 - u^2 t_2^2 + v^2 t_1 t_3$ is a unit in C/JC and so $x^2 = 0$, which claims that $(x, y, z)^2 = 0$, because $y = ux$ and $z = vx$.

We put $I = (f, g, h)S + Q^2$, where $f = YT_4 + XT_1 T_2$, $g = ZT_4 - XT_1^2$ and $h = YT_1 + ZT_2$. Then $IC = JC$ by Claim 1. We want to show that I is Q-primary and S/I is not Cohen-Macaulay, which will complete the proof of (3.8) because $I = \mathfrak{a}$.

Let us consider the exact sequence

($\sharp$) $$0 \longrightarrow Q/I \longrightarrow S/I \longrightarrow S/Q \longrightarrow 0.$$

CLAIM 2.　$\mathrm{rank}_{S/Q} \, Q/I = 1$.

Proof of Claim 2.　We shall show that $\ell_C(QC/IC) = 1$. Notice that $\ell_C(QC/IC) \leq 1$, because $QC \supseteq IC \supseteq Q^2C$ and $QC = (X, f, g)C$. Hence it is enough to show that $QC \neq IC$. Assume the contrary. Then $QC = (f, g, h)C$ by Nakayama's lemma, which is impossible because

$$\begin{bmatrix} f \\ g \\ h \end{bmatrix} = U \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{where} \quad U = \begin{bmatrix} T_1 T_2 & T_4 & 0 \\ -T_1^2 & 0 & T_4 \\ 0 & T_1 & T_2 \end{bmatrix}$$

and because $\det U = 0$.

We consider the following complex

$$0 \longrightarrow S/Q \xrightarrow{\varphi_3} (S/Q)^3 \xrightarrow{\varphi_2} (S/Q)^2 \xrightarrow{\varphi_1} S/Q \, ,$$

$$\begin{bmatrix} T_2 \\ T_1 \\ -T_4 \end{bmatrix} \qquad \begin{bmatrix} -T_1^2 & T_1 T_2 & 0 \\ 0 & T_4 & T_1 \\ T_4 & 0 & T_2 \end{bmatrix} \qquad [T_4 \quad -T_1 T_2 \quad T_1^2]$$

which is exact by [1, Theorem]. Let $\varepsilon \colon (S/Q)^3 \to Q/I$ be the S/Q-homomorphism such that $\varepsilon(e_1) = X \bmod I$, $\varepsilon(e_2) = Y \bmod I$ and $\varepsilon(e_3) = Z \bmod I$, where e_1, e_2 and e_3 are the standard basis of $(S/Q)^3$. Then since $\varepsilon \circ \varphi_2 = 0$, there exists an epimorphism $\rho \colon (T_4, -T_1 T_2, T_1^2)(S/Q) \to Q/I$ such that $\rho \circ \varphi_1 = \varepsilon$. Then ρ is an isomorphism by Claim 2. Therefore we have that Q/I is embedded, via ρ, in S/Q and that proj. $\dim_{S/Q} Q/I = 2$. Thus we see that $\mathrm{Ass}_S \, Q/I = \{Q\}$ and $\mathrm{depth}_{S_N}(Q/I)_N = 2$, where $N = \mathfrak{m}S + (T_1, T_2, T_3, T_4)S$ (notice that S/Q is a polynomial ring with four variables over the field $k = A/\mathfrak{m}$). Hence by the exact sequence (♯) we get $\mathrm{Ass}_S S/I = \{Q\}$ and $\mathrm{depth}_{S_N}(S/I)_N = 2$, so that S/I is not a Cohen-Macaulay ring because $\dim(S/I)_N = 4$. This completes the proof of (3.8).

§4.　The generation of $R_s(\mathfrak{p})$ in degree two

Let A be a regular local ring of $\dim A = 3$ and $\mathfrak{p}$ a prime ideal in A of $\dim A/\mathfrak{p} = 1$. In this general situation we shall characterize prime ideals $\mathfrak{p}$ whose symbolic Rees algebras $R_s(\mathfrak{p})$ are generated by $\mathfrak{p}t$ and $\mathfrak{p}^{(2)}t^2$. The first result is

THEOREM (4.1).　*Suppose that $A/\mathfrak{m}$ is infinite. Then the following conditions are equivalent.*

(1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$.

(2) $\ell_A(A/(f, g, x)) = 4 \cdot \ell_A(A/(x) + \mathfrak{p})$ for some $f, g \in \mathfrak{p}^{(2)}$ and $x \in \mathfrak{m} \backslash \mathfrak{p}$.

(3) $(f, g): \mathfrak{p}^{(2)} = \mathfrak{p}$ for some $f, g \in \mathfrak{p}^{(2)}$.

(4) $(f, g) \supseteq \mathfrak{p}^{(3)}$ for some $f, g \in \mathfrak{p}^{(2)}$.

When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. See [5, Corollaries (3.8) and (3.9)] for the implication $(2) \Rightarrow (1)$ and the last assertion.

$(1) \Rightarrow (2)$ As $\mathfrak{p}^{(n)} = \mathfrak{p}\mathfrak{p}^{(n-1)} + \mathfrak{p}^{(2)}\mathfrak{p}^{(n-2)}$ $(n \geq 1)$, we get $\mathfrak{p}^{(2n)} = [\mathfrak{p}^{(2)}]^n$ and $\mathfrak{p}^{(2n+1)} = \mathfrak{p}[\mathfrak{p}^{(2)}]^n$ for all $n \geq 1$. Hence depth $A/[\mathfrak{p}^{(2)}]^n = 1$ for any $n \geq 1$ and so by [3, Corollary, (i)] we may choose $f, g \in \mathfrak{p}^{(2)}$ so that $[\mathfrak{p}^{(2)}]^{r+1} = (f, g)[\mathfrak{p}^{(2)}]^r$ for some $r \geq 0$. Therefore we get the condition (2) by the proof of [11, Theorem 3.1].

$(2) \Rightarrow (4)$ See [5, Proposition (3.4)].

$(4) \Rightarrow (2)$ Let $B = A_\mathfrak{p}$ and $\mathfrak{n} = \mathfrak{p}A_\mathfrak{p}$. Then $\mathfrak{n}^2 \neq (f, g)B \supseteq \mathfrak{n}^3$ (recall that $\mu_A(\mathfrak{n}^2) = 3$). Therefore we get $\ell_A(\mathfrak{n}^2/(f, g)B) = 1$ as $B/(f, g)B$ is a Gorenstein ring. Consequently $\ell_B(B/(f, g)B) = 4$ and so by the additive formula of multiplicity (cf. [18, p. 126]) we have

$$\begin{aligned}
\ell_A(A/(f, g, x)) &= e_{xA}(A/(f, g)) \\
&= \ell_B(B/(f, g)B) \cdot e_{xA}(A/\mathfrak{p}) \\
&= 4 \cdot \ell_A(A/(x) + \mathfrak{p})
\end{aligned}$$

for any $x \in \mathfrak{m} \backslash \mathfrak{p}$. Thus we get the condition (2).

$(4) \Rightarrow (3)$ Notice that (f, g) is a $\mathfrak{p}$-primary ideal and $\mathfrak{p}^{(2)} \not\subset (f, g)$. Then we have $(f, g): \mathfrak{p}^{(2)} \subseteq \mathfrak{p}$ so that $(f, g): \mathfrak{p}^{(2)} = \mathfrak{p}$, because $(f, g): \mathfrak{p}^{(2)} \supseteq \mathfrak{p}$ by the assumption (4).

$(3) \Rightarrow (4)$ As $\mathfrak{p}^{(2)} \supseteq (f, g) \supseteq \mathfrak{p}\mathfrak{p}^{(2)}$, we have that $(f, g) \supseteq \mathfrak{p}^3$ and $\mathfrak{p} = \sqrt{(f, g)}$. Hence $(f, g) \supseteq \mathfrak{p}^{(3)}$, because (f, g) is a $\mathfrak{p}$-primary ideal of A.

For prime ideals $\mathfrak{p}$ with $\mu_A(\mathfrak{p}^{(2)}/\mathfrak{p}^2) \leq 1$, we can improve Theorem (4.1) as follows.

THEOREM (4.2).   *Suppose that $A/\mathfrak{m}$ is infinite. Then the following conditions are equivalent.*

(1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ and $\mu_A(\mathfrak{p}^{(2)}/\mathfrak{p}^2) \leq 1$.

(2) $\ell_A(A/(f, g, x)) = 2 \cdot \ell_A(A/(x) + \mathfrak{p})$ for some $f \in \mathfrak{p}$, $g \in \mathfrak{p}^{(2)}$ and $x \in \mathfrak{m} \backslash \mathfrak{p}$.

(3) $\mathfrak{p}^{(2)} = f\mathfrak{p} + (g)$ for some $f \in \mathfrak{p}$ and $g \in \mathfrak{p}^{(2)}$.

(4) $(f, g): \mathfrak{p} = \mathfrak{p}$ for some $f \in \mathfrak{p}$ and $g \in \mathfrak{p}^{(2)}$.

When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. $(1) \Rightarrow (2)$ By (4.1) we can choose $\alpha, \beta \in \mathfrak{p}^{(2)}$ so that $\ell_A(A/(\alpha, \beta, x))$ $= 4 \cdot \ell_A(A/(x) + \mathfrak{p})$ for some $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then $\mathfrak{p}^{(3)} = (\alpha, \beta)\mathfrak{p}$ by [5, Proposition (3.4), Lemma (3.5)]. If $(\alpha, \beta) \subseteq \mathfrak{m}\mathfrak{p}^{(2)} + \mathfrak{p}^2$, we have $\mathfrak{p}^{(3)} \subseteq \mathfrak{m}\mathfrak{p}^{(3)} + \mathfrak{p}^3$ so that $\mathfrak{p}^{(3)} = \mathfrak{p}^3$ by Nakayama's lemma. Then as $\mu_A(\mathfrak{p}) = 2$ by [12, (2.5) Corollary], the assertion (2) is clear. Therefore we may assume $\beta \notin \mathfrak{m}\mathfrak{p}^{(2)} + \mathfrak{p}^2$. Let $g = \beta$. Then we have $\mathfrak{p}^{(2)} = (g) + \mathfrak{p}^2$ as $\mu_A(\mathfrak{p}^{(2)}/\mathfrak{p}^2) \leq 1$ by our assumption. We put $B = A/gA$ and $P = \mathfrak{p}/gA$. Then as $P^2 = \mathfrak{p}^{(2)}/gA$, we have that P^2 is P-primary. Because $[\mathfrak{p}^{(2)}]^2 = (\alpha, g)\mathfrak{p}^{(2)}$ (cf. The proof of [11, Theorem 3.1]), we get $P^4 = \alpha P^2$ whence $\lambda(P) = 1$. Choose $f \in \mathfrak{p}$ so that $P^{r+1} = fP^r$ for some $r \geq 0$. Then as $v(B_P) = 2 = e(B_P)$ (notice that $g \in \mathfrak{p}^{(2)} \setminus \mathfrak{p}^{(3)}$, cf. [5, Proposition (3.7), (3)]), we see $P^2 B_P = fPB_P$ by [17, 3.8. Theorem] so that $P^2 = fP$, because P^2 and fP are P-primary ideals of B. Hence $\mathfrak{p}^{(2)} \subseteq (f, g) \subseteq \mathfrak{p}$ and we have $\mathfrak{p}^{(2)} + (f) = (f, g)$. Thus (f, g) is a $\mathfrak{p}$-primary ideal and $f \notin \mathfrak{p}^{(2)}$, whence

$$
\begin{aligned}
\ell_A(A/(f, g, x)) &= e_{xA}(A/(f) + \mathfrak{p}^{(2)}) \\
&= \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/fA_\mathfrak{p} + \mathfrak{p}^2 A_\mathfrak{p}) \cdot e_{xA}(A/\mathfrak{p}) \\
&= 2 \cdot \ell_A(A/(x) + \mathfrak{p})
\end{aligned}
$$

for any $x \in \mathfrak{m} \setminus \mathfrak{p}$.

$(2) \Rightarrow (1)$ See [5, Propositions (3.4) and (3.7)].

$(2) \Rightarrow (3)$ As $\mathfrak{p}^{(2)} \subseteq (f, g)$ by [5, Proposition (3.4)], we get $\mathfrak{p}^{(2)} = f\mathfrak{p} + (g)$ by [5, Lemma (3.5)].

$(3) \Rightarrow (4)$ As $(f, g) \supseteq \mathfrak{p}^{(2)}$, (f, g) is $\mathfrak{p}$-primary and $(f, g) : \mathfrak{p} \supseteq \mathfrak{p}$. Therefore $(f, g) : \mathfrak{p} = \mathfrak{p}$, because $\mathfrak{p} \neq (f, g)$.

$(4) \Rightarrow (2)$ As $\mathfrak{p}^2 \subseteq (f, g) \subseteq \mathfrak{p}$, we see that (f, g) is $\mathfrak{p}$-primary. Therefore $\mathfrak{p}^{(2)} \subseteq (f, g)$ so that $\mathfrak{p}^{(2)} + (f) = (f, g)$, which implies $f \notin \mathfrak{p}^{(2)}$. Hence we have $\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/fA_\mathfrak{p} + \mathfrak{p}^2 A_\mathfrak{p}) = 2$ and so $\ell_A(A/(f, g, x)) = 2 \cdot \ell_A(A/(x) + \mathfrak{p})$ for any $x \in \mathfrak{m} \setminus \mathfrak{p}$. Thus we get the assertion (2).

In the rest of this section let A, $\mathfrak{p}$ and M be as in Section 2. Then by (4.2) we get the following

COROLLARY (4.3) (cf. [9, Corollary 2.12]). *The following conditions are equivalent.*

(1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$.

(2) *M has type* (I) *and* (i) $\beta = \beta'$ *or* (ii) $\alpha = \alpha'$ *and* $\gamma = \gamma'$.
When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. $(1) \Rightarrow (2)$ Since $\mu_A(\mathfrak{p}^{(2)}/\mathfrak{p}^2) \leq 1$ by (2.2) and (2.3), we get $\mu_A(\mathfrak{p}^{(2)})$

≤ 4 by (3) of (4.2) so that M has type (I) and $\beta = \beta'$ or $\gamma = \gamma'$ by (2.4). Suppose that $\beta < \beta'$ and $\gamma = \gamma'$. We shall prove that $\alpha = \alpha'$. Since we get

$$(Y) + \mathfrak{p}\mathfrak{p}^{(2)} = (Y) + (Z^{5\gamma}, X^{\alpha}Z^{4\gamma}, X^{3\alpha}Z^{3\gamma}, X^{3\alpha+\alpha'}Z^{2\gamma}, X^{3\alpha+2\alpha'}Z^{\gamma}, X^{3\alpha+3\alpha'})$$
$$\text{if } 2\alpha < \alpha',$$
$$= (Y) + (Z^{5\gamma}, X^{\alpha}Z^{4\gamma}, X^{\alpha+\alpha'}Z^{3\gamma}, X^{3\alpha+\alpha'}Z^{2\gamma}, X^{3\alpha+2\alpha'}Z^{\gamma}, X^{3\alpha+3\alpha'})$$
$$\text{if } 2\alpha \geq \alpha'$$

by (2.2), we obtain

$$\ell_A(A/(Y) + \mathfrak{p}\mathfrak{p}^{(2)}) = 13\alpha\gamma + 6\alpha'\gamma \quad \text{if } 2\alpha < \alpha',$$
$$= 11\alpha\gamma + 7\alpha'\gamma \quad \text{if } 2\alpha \geq \alpha'.$$

On the other hand the condition (1) implies $\mathfrak{p}\mathfrak{p}^{(2)} = \mathfrak{p}^{(3)}$ so that

$$\ell_A(A/(Y) + \mathfrak{p}\mathfrak{p}^{(2)}) = \ell_A(A/(Y) + \mathfrak{p}^{(3)})$$
$$= e_{Y\,A}(A/\mathfrak{p}^{(3)})$$
$$= \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}^3 A_\mathfrak{p}) \cdot \ell_A(A/(Y) + \mathfrak{p})$$
$$= 6(2\alpha\gamma + \alpha'\gamma).$$

Thus we have

$$12\alpha\gamma + 6\alpha'\gamma = 13\alpha\gamma + 6\alpha'\gamma \quad \text{if } 2\alpha < \alpha',$$
$$= 11\alpha\gamma + 7\alpha'\gamma \quad \text{if } 2\alpha \geq \alpha'.$$

Hence $2\alpha \geq \alpha'$ and $\alpha\gamma = \alpha'\gamma$ so that $\alpha = \alpha'$, as required.

$(2) \Rightarrow (1)$ First we assume $\beta = \beta'$. Then $\alpha < \alpha'$ or $\gamma < \gamma'$ by (2.1). Let d_2 be the element obtained by (2.2). If $\alpha < \alpha'$ (resp. $\gamma < \gamma'$), then $d_2 \equiv -Z^{\gamma+2\gamma'} \bmod (X)$ (resp. $d_2 \equiv -X^{\alpha+2\alpha'} \bmod (Z)$) so that

$$\ell_A(A/(c, d_2, X)) = \ell_A(A/(Y^{2\beta}, Z^{\gamma+2\gamma'}, X))$$
$$= 2\beta\gamma + 4\beta\gamma'$$
$$= 2 \cdot \ell_A(A/(X) + \mathfrak{p})$$
$$(\text{resp. } \ell_A(A/(c, d_2, Z)) = \ell_A(A/(Y^{2\beta}, X^{\alpha+2\alpha'}, Z))$$
$$= 2\alpha\beta + 4\alpha'\beta$$
$$= 2 \cdot \ell_A(A/(Z) + \mathfrak{p})),$$

whence $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ by (4.2).

Next assume that $\alpha = \alpha'$ and $\gamma = \gamma'$. Then $\beta < \beta'$ by (2.1). Hence by (2.2) we get $d_2 \equiv -Z^{3\gamma} \bmod (Y)$ so that

$$\ell_A(A/(b, d_2, Y)) = \ell_A(A/(X^{2\alpha}, Z^{3\gamma}, Y))$$
$$= 6\alpha\gamma$$
$$= 2 \cdot \ell_A(A/(Y) + \mathfrak{p}).$$

Thus $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ by (4.2), which completes the proof of (4.3).

§ 5.   The third symbolic power $\mathfrak{p}^{(3)}$

The purpose of this section is to investigate the structure of $\mathfrak{p}^{(3)}$ for the prime ideals $\mathfrak{p}$ generated by the maximal minors of matrices M of the form

$$M = \begin{bmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{bmatrix},$$

where X, Y, Z is a regular system of parameters for A and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers. We put $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}$, $b = X^{\alpha+\alpha'} - Y^\beta Z^{\gamma'}$ and $c = Y^{\beta+\beta'} - X^\alpha Z^\gamma$.

We begin with the case where M has type (I). By the equation $Y^\beta a + Z^\gamma b + X^{\alpha'}c = 0$ and those given in (2.2), we have

$$b(X^\alpha d_2 + Y^{\beta'-\beta}b^2) = Z^{\gamma'-\gamma}abc = Z^{\gamma'-\gamma}c(Y^\beta d_2 + X^{\alpha'-\alpha}Z^{\gamma'-\gamma}c^2),$$

$$ac(-Y^\beta a - X^{\alpha'}c) = Z^\gamma abc = Z^{2\gamma-\gamma'}b(X^\alpha d_2 + Y^{\beta'-\beta}b^2) \qquad \text{if } 2\gamma \geq \gamma',$$

$$Z^{\gamma'-2\gamma}ac(-Y^\beta a - X^{\alpha'}c) = Z^{\gamma'-\gamma}abc = b(X^\alpha d_2 + Y^{\beta'-\beta}b^2) \qquad \text{if } 2\gamma < \gamma'$$

so that

(5.1) $X^\alpha bd_2 - X^{\alpha'-\alpha}Z^{2\gamma'-2\gamma}c^3 = Y^\beta Z^{\gamma'-\gamma}cd_2 - Y^{\beta'-\beta}b^3,$

(5.2) $X^\alpha Z^{2\gamma-\gamma'}bd_2 + X^{\alpha'}ac^2 = -Y^\beta a^2 c - Y^{\beta'-\beta}Z^{2\gamma-\gamma'}b^3 \qquad \text{if } 2\gamma \geq \gamma',$

(5.3) $X^\alpha bd_2 + X^{\alpha'}Z^{\gamma'-2\gamma}ac^2 = -Y^\beta Z^{\gamma'-2\gamma}a^2 c - Y^{\beta'-\beta}b^3 \qquad \text{if } 2\gamma < \gamma'.$

THEOREM (5.4).   *Suppose that M has type* (I).

(1)   *Assume $2\alpha < \alpha'$ or $2\gamma < \gamma'$. Then there exists $d_3 \in \mathfrak{p}^{(3)}$ such that* $\mathfrak{p}^{(3)} = (d_3) + \mathfrak{p}\mathfrak{p}^{(2)}$ *and*

$$d_3 \equiv Z^{\gamma+3\gamma'} \bmod (X) \qquad\qquad \text{if } 2\alpha < \alpha' \text{ and } 2\beta < \beta',$$
$$d_3 \equiv Y^{2\beta-\beta'}Z^{\gamma+3\gamma'} \bmod (X) \qquad \text{if } 2\alpha < \alpha' \text{ and } 2\beta \geq \beta',$$
$$d_3 \equiv -X^{2\alpha+3\alpha'}Y^{\beta'-2\beta} \bmod (Z) \quad \text{if } 2\beta < \beta' \text{ and } 2\gamma < \gamma',$$
$$d_3 \equiv -X^{2\alpha+3\alpha'} \bmod (Z) \qquad\qquad \text{if } 2\beta \geq \beta' \text{ and } 2\gamma < \gamma'.$$

(2)   *Assume $2\alpha \geq \alpha'$ and $2\gamma \geq \gamma'$, Then there exist $d_3 \in \mathfrak{p}^{(3)}$ and $d_3' \in \mathfrak{p}^{(3)}$ such that* $\mathfrak{p}^{(3)} = (d_3, d_3') + \mathfrak{p}\mathfrak{p}^{(2)}$ *and*

$$d_3 \equiv X^{2\alpha-\alpha'}Z^{\gamma+3\gamma'} \bmod (Y) \quad and \quad d_3' \equiv Z^{3\gamma+2\gamma'} \bmod (Y) \quad if\ 2\beta < \beta',$$

$$d_3 \equiv -Y^{4\beta+2\beta'}Z^{2\gamma'-2\gamma} \bmod (X) \quad and \quad d_3' \equiv Y^{2\beta-\beta'}Z^{3\gamma+2\gamma'} \bmod (X)$$
$$if\ 2\alpha > \alpha' > \alpha\ and\ 2\beta \geq \beta',$$

$$d_3 \equiv -X^{4\alpha+2\alpha'} \bmod (Y) \quad and \quad d_3' \equiv -X^{2\alpha+3\alpha'}Z^{2\gamma-\gamma'} \bmod (Y)$$
$$if\ 2\beta > \beta' > \beta,$$

$$d_3 \equiv -X^{4\alpha+2\alpha'} \bmod (Z) \quad and \quad d_3' \equiv -X^{2\alpha'-\alpha}Y^{3\beta+2\beta'} \bmod (Z)$$
$$if\ 2\beta \geq \beta'\ and\ 2\gamma > \gamma' > \gamma,$$

$$d_3 \equiv -X^{2\alpha+3\alpha'} \bmod (Z) \quad and \quad d_3' \in \mathfrak{p}\mathfrak{p}^{(2)} \quad if\ 2\alpha = \alpha'\ and\ 2\beta = \beta',$$

$$d_3 \in \mathfrak{p}\mathfrak{p}^{(2)} \quad and \quad d_3' \equiv Z^{7\gamma} - X^{5\alpha} \bmod (Y)$$
$$if\ \alpha = \alpha',\ 2\beta = \beta'\ and\ 2\gamma = \gamma'.$$

Remark (5.5). When M is not of any case stated in (5.4), M satisfies the condition (2) of (4.3) so that $\mathfrak{p}^{(3)} = \mathfrak{p}\mathfrak{p}^{(2)}$.

Proof of Theorem (5.4).

(1) First we assume $2\alpha < \alpha'$. By (5.1) we have

$$X^\alpha(bd_2 - X^{\alpha'-2\alpha}Z^{2\gamma'-2\gamma}c^3) = Y^\beta(Z^{\gamma'-\gamma}cd_2 - Y^{\beta'-2\beta}b^3) \quad if\ 2\beta < \beta',$$
$$= Y^{\beta'-\beta}(Y^{2\beta-\beta'}Z^{\gamma'-\gamma}cd_2 - b^3) \quad if\ 2\beta \geq \beta'.$$

Hence there exists $d_3 \in \mathfrak{p}^{(3)}$ such that

$$bd_2 - X^{\alpha'-2\alpha}Z^{2\gamma'-2\gamma}c^3 = Y^\beta d_3 \quad if\ 2\beta < \beta',$$
$$= Y^{\beta'-\beta}d_3 \quad if\ 2\beta \geq \beta'.$$

As $\alpha < \alpha'$, by (2.1) $d_2 \equiv -Z^{\gamma+2\gamma'} \bmod (X)$ and so

$$d_3 \equiv Z^{\gamma+3\gamma'} \bmod(X) \quad if\ 2\beta < \beta',$$
$$\equiv Y^{2\beta-\beta'}Z^{\gamma+3\gamma'} \bmod (X) \quad if\ 2\beta \geq \beta'.$$

We put $I := (d_3) + \mathfrak{p}\mathfrak{p}^{(2)}\ (\supseteq \mathfrak{p}^3)$. Then if $2\beta < \beta'$, we have

$$(X) + I = (X, Z^{\gamma+3\gamma'}, Y^{3\beta}Z^{3\gamma'}, Y^{\beta+\beta'}Z^{\gamma+2\gamma'}, Y^{3\beta+\beta'}Z^{2\gamma'},$$
$$Y^{2\beta+2\beta'}Z^{\gamma+\gamma'}, Y^{3\beta+2\beta'}Z^{\gamma'}, Y^{3\beta+3\beta'}).$$

Hence $\ell_A(A/(X) + I) = 6\beta\gamma + 6\beta\gamma' + 6\beta'\gamma'$. On the other hand we have by the additive formula [18, p. 126] of multiplicity that

$$\ell_A(A/(X) + \mathfrak{p}^{(3)}) = e_{XA}(A/\mathfrak{p}^{(3)})$$
$$= \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}^3 A_\mathfrak{p}) \cdot e_{XA}(A/\mathfrak{p})$$
$$= 6 \cdot \ell_A(A/(X) + \mathfrak{p})$$
$$= 6\beta\gamma + 6\beta\gamma' + 6\beta'\gamma'.$$

Thus we get $\ell_A(A/(X) + I) = \ell_A(A/(X) + \mathfrak{p}^{(3)})$ and consequently $(X) + I = (X) + \mathfrak{p}^{(3)}$. Hence $\mathfrak{p}^{(3)} = I + (X) \cap \mathfrak{p}^{(3)}$, so we have $\mathfrak{p}^{(3)} = I$ by Nakayama's lemma. Similarly we get $\mathfrak{p}^{(3)} = I$ for the case $2\beta \geq \beta'$, too.

Secondly, assume $2\gamma < \gamma'$. Then by (5.3) we have

$$
\begin{aligned}
X^\alpha(bd_2 + X^{\alpha'-\alpha}Z^{\gamma'-2\gamma}ac^2) &= Y^\beta(-Z^{\gamma'-2\gamma}a^2c - Y^{\beta'-2\beta}b^3) && \text{if } 2\beta < \beta', \\
&= Y^{\beta'-\beta}(-Y^{2\beta-\beta'}Z^{\gamma'-2\gamma}a^2c - b^3) && \text{if } 2\beta \geq \beta'.
\end{aligned}
$$

Hence there exists $d_3 \in \mathfrak{p}^{(3)}$ satisfying

$$
\begin{aligned}
bd_2 + X^{\alpha'-\alpha}Z^{\gamma'-2\gamma}ac^2 &= Y^\beta d_3 && \text{if } 2\beta < \beta', \\
&= Y^{\beta'-\beta}d_3 && \text{if } 2\beta \geq \beta'
\end{aligned}
$$

so that

$$
\begin{aligned}
d_3 &\equiv -X^{2\alpha+3\alpha'}Y^{\beta'-2\beta} \bmod (Z) && \text{if } 2\beta < \beta', \\
&\equiv -X^{2\alpha+3\alpha'} \bmod (Z) && \text{if } 2\beta \geq \beta'.
\end{aligned}
$$

Let $I = (d_3) + \mathfrak{p}\mathfrak{p}^{(2)}$. Then similarly as in the proof for the case where $2\alpha < \alpha'$, we get

$$
\begin{aligned}
\ell_A(A/(Z) + I) &= 6\alpha'\beta + 6\alpha'\beta' + 6\alpha\beta' \\
&= \ell_A(A/(Z) + \mathfrak{p}^{(3)}),
\end{aligned}
$$

whence $I = \mathfrak{p}^{(3)}$ by Nakayama's lemma.

(2) Since $2\alpha \geq \alpha'$, we have by (5.1) that

$$
\begin{aligned}
X^{\alpha'-\alpha}(X^{2\alpha-\alpha'}bd_2 - Z^{2\gamma'-2\gamma}c^3) &= Y^\beta(Z^{\gamma'-\gamma}cd_2 - Y^{\beta'-2\beta}b^3) && \text{if } 2\beta < \beta', \\
&= Y^{\beta'-\beta}(Y^{2\beta-\beta'}Z^{\gamma'-\gamma}cd_2 - b^3) && \text{if } 2\beta \geq \beta'.
\end{aligned}
$$

On the other hand since $2\gamma \geq \gamma'$, we get by (5.2) that

$$
\begin{aligned}
X^\alpha(Z^{2\gamma-\gamma'}bd_2 + X^{\alpha'-\alpha}ac^2) &= Y^\beta(-a^2c - Y^{\beta'-2\beta}Z^{2\gamma-\gamma'}b^3) && \text{if } 2\beta < \beta'. \\
&= Y^{\beta'-\beta}(-Y^{2\beta-\beta'}a^2c - Z^{2\gamma-\gamma'}b^3) && \text{if } 2\beta \geq \beta'.
\end{aligned}
$$

Hence there exist $d_3 \in \mathfrak{p}^{(3)}$ and $d_3' \in \mathfrak{p}^{(3)}$ such that

$$
\begin{aligned}
X^{\alpha'-\alpha}d_3 &= Z^{\gamma'-\gamma}cd_2 - Y^{\beta'-2\beta}b^3 && \text{if } 2\beta < \beta', \\
&= Y^{2\beta-\beta'}Z^{\gamma'-\gamma}cd_2 - b^3 && \text{if } 2\beta \geq \beta', \\
X^{2\alpha-\alpha'}bd_2 - Z^{2\gamma'-2\gamma}c^3 &= Y^\beta d_3 && \text{if } 2\beta < \beta', \\
&= Y^{\beta'-\beta}d_3 && \text{if } 2\beta \geq \beta',
\end{aligned}
$$

and

$$X^\alpha d_3' = -a^2 c - Y^{\beta' - 2\beta} Z^{2\gamma - \gamma'} b^3 \qquad \text{if } 2\beta < \beta',$$
$$= -Y^{2\beta - \beta'} a^2 c - Z^{2\gamma - \gamma'} b^3 \qquad \text{if } 2\beta \geq \beta',$$
$$Z^{2\gamma - \gamma'} b d_2 + X^{\alpha' - \alpha} a c^2 = Y^\beta d_3' \qquad \text{if } 2\beta < \beta',$$
$$= Y^{\beta' - \beta} d_3' \qquad \text{if } 2\beta \geq \beta'.$$

Then putting $I = (d_3, d_3') + \mathfrak{p}\mathfrak{p}^{(2)}$, we will get, similarly as in the proof of the assertion (1), that $I = \mathfrak{p}^{(3)}$ for each case in the statement (2). The detail shall be left to the readers.

COROLLARY (5.6). *Suppose that M has type* (I). *Then*

$$\mu_A(\mathfrak{p}^{(3)}/\mathfrak{p}\mathfrak{p}^{(2)}) \leq 1 \qquad \text{if } 2\alpha \leq \alpha' \text{ or } 2\gamma \leq \gamma',$$
$$\leq 2 \qquad \text{if } 2\alpha > \alpha' \text{ and } 2\gamma > \gamma'.$$

Let us now assume that M has type (II). Then since $Y^\beta a + Z^\gamma b + X^{\alpha'} c = 0$, we get by the equations given in (2.3) that

$$c(Z^{\gamma' - \gamma} a c - Y^{\beta' - \beta} b^2) = X^{\alpha'} c d_2 = d_2(-Y^\beta a - Z^\gamma b),$$
$$a(X^{\alpha - \alpha'} a b - Z^{\gamma' - \gamma} c^2) = Y^\beta a d_2 = d_2(-Z^\gamma b - X^{\alpha'} c) \quad \text{and}$$
$$b(Y^{\beta' - \beta} b c - X^{\alpha - \alpha'} a^2) = Z^\gamma b d_2 = d_2(-Y^\beta a - X^{\alpha'} c).$$

Hence we have

(5.7) $$Y^{\beta' - \beta} b^2 c - Y^\beta a d_2 = Z^{\gamma' - \gamma} a c^2 + Z^\gamma b d_2,$$

(5.8) $$Z^{\gamma' - \gamma} a c^2 - Z^\gamma b d_2 = X^{\alpha - \alpha'} a^2 b + X^{\alpha'} c d_2 \quad \text{and}$$

(5.9) $$X^{\alpha - \alpha'} a^2 b - X^{\alpha'} c d_2 = Y^{\beta' - \beta} b^2 c + Y^\beta a d_2.$$

We furthermore assume that

(♯) $$\alpha \neq 2\alpha', \quad 2\beta \neq \beta' \quad \text{or} \quad 2\gamma \neq \gamma'$$

(cf. (2.1), (3)). Then after suitable permutations of columns, M is assumed to satisfy one of the followings:

(i) $2\alpha' < \alpha,\ 2\beta < \beta'$ and $2\gamma < \gamma'$;

(ii) $2\alpha' < \alpha,\ 2\beta < \beta'$ and $2\gamma \geq \gamma'$;

(iii) $2\alpha' < \alpha,\ 2\beta \geq \beta'$ and $2\gamma \geq \gamma'$;

(iv) $2\alpha' > \alpha,\ 2\beta \geq \beta'$ and $2\gamma \geq \gamma'$.

THEOREM (5.10). *Suppose that M has type* (II) *and satisfies one of the above conditions* (i), (ii), (iii) *and* (iv). *Then we have the following assertions.*

(1) *If $\frac{1}{2} \in A$, then there exist d_3, d_3' and d_3'' in $\mathfrak{p}^{(3)}$ such that*

$$d_3 \equiv 2Y^{\beta+2\beta'}Z^{2\gamma'-\gamma} \bmod (X), \quad d_3 \equiv -Y^{2\beta+3\beta'}Z^{\gamma'-2\gamma} \bmod (X) \quad and$$

$$d_3'' \equiv Y^{\beta+3\beta'}Z^{\gamma'-\gamma} \bmod (X), \qquad if \ M \ is \ of \ case \ (i);$$

$$d_3 \equiv 2Y^{\beta+2\beta'}Z^{\gamma+\gamma'} \bmod (X), \quad d_3' \equiv -Y^{2\beta+3\beta'} \bmod (X) \quad and$$

$$d_3'' \equiv Y^{\beta+3\beta'}Z^{\gamma'-\gamma} \bmod (X), \qquad if \ M \ is \ of \ case \ (ii);$$

$$d_3 \equiv 2Y^{3\beta+\beta'}Z^{\gamma+\gamma'} \bmod (X), \quad d_3' \equiv -Y^{2\beta+3\beta'} \bmod (X) \quad and$$

$$d_3'' \equiv Y^{3\beta+2\beta'}Z^{\gamma'-\gamma} \bmod (X), \qquad if \ M \ is \ of \ case \ (iii);$$

$$d_3 \equiv 2Y^{3\beta+\beta'}Z^{\gamma+\gamma'} \bmod (X), \quad d_3' \equiv -Y^{\beta}Z^{3\gamma+2\gamma'} \bmod (X) \quad and$$

$$d_3'' \equiv -Y^{2\beta-\beta'}Z^{2\gamma+3\gamma'} \bmod (X), \qquad if \ M \ is \ of \ case \ (iv).$$

We furthermore have $\mathfrak{p}^{(3)} = (d_3, d_3', d_3'') + \mathfrak{p}\mathfrak{p}^{(2)}$.

(2) *If* $\mathrm{ch}\, A = 2$, *then there exist* $e_3 \in \mathfrak{p}^{(3)}$ *such that* $\mathfrak{p}^{(3)} = (e_3) + \mathfrak{p}\mathfrak{p}^{(2)}$ *and*

$$e_3 \equiv Y^{\beta+3\beta'}Z^{\gamma'-2\gamma} \bmod (X) \qquad if \ M \ is \ of \ case \ (i);$$

$$e_3 \equiv Y^{\beta+3\beta'} \bmod (X) \qquad if \ M \ is \ of \ case \ (ii);$$

$$e_3 \equiv -Y^{3\beta+2\beta'} \bmod (X) \qquad if \ M \ is \ of \ case \ (iii);$$

$$e_3 \equiv -Y^{2\beta-\beta'}Z^{3\gamma+2\gamma'} \bmod (X) \quad if \ M \ is \ of \ case \ (iv).$$

Proof. (1) Let us assume the case (i). Then by (5.7), (5.8) and (5.9) we have

$$Y^{\beta}(Y^{\beta'-2\beta}b^2c - ad_2) = Z^{\gamma}(z^{\gamma'-2\gamma}ac^2 + bd_2),$$

$$Z^{\gamma}(Z^{\gamma'-2\gamma}ac^2 - bd_2) = X^{\alpha'}(X^{\alpha-2\alpha'}a^2b + cd_2) \quad and$$

$$X^{\alpha'}(X^{\alpha-2\alpha'}a^2b - cd_2) = Y^{\beta}(Y^{\beta'-2\beta}b^2c + ad_2).$$

Hence there exist d_3, d_3' and d_3'' in $\mathfrak{p}^{(3)}$ such that

$$Y^{\beta}d_3 = Z^{\gamma'-2\gamma}ac^2 + bd_2,$$

$$Z^{\gamma}d_3' = X^{\alpha-2\alpha'}a^2b + cd_2 \quad and$$

$$Y^{\beta}d_3'' = X^{\alpha-2\alpha'}a^2b - cd_2.$$

Thus

$$d_3 \equiv 2Y^{\beta+2\beta'}Z^{2\gamma'-\gamma} \bmod (X),$$

$$d_3' \equiv -Y^{2\beta+3\beta'}Z^{\gamma'-2\gamma} \bmod (X) \quad and$$

$$d_3'' \equiv Y^{\beta+3\beta'}Z^{\gamma'-\gamma} \bmod (X).$$

Let $I = (d_3, d_3', d_3'') + \mathfrak{p}\mathfrak{p}^{(2)}$. Then we have

$$(X) + I = (X) + (Z^{3\gamma+3\gamma'}, \ Y^{\beta}Z^{2\gamma+3\gamma'}, \ Y^{2\beta}Z^{\gamma+3\gamma'}, \ Y^{3\beta}Z^{3\gamma'},$$

$$Y^{\beta+\beta'}Z^{2\gamma+2\gamma'}, \ Y^{2\beta+\beta'}Z^{\gamma+2\gamma'}, \ Y^{3\beta+\beta'}Z^{2\gamma'}, \ Y^{\beta+2\beta'}Z^{2\gamma'-\gamma},$$

$$Y^{2\beta+2\beta'}Z^{\gamma+\gamma'}, \ Y^{3\beta+2\beta'}Z^{\gamma'}, \ Y^{\beta+3\beta'}Z^{\gamma'-\gamma}, \ Y^{2\beta+3\beta'}Z^{\gamma'-2\gamma},$$

$$Y^{3\beta+3\beta'}).$$

Hence $\ell_A(A/(X) + I) = 6(\beta\gamma + \beta\gamma' + \beta'\gamma') = \ell_A(A/(X) + \mathfrak{p}^{(3)})$, which implies $(X) + I = (X) + \mathfrak{p}^{(3)}$ so that $\mathfrak{p}^{(3)} = I$ by Nakayama's lemma. The same proof works for the other case, too. For instance, in the case (ii), we get by (5.7), (5.8) and (5.9) that

$$Y^\beta(Y^{\beta'-2\beta}b^2c - ad_2) = Z^{\gamma'-\gamma}(ac^2 + Z^{2\gamma-\gamma'}bd_2),$$
$$Z^{\gamma'-\gamma}(ac^2 - Z^{2\gamma-\gamma'}bd_2) = X^{\alpha'}(X^{\alpha-2\alpha'}a^2b + cd_2) \quad \text{and}$$
$$X^{\alpha'}(X^{\alpha-2\alpha'}a^2b - cd_2) = Y^\beta(Y^{\beta'-2\beta}b^2c + ad_2).$$

Hence there exist d_3, d_3' and d_3'' in $\mathfrak{p}^{(3)}$ satisfying

$$Y^\beta d_3 = ac^2 + Z^{2\gamma-\gamma'}bd_2,$$
$$Z^{\gamma'-\gamma}d_3' = X^{\alpha-2\alpha'}a^2b + cd_2 \quad \text{and}$$
$$Y^\beta d_3'' = X^{\alpha-2\alpha'}a^2b - cd_2$$

so that

$$d_3 \equiv 2Y^{\beta+2\beta'}Z^{\gamma+\gamma'} \bmod (X),$$
$$d_3' \equiv -Y^{2\beta+3\beta'} \bmod (X) \quad \text{and}$$
$$d_3'' \equiv Y^{\beta+3\beta'}Z^{\gamma'-\gamma} \bmod (X).$$

And therefore we can prove $\mathfrak{p}^{(3)} = (d_3, d_3', d_3'') + \mathfrak{p}\mathfrak{p}^{(2)}$ by showing that $\ell_A(A/(X) + \mathfrak{p}^{(3)}) = \ell_A(A/(X, d_3, d_3', d_3'') + \mathfrak{p}\mathfrak{p}^{(2)})$ similarly as above. We would like to leave to the readers the rest of Proof of Theorem (5.10), (1).

(2) Notice that

(5.11) $$X^\alpha a + Y^{\beta'}b + Z^{\gamma'}c = 0 \quad \text{and}$$

(5.12) $$Y^\beta a + Z^\gamma b + X^{\alpha'}c = 0.$$

Then since ch $A = 2$, by (5.12) we get

(5.13) $$Y^{2\beta}a^2 + Z^{2\gamma}b^2 + X^{2\alpha'}c^2 = 0.$$

Suppose the case (i). Then by (5.11) and (5.13) we have

$$(Y^{2\beta}a^2 + X^{2\alpha'}c^2)Z^{\gamma'-2\gamma}c = -Z^{\gamma'}b^2c$$
$$= (X^\alpha a + Y^{\beta'}b)b^2,$$

whence

$$X^{2\alpha'}Z^{\gamma'-2\gamma}c^3 - X^\alpha ab^2 = Y^{\beta'}b^3 - Y^{2\beta}Z^{\gamma'-2\gamma}a^2c.$$

Since $2\alpha' < \alpha$ and $2\beta < \beta'$ by our assumption, we get

$$X^{2\alpha'}(Z^{\gamma'-2\gamma}c^3 - X^{\alpha-2\alpha'}ab^2) = Y^{2\beta}(Y^{\beta'-2\beta}b^3 - Z^{\gamma'-2\gamma}a^2c),$$

from which we find $e_3 \in \mathfrak{p}^{(3)}$ satisfying

$$Y^{2\beta}e_3 = Z^{\gamma'-2\gamma}c^3 - X^{\alpha-2\alpha'}ab^2 .$$

Notice that $e_3 \equiv Y^{\beta+3\beta'}Z^{\gamma'-2\gamma} \bmod (X)$. Then because

$$\ell_A(A/(X, e_3) + \mathfrak{p}\mathfrak{p}^{(2)}) = 6(\beta\gamma + \beta\gamma' + \beta'\gamma')$$
$$= \ell_A(A/(X) + \mathfrak{p}^{(3)}) ,$$

we see $(X, e_3) + \mathfrak{p}\mathfrak{p}^{(2)} = (X) + \mathfrak{p}^{(3)}$ so that $(e_3) + \mathfrak{p}\mathfrak{p}^{(2)} = \mathfrak{p}^{(3)}$. A similar proof works for the other cases (ii), (iii) and (iv), whose detail shall be left to the readers.

§ 6. The generation of $R_s(\mathfrak{p})$ in degree three

The purpose of this section is to characterize the prime ideals $\mathfrak{p}$ whose symbolic Rees algebras $R_s(\mathfrak{p})$ are generated in degree three. Let us maintain the same assumption as in Section 5.

First we shall discuss the case where M has type (I) and the results of this case are summarized into the next

THEOREM (6.1). *Suppose that M has type* (I). *Then the following assertions are equivalent.*
  (1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.
  (2) *M satisfies one of the following conditions.*
   (i) $2\alpha \leq \alpha'$ *and* $2\beta = \beta'$.
   (ii) $2\beta = \beta'$ *and* $2\gamma \leq \gamma'$.
   (iii) $2\alpha = \alpha'$ *and* $\gamma = \gamma'$.
   (iv) $2\alpha > \alpha'$, $2\beta \geq \beta'$ *and* $\gamma = \gamma'$.
   (v) $\beta = \beta'$.
   (vi) $\alpha = \alpha'$ *and* $\gamma = \gamma'$.
When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

We divide the proof of (6.1) according to the cases stated in (5.4). The proof will be finished by the end of Proposition (6.11).

PROPOSITION (6.2). *Suppose that M is of type* (I) *and satisfies $2\alpha < \alpha'$ and $2\beta < \beta'$. Then $R_s(\mathfrak{p}) \neq A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.*

Proof. It is enough to show that $\mathfrak{p}^{(4)} \neq [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$. Assume the contrary and we have $\ell_A(A/(X) + \mathfrak{p}^{(4)}) = \ell_A(A/(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)})$. On the other hand by (2.2) and (5.4) we get

$$(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} = (X) + (Z^{2\gamma+4\gamma'},\ Y^\beta Z^{\gamma+4\gamma'},\ Y^{4\beta} Z^{4\gamma'},\ Y^{\beta+\beta'} Z^{\gamma+3\gamma'},$$
$$Y^{4\beta+\beta'} Z^{3\gamma'},\ Y^{2\beta+2\beta'} Z^{\gamma+2\gamma'},\ Y^{4\beta+2\beta'} Z^{2\gamma'},\ Y^{3\beta+3\beta'} Z^{\gamma+\gamma'},$$
$$Y^{4\beta+3\beta'} Z^{\gamma'},\ Y^{4\beta+4\beta'}).$$

Hence if $3\beta > \beta'$, we have

$$(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} = (X) + (Z^{2\gamma+4\gamma'},\ Y^\beta Z^{\gamma+4\gamma'},\ Y^{\beta+\beta'} Z^{\gamma+3\gamma'},\ Y^{4\beta+\beta'} Z^{3\gamma'},$$
$$Y^{2\beta+2\beta'} Z^{\gamma+2\gamma'},\ Y^{4\beta+2\beta'} Z^{2\gamma'},\ Y^{3\beta+3\beta'} Z^{\gamma+\gamma'},\ Y^{4\beta+3\beta'} Z^{\gamma'},$$
$$Y^{4\beta+4\beta'}).$$

because $Y^{4\beta} Z^{4\gamma'} \in (Y^{\beta+\beta'} Z^{\gamma+3\gamma'})$. Therefore, counting the number of monomials of Y and Z that are not in $(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$, we find

$$\ell_A(A/(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}) = 11\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' \qquad \text{if } 3\beta \le \beta',$$
$$= 8\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' + \beta'\gamma \quad \text{if } 3\beta > \beta',$$

while

$$\ell_A(A/(X) + \mathfrak{p}^{(4)}) = \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}^4 A_\mathfrak{p}) \cdot \ell_A(A/(X) + \mathfrak{p})$$
$$= 10(\beta\gamma + \beta\gamma' + \beta'\gamma')$$

by the additive formula [18, p. 126] of multiplicity. Thus we have

$$10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 11\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' \qquad \text{if } 3\beta \le \beta',$$
$$= 8\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' + \beta'\gamma \quad \text{if } 3\beta > \beta',$$

from which we conclude that $\beta\gamma = 0$ or $2\beta = \beta'$. This contradicts our assumption that $2\beta < \beta'$. Hence $\mathfrak{p}^{(4)} \ne [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$.

PROPOSITION (6.3). *Suppose that M is of type* (I) *and satisfies $2\alpha < \alpha'$ and $2\beta \ge \beta'$. Then $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ and the following conditions are equivalent.*

(1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t,\ \mathfrak{p}^{(2)}t^2,\ \mathfrak{p}^{(3)}t^3]$.

(2) $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)}$.

(3) $2\beta = \beta'$ *or* $\beta = \beta'$.

When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. By (2.2) and (5.4) we have

$$(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} = (X) + (Z^{2\gamma+4\gamma'},\ Y^{3\beta-\beta'} Z^{\gamma+4\gamma'},\ Y^{\beta+\beta'} Z^{2\gamma+3\gamma'},\ Y^{3\beta} Z^{\gamma+3\gamma'},$$
$$Y^{2\beta+2\beta'} Z^{\gamma+2\gamma'},\ Y^{4\beta+2\beta'} Z^{2\gamma'},\ Y^{3\beta+3\beta'} Z^{\gamma+\gamma'},\ Y^{4\beta+3\beta'} Z^{\gamma'},$$
$$Y^{4\beta+4\beta'}).$$

whence

$$\ell_A(A/(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}) = 10\beta\gamma + 10\beta\gamma' + 10\beta'\gamma'.$$

Because $\ell_A(A/(X) + \mathfrak{p}^{(4)}) = 10\beta\gamma + 10\beta\gamma' + 10\beta'\gamma'$ (cf. Proof of (6.2)), we get $(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} = (X) + \mathfrak{p}^{(4)}$ so that $[\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} = \mathfrak{p}^{(4)}$ as required.

$(1) \Rightarrow (2)$ This is obvious.

$(2) \Rightarrow (3)$ Since by (2.2) and (5.4) we explicitly have a system of generators for the ideal $(X) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)}$, we get

$$\begin{aligned}
\ell_A(A/(X) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)}) &= 17\beta\gamma + 15\beta\gamma' + 15\beta'\gamma' - \beta'\gamma &&\text{if } 3\beta < 2\beta', \\
&= 14\beta\gamma + 15\beta\gamma' + 15\beta'\gamma' + \beta'\gamma &&\text{if } 3\beta \geq 2\beta'.
\end{aligned}$$

As $\ell_A(A/(X) + \mathfrak{p}^{(5)}) = \ell_A(A/(X) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)})$ by our assumption and as

$$\begin{aligned}
\ell_A(A/(X) + \mathfrak{p}^{(5)}) &= \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}^5 A_\mathfrak{p}) \cdot \ell_A(A/(X) + \mathfrak{p}) \\
&= 15(\beta\gamma + \beta\gamma' + \beta'\gamma'),
\end{aligned}$$

we have

$$\begin{aligned}
15(\beta\gamma + \beta\gamma' + \beta'\gamma') &= 17\beta\gamma + 15\beta\gamma' + 15\beta'\gamma' - \beta'\gamma &&\text{if } 3\beta < 2\beta', \\
&= 14\beta\gamma + 15\beta\gamma' + 15\beta'\gamma' + \beta'\gamma &&\text{if } 3\beta \geq 2\beta'.
\end{aligned}$$

Hence $2\beta = \beta'$ or $\beta = \beta'$ by these equations.

$(3) \Rightarrow (1)$ By (4.3) we have only to consider the case where $2\beta = \beta'$. Let d_3 be the element in (1) of (5.4). Then by [11, Theorem 3.1] $R_s(\mathfrak{p})$ is a Noetherian ring, because

$$\begin{aligned}
\ell_A(A/(X, c, d_3)) &= \ell_A(A/(X, Y^{3\beta}, Z^{\gamma + 3\gamma'})) \\
&= 3(\beta\gamma + 3\beta\gamma') \\
&= 1 \cdot 3 \cdot \ell_A(A/(X) + \mathfrak{p}).
\end{aligned}$$

We want to check that $R_s(\mathfrak{p})$ is a Gorenstein ring. Let $I = (c) + \mathfrak{p}^{(2)}$. Then since

$$(X) + I = (X) + (Z^{\gamma + 2\gamma'}, Y^{2\beta}Z^{2\gamma'}, Y^{3\beta}),$$

we have $\ell_A(A/(X) + I) = 2\beta\gamma + 6\beta\gamma'$, while

$$\begin{aligned}
e_{XA}(A/I) &= \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/cA_\mathfrak{p} + \mathfrak{p}^2 A_\mathfrak{p}) \cdot \ell_A(A/(X) + \mathfrak{p}) \\
&= 2(\beta\gamma + 3\beta\gamma').
\end{aligned}$$

Consequently we get $\ell_A(A/(X) + I) = e_{XA}(A/I)$, whence the ring A/I is Cohen-Macaulay. Thus $R_s(\mathfrak{p})$ is a Gorenstein ring (cf. [5, Theorem 1.1]), which completes the proof of (6.3).

PROPOSITION (6.4). *Suppose that M is of type* (I) *and satisfies* $2\beta < \beta'$ *and* $2\gamma < \gamma'$. *Then* $R_s(\mathfrak{p}) \neq A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.

Proof. We will show that $\mathfrak{p}^{(4)} \neq [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$. Assume the contrary and we have $\ell_A(A/(Z) + \mathfrak{p}^{(4)}) = \ell_A(A/(Z) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)})$. Notice that

$$\begin{aligned} \ell_A(A/(Z) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}) &= 10\alpha'\beta + 10\alpha'\beta' + 11\alpha\beta' - 2\alpha\beta && \text{if } \beta' < 3\beta, \\ &= 10\alpha'\beta + 10\alpha\beta' + 10\alpha'\beta' - \alpha\beta && \text{if } \beta' \geq 3\beta \end{aligned}$$

by (2.2) and (5.4). Then since $\ell_A(A/(Z) + \mathfrak{p}^{(4)}) = 10\alpha'\beta + 10\alpha'\beta' + 10\alpha\beta'$, we get

$$\begin{aligned} 10\alpha'\beta + 10\alpha'\beta' + 10\alpha\beta' &= 10\alpha'\beta + 10\alpha'\beta' + 11\alpha\beta' - 2\alpha\beta && \text{if } \beta' < 3\beta, \\ &= 10\alpha'\beta + 10\alpha\beta' + 10\alpha'\beta' - \alpha\beta && \text{if } \beta' \geq 3\beta. \end{aligned}$$

Hence $2\beta = \beta'$, which contradicts our assumption that $2\beta < \beta'$.

The proof of the next proposition is quite similar as that of (6.3) so that we would like to leave it to the readers.

PROPOSITION (6.5). *Suppose that M is of type* (I) *and satisfies* $2\beta \geq \beta'$ *and* $2\gamma < \gamma'$. *Then* $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ *and the following conditions are equivalent.*

    (1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.

    (2) $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)}$.

    (3) $2\beta = \beta'$ *or* $\beta = \beta'$.

When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

PROPOSITION (6.6). *Suppose that M is of type* (I) *and satisfies* $2\alpha \geq \alpha'$, $2\beta < \beta'$ *and* $2\gamma \geq \gamma'$. *Then the following conditions are equivalent.*

    (1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.

    (2) $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ *and* $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$.

    (3) *M satisfies one of the following conditions*:

       (i) $\alpha = \alpha'$ *and* $\gamma = \gamma'$.

      (ii) $2\alpha = \alpha'$ *and* $\gamma = \gamma'$.

When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Since $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$, we get by (2.2) and (5.4) that

$$\begin{aligned} (\sharp) \qquad (Y) + \mathfrak{p}^{(4)} = (Y) + (&Z^{2\gamma+4\gamma'}, \; X^\alpha Z^{4\gamma+2\gamma'}, \; X^{3\alpha-\alpha'}Z^{2\gamma+3\gamma'}, \; X^{2\alpha}Z^{3\gamma+2\gamma'}, \\ &X^{3\alpha}Z^{4\gamma+\gamma'}, \; X^{3\alpha}Z^{\gamma+3\gamma'}, \; X^{2\alpha+\alpha'}Z^{2\gamma+2\gamma'}, \; X^{4\alpha}Z^{4\gamma}, \; X^{2\alpha+2\alpha'}Z^{\gamma+2\gamma'}, \\ &X^{4\alpha+\alpha'}Z^{3\gamma}, \; X^{4\alpha+2\alpha'}Z^{2\gamma}, \; X^{4\alpha+3\alpha'}Z^\gamma, \; X^{4\alpha+4\alpha'}). \end{aligned}$$

where we can delete

$$X^{3\alpha}Z^{\gamma+3\gamma'} \quad \text{and} \quad X^{2\alpha+2\alpha'}Z^{\gamma+2\gamma'} \qquad \text{if } 3\gamma < 2\gamma',$$
$$X^{3\alpha}Z^{4\gamma+\gamma'} \qquad\qquad\qquad\qquad \text{if } 3\gamma \geq 2\gamma',$$

out of the system of generators for the ideal $(Y) + \mathfrak{p}^{(4)}$. Hence

$$\ell_A(A/(Y) + \mathfrak{p}^{(4)}) = 11\alpha\gamma + 10\alpha\gamma' + 10\alpha'\gamma \qquad \text{if } 3\gamma < 2\gamma',$$
$$= 8\alpha\gamma + 12\alpha\gamma' + 10\alpha'\gamma \qquad \text{if } 3\gamma \geq 2\gamma'.$$

On the other hand, as $\ell_A(A/(Y) + \mathfrak{p}^{(4)}) = e_{YA}(A/\mathfrak{p}^{(4)}) = 10(\alpha\gamma + \alpha\gamma' + \alpha'\gamma)$, we get

$$10(\alpha\gamma + \alpha\gamma' + \alpha'\gamma) = 11\alpha\gamma + 10\alpha\gamma' + 10\alpha'\gamma \qquad \text{if } 3\gamma < 2\gamma',$$
$$= 8\alpha\gamma + 12\alpha\gamma' + 10\alpha'\gamma \qquad \text{if } 3\gamma \geq 2\gamma'.$$

Hence $\gamma = \gamma'$. Thus by (2.2), (5.4) and ($\sharp$) we have

$$\begin{aligned}
(Y) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)} = (Y) + (&Z^{8\gamma},\ X^{2\alpha-\alpha'}Z^{7\gamma},\ X^{4\alpha-\alpha'}Z^{6\gamma},\ X^{\alpha+\alpha'}Z^{6\gamma}, \\
&X^{4\alpha}Z^{5\gamma},\ X^{4\alpha+\alpha'}Z^{4\gamma},\ X^{3\alpha+3\alpha'}Z^{3\gamma},\ X^{5\alpha+3\alpha'}Z^{2\gamma}, \\
&X^{5\alpha+4\alpha'}Z^{\gamma},\ X^{5\alpha+5\alpha'}).
\end{aligned}$$

where we can delete

$$X^{4\alpha-\alpha'}Z^{6\gamma} \qquad \text{if } 3\alpha \geq 2\alpha',$$
$$X^{\alpha+\alpha'}Z^{6\gamma} \qquad \text{if } 3\alpha < 2\alpha',$$

out of the system of generators for the ideal $(Y) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$. Therefore we find

$$\ell_A(A/(Y) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}) = 29\alpha\gamma + 16\alpha'\gamma \qquad \text{if } 3\alpha \geq 2\alpha',$$
$$= 32\alpha\gamma + 14\alpha'\gamma \qquad \text{if } 3\alpha < 2\alpha'.$$

Since $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$ by our assumption and since $\ell_A(A/(Y) + \mathfrak{p}^{(5)}) = 30\alpha\gamma + 15\alpha'\gamma$, we see

$$30\alpha\gamma + 15\alpha'\gamma = 29\alpha\gamma + 16\alpha'\gamma \qquad \text{if } 3\alpha \geq 2\alpha',$$
$$= 32\alpha\gamma + 14\alpha'\gamma \qquad \text{if } 3\alpha < 2\alpha'.$$

The first (resp. second) equation implies $\alpha = \alpha'$ (resp. $2\alpha = \alpha'$).

(3) $\Rightarrow$ (1) We may assume the case (ii) (cf. (4.3)). Let d_3 be the element in (5.4), (2). Then since $\ell_A(A/(Y, d_3, b)) = 1 \cdot 3 \cdot \ell_A(A/(Y) + \mathfrak{p})$ and since $A/(b) + \mathfrak{p}^{(2)}$ is a Cohen-Macaulay ring, we have by [5, Theorem (1.1)] that the A-algebra $R_s(\mathfrak{p})$ is a Gorenstein ring generated in degree three.

PROPOSITION (6.7). *Suppose that M is of type* (I) *and satisfies $2\alpha >$*
$\alpha' > \alpha$, $2\beta \geq \beta'$ and $2\gamma \geq \gamma'$. Then the following conditions are equivalent.

(1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.

(2) $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ *and* $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$.

(3) *M satisfies one of the following conditions.*

  (i) $\beta = \beta'$.

  (ii) $\gamma = \gamma'$.

  (iii) $2\beta = \beta'$ *and* $2\gamma = \gamma'$.

When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. (1) $\Rightarrow$ (2)   This is obvious.

(2) $\Rightarrow$ (3)   Since $\ell_A(A/(X) + \mathfrak{p}^{(4)}) = \ell_A(A/(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)})$, we get

$$\ell_A(A/(X) + \mathfrak{p}^{(4)}) = 11\beta\gamma + 9\beta\gamma' + 11\beta'\gamma' - \beta'\gamma \quad \text{if } 3\gamma \geq 2\gamma',$$
$$= 8\beta\gamma + 11\beta\gamma' + 9\beta'\gamma' + 2\beta'\gamma \quad \text{if } 3\gamma < 2\gamma',$$

while $\ell_A(A/(X) + \mathfrak{p}^{(4)}) = e_{XA}(A/\mathfrak{p}^{(4)}) = 10(\beta\gamma + \beta\gamma' + \beta'\gamma')$. Hence we have

$$10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 11\beta\gamma + 9\beta\gamma' + 11\beta'\gamma' - \beta'\gamma \quad \text{if } 3\gamma \geq 2\gamma',$$
$$= 8\beta\gamma + 11\beta\gamma' + 9\beta'\gamma' + 2\beta'\gamma \quad \text{if } 3\gamma < 2\gamma',$$

where the first (resp. second) equation implies that $(\beta' - \beta)(\gamma' - \gamma) = 0$
(resp. $(\beta' - \beta)(2\gamma - \gamma') = 0$) so that $\beta = \beta'$ or $\gamma = \gamma'$ (resp. $\beta = \beta'$ or $2\gamma = \gamma'$).

If $2\gamma = \gamma'$, we have

$$\ell_A(A/(X) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}) = 47\beta\gamma + 29\beta'\gamma \quad \text{if } 3\beta < 2\beta',$$
$$= 44\beta\gamma + 31\beta'\gamma \quad \text{if } 3\beta \geq 2\beta'.$$

Therefore since $\ell_A(A/(X) + \mathfrak{p}^{(5)}) = \ell_A(A/(X) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)})$ by our assumption and since $\ell_A(A/(X) + \mathfrak{p}^{(5)}) = e_{XA}(A/\mathfrak{p}^{(5)}) = 15(3\beta\gamma + 2\beta'\gamma)$, we get

$$15(3\beta\gamma + 2\beta'\gamma) = 47\beta\gamma + 29\beta'\gamma \quad \text{if } 3\beta < 2\beta',$$
$$= 4\beta\gamma + 31\beta'\gamma \quad \text{if } 3\beta \geq 2\beta'.$$

The first (resp. second) equation implies $2\beta = \beta'$ (resp. $\beta = \beta'$), whence the assertion (3).

(3) $\Rightarrow$ (1)   We may assume the cases (ii) and (iii) (cf. (4.3)). First assume the case (ii) and let d_2 and d_3 be the elements obtained by (2.2) and (2) of (5.4). Then we have

$$\ell_A(A/(d_2, d_3, X)) = \ell_A(A/(Z^{3\gamma}, Y^{4\beta + 2\beta'}, X))$$
$$= 3\gamma(4\beta + 2\beta')$$
$$= 2 \cdot 3 \cdot \ell_A(A/(X) + \mathfrak{p}).$$

Hence $R_s(\mathfrak{p})$ is Noetherian (cf. [11, Theorem (3.1)]). We put $I = (d_2) + \mathfrak{p}^{(3)}$. Then since

$$(X) + I = (X) + (Z^{3\gamma}, Y^{3\beta+\beta'}Z^{2\gamma}, Y^{3\beta+2\beta'}Z^{\gamma}, Y^{4\beta+2\beta'})$$

by (5.4), we get $\ell_A(A/(X) + I) = 10\beta\gamma + 5\beta'\gamma = e_{XA}(A/I)$ so that A/I is Cohen-Macaulay. Thus $R_s(\mathfrak{p})$ is a Gorenstein ring generated in degree three (cf. [5, Theorem (1.1)]).

Secondly we consider the case (iii). Let d_3' be the element in (2) of (5.4). Then $\ell_A(A/(d_3', c, X)) = 1 \cdot 3 \cdot \ell_A(A/(X) + \mathfrak{p})$ and $A/(c) + \mathfrak{p}^{(2)}$ is Cohen-Macaulay. Hence $R_s(\mathfrak{p})$ is a Gorenstein ring generated in degree three.

PROPOSITION (6.8). *Suppose that M is of type* (I) *and satisfies* $2\alpha \geq \alpha'$, $2\beta > \beta' > \beta$ *and* $2\gamma \geq \gamma'$. *Then the following conditions are equivalent.*
 (1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.
 (2) $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ *and* $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$.
 (3) $\gamma = \gamma'$.
When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. $(1) \Rightarrow (2)$ This is obvious.
 $(2) \Rightarrow (3)$ Similarly as in Proof of (6.7) we get

$$\ell_A(A/(Y) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}) = 8\alpha\gamma + 12\alpha\gamma' + 11\alpha'\gamma - \alpha'\gamma' \quad \text{if } 3\gamma \geq 2\gamma',$$
$$= 14\alpha\gamma + 8\alpha\gamma' + 8\alpha'\gamma + \alpha'\gamma' \quad \text{if } 3\gamma < 2\gamma'.$$

Since $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$, we have

$$10(\alpha\gamma + \alpha\gamma' + \alpha'\gamma) = 8\alpha\gamma + 12\alpha\gamma' + 11\alpha'\gamma - \alpha'\gamma' \quad \text{if } 3\gamma \geq 2\gamma',$$
$$= 14\alpha\gamma + 8\alpha\gamma' + 8\alpha'\gamma + \alpha'\gamma' \quad \text{if } 3\gamma < 2\gamma'.$$

The first (resp. second) equation implies $(2\alpha - \alpha')(\gamma - \gamma') = 0$ (resp. $(2\alpha - \alpha')$ $(2\gamma - \gamma') = 0$) and so $2\alpha = \alpha'$ or $\gamma = \gamma'$ (resp. $2\alpha = \alpha'$ or $2\gamma = \gamma'$). We want to show that the equality $2\gamma = \gamma'$ is impossible and that $\gamma = \gamma'$, if $2\alpha = \alpha'$. First assume that $2\alpha = \alpha'$. Then we have

$$\ell_A(A/(Y) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}) = 43\alpha\gamma + 17\alpha\gamma' \quad \text{if } 3\gamma \geq 2\gamma',$$
$$= 46\alpha\gamma + 15\alpha\gamma' \quad \text{if } 3\gamma < 2\gamma'.$$

Hence by the assumption that $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$ we get

$$45\alpha\gamma + 15\alpha\gamma' = 43\alpha\gamma + 17\alpha\gamma' \quad \text{if } 3\gamma \geq 2\gamma',$$
$$= 46\alpha\gamma + 15\alpha\gamma' \quad \text{if } 3\gamma < 2\gamma',$$

whence $\gamma = \gamma'$. Now assume that $2\gamma = \gamma'$. Then

$$\ell_A(A/(Y) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}) = 46\alpha\gamma + 15\alpha'\gamma,$$

and so

$$45\alpha\gamma + 15\alpha'\gamma = 46\alpha\gamma + 15\alpha'\gamma,$$

which implies $\alpha\gamma = 0$. Thus $2\gamma \neq \gamma'$.

$(3) \Rightarrow (1)$ Let d_2 and d_3 be the elements obtained by (2.2) and (2) of (5.4). Then $\ell_A(A/(d_2, d_3, Y)) = 2 \cdot 3 \cdot \ell_A(A/(Y) + \mathfrak{p})$ and $A/(d_2) + \mathfrak{p}^{(3)}$ is Cohen-Macaulay. Therefore $R_s(\mathfrak{p})$ is a Gorenstein ring generated in degree three by [5, Theorem (1.1)].

PROPOSITION (6.9). *Suppose that M is of type* (I) *and satisfies $2\alpha \geq \alpha'$, $2\beta \geq \beta'$ and $2\gamma > \gamma' > \gamma$. Then the following conditions are equivalent.*
  (1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.
  (2) $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ *and* $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$.
  (3) *M satisfies one of the following conditions.*
    (i) $\beta = \beta'$.
    (ii) $2\alpha = \alpha'$ *and* $2\beta = \beta'$.
When this is the case, $R_s(\mathfrak{p})$ is a Gorenstein ring.

Proof. $(1) \Rightarrow (2)$ This is obvious.
  $(2) \Rightarrow (3)$ As $\ell_A(A/(Z) + \mathfrak{p}^{(4)}) = \ell_A(A/(Z) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)})$, we get

$$10(\alpha'\beta + \alpha'\beta' + \alpha\beta') = 11\alpha'\beta + 9\alpha'\beta' + 12\alpha\beta' - 2\alpha\beta,$$

whence $(2\alpha - \alpha')(\beta - \beta') = 0$. Thus $2\alpha = \alpha'$ or $\beta = \beta'$. Assume $2\alpha = \alpha'$. Then since $\mathfrak{p}^{(5)} = \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}$, we have

$$\begin{aligned}
30\alpha\beta + 45\alpha\beta' &= 29\alpha\beta + 46\alpha\beta' \quad &\text{if } 3\beta \geq 2\beta', \\
&= 32\alpha\beta + 44\alpha\beta' \quad &\text{if } 3\beta < 2\beta'.
\end{aligned}$$

The first (resp. second) equation induces $\beta = \beta'$ (resp. $2\beta = \beta'$) as required.
  $(3) \Rightarrow (1)$ We may assume the case (ii). Let d_3 be the element in (2) of (5.4). Then $\ell_A(A/(d_3, c, Z)) = 1 \cdot 3 \cdot \ell_A(A/(Z) + \mathfrak{p})$ and $A/(c) + \mathfrak{p}^{(2)}$ is Cohen-Macaulay. Hence $R_s(\mathfrak{p})$ is a Gorenstein ring generated in degree three (cf. [5, Theorem (1.1)]).

The next assertion is similarly proved as the implication $(3) \Rightarrow (1)$ in (6.9).

PROPOSITION (6.10). *Suppose that M is of type* (I) *and satisfies $2\alpha = \alpha'$ and $2\beta = \beta'$. Then $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$ and it is a Gorenstein ring.*

PROPOSITION (6.11). *Suppose that M is of type* (I) *and satisfies* $\alpha = \alpha'$, $2\beta = \beta'$ *and* $2\gamma = \gamma'$. *Then* $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$ *and it is a Gorenstein ring.*

Proof. Let d_3' be the element in (2) of (5.4). Then $\ell_A(A/(d_3', c, Y)) = 1 \cdot 3 \cdot \ell_A(A/(Y) + \mathfrak{p})$ and $A/(c) + \mathfrak{p}^{(2)}$ is Cohen-Macaulay. Hence $R_s(\mathfrak{p})$ is a Gorenstein ring generated in degree three.

For the rest of this section we assume that A contains a field k and that M has type (II). The purpose is to prove the following

THEOREM (6.12). *Suppose that M has type* (II) *and satisfies the condition*

(♯) $\alpha \neq 2\alpha'$, $2\beta \neq \beta'$ *or* $2\gamma \neq \gamma'$.

Then the following assertions are equivalent.

(1) $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$.

(2) *The characteristic of k is equal to* 2 *and M satisfies one of the following conditions.*

   (i) $2\alpha' \leq \alpha$, $2\beta \geq \beta'$ *and* $2\gamma = \gamma'$.

   (ii) $2\alpha' \geq \alpha$, $2\beta = \beta'$ *and* $2\gamma \leq \gamma'$.

   (iii) $2\alpha' = \alpha$, $2\beta \leq \beta'$ *and* $2\gamma \geq \gamma'$.

When this is the case, $R_s(\mathfrak{p})$ is not a Cohen-Macaulay ring.

We divide the proof of Theorem (6.12) into the cases stated in (5.10). The proof will be completed by (6.16), summarizing the results of (6.13), (6.14), (6.15) and (6.16).

We begin with the following

PROPOSITION (6.13). *Suppose that M has type* (II) *and satisfies* $2\alpha' < \alpha$, $2\beta < \beta'$ *and* $2\gamma < \gamma'$. *Then* $[\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} \neq \mathfrak{p}^{(4)}$.

Proof. First assume that ch $k \neq 2$. Since by (2.3) and (5.10) we explicitly have a system of generators for the ideal $(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$, we get

$$\ell_A(A/(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}) = 16\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' \quad \text{if } 3\beta \leq \beta' \text{ and } 3\gamma \leq \gamma',$$
$$= 7\beta\gamma + 13\beta\gamma' + 10\beta'\gamma' \quad \text{if } 3\beta \leq \beta' \text{ and } 3\gamma > \gamma',$$
$$= 7\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' + 3\beta'\gamma$$
$$\text{if } 3\beta > \beta' \text{ and } 3\gamma \leq \gamma',$$
$$= 7\beta\gamma + 10\beta\gamma' + 11\beta'\gamma' \quad \text{if } 3\beta > \beta' \text{ and } 3\gamma > \gamma'.$$

Therefore if $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$, then since $\ell_A(A/(X) + \mathfrak{p}^{(4)}) = 10(\beta\gamma + \beta\gamma'$

$+ \beta'\gamma'$), we have by the above equalities that

$$\beta\gamma = 0 \qquad \text{if } 3\beta \leq \beta' \text{ and } 3\gamma \leq \gamma',$$
$$\gamma = \gamma' \qquad \text{if } 3\beta \leq \beta' \text{ and } 3\gamma > \gamma',$$
$$\beta = \beta' \qquad \text{if } 3\beta > \beta' \text{ and } 3\gamma \leq \gamma' \text{ and}$$
$$3\beta\gamma = \beta'\gamma' > 4\beta\gamma \qquad \text{if } 3\beta > \beta' \text{ and } 3\gamma > \gamma',$$

all of which are impossible. Hence $\mathfrak{p}^{(4)} \neq [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$.

Secondly assume that ch $k = 2$ and that $[\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} = \mathfrak{p}^{(4)}$. Then by (2.3) and (5.10) we get

$$10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 13\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' \qquad \text{if } 3\beta \leq \beta' \text{ and } 3\gamma \leq \gamma',$$
$$= 7\beta\gamma + 12\beta\gamma' + 10\beta'\gamma' \qquad \text{if } 3\beta \leq \beta' \text{ and } 3\gamma > \gamma',$$
$$= 7\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' + 2\beta'\gamma \qquad \text{if } 3\beta > \beta' \text{ and } 3\gamma \leq \gamma',$$
$$= 10\beta\gamma + 9\beta\gamma' + 11\beta'\gamma' - \beta'\gamma \qquad \text{if } 3\beta > \beta' \text{ and } 3\gamma > \gamma',$$

all of which contradict the standard assumption on β and γ. Hence $\mathfrak{p}^{(4)} \neq [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$.

PROPOSITION (6.14). *Suppose that M has type* (II) *and satisfies* $2\alpha' < \alpha$, $2\beta < \beta'$ *and* $2\gamma \geq \gamma'$. *Then* $[\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} \neq \mathfrak{p}^{(4)}$.

Proof. Assume that $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$. Then if ch $k \neq 2$, we have

$$10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 9\beta\gamma + 12\beta\gamma' + 10\beta'\gamma' \qquad \text{if } 3\beta \leq \beta',$$
$$= 15\beta\gamma + 6\beta\gamma' + 12\beta'\gamma' - 2\beta'\gamma \qquad \text{if } 3\beta > \beta',$$

which contradict the assumption on β and γ. Therefore the characteristic of k has to be 2. But when ch $k = 2$, we have the equalities

$$10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 11\beta\gamma + 10\beta\gamma' + 10\beta'\gamma' \qquad \text{if } 3\beta \leq \beta',$$
$$= 14\beta\gamma + 7\beta\gamma' + 11\beta'\gamma' - \beta'\gamma \qquad \text{if } 3\beta > \beta',$$

both of which again contradict the assumption on β and γ, too. Thus we have $\mathfrak{p}^{(4)} \neq [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$.

PROPOSITION (6.15). *Suppose that M has type* (II) *and satisfies* $2\alpha' < \alpha$, $2\beta \geq \beta'$ *and* $2\gamma \geq \gamma'$. *Then*

(1) $[\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} \neq \mathfrak{p}^{(4)}$ *if* ch $k \neq 2$.

(2) *Suppose that* ch $k = 2$. *Then* $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$ *if and only if* $2\gamma = \gamma'$. *When this is the case,* $R_s(\mathfrak{p})$ *is not a Cohen-Macaulay ring.*

Proof. (1) Assume that $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$. Then we have

$$10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 13\beta\gamma + 8\beta\gamma' + 11\beta'\gamma' - \beta'\gamma \qquad \text{if } 3\beta < 2\beta',$$
$$= 7\beta\gamma + 11\beta\gamma' + 9\beta'\gamma' + 3\beta'\gamma \qquad \text{if } 3\beta \geq 2\beta'.$$

Both the equations contradict the assumption on β and γ, whence $\mathfrak{p}^{(4)} \neq [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$.

(2) Suppose that $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$. Then $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ so that

$$10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 12\beta\gamma + 9\beta\gamma' + 10\beta'\gamma' \qquad \text{if } 3\beta < 2\beta'.$$
$$= 6\beta\gamma + 12\beta\gamma' + 8\beta'\gamma' + 4\beta'\gamma \qquad \text{if } 3\beta \geq 2\beta',$$

whence we have $2\gamma = \gamma'$. Conversely assume that $2\gamma = \gamma'$ and let e_3 be the element in (2) of (5.10). Then since

$$\ell_A(A/(e_3, a, X)) = \ell_A(A/(Y^{3\beta + 2\beta'}, Z^{3\gamma}, X))$$
$$= 3\gamma(3\beta + 2\beta')$$
$$= 1 \cdot 3 \cdot \ell_A(A/(X) + \mathfrak{p}),$$

we get $R_s(\mathfrak{p}) = A[\mathfrak{p}t, d_2t^2, e_3t^3]$ by [5, Corollary (3.8)]. The ring $R_s(\mathfrak{p})$ is not Cohen-Macaulay (cf. [5, Theorem (1.1)] and [20, Corollary (3.4)]), because $A/(a) + \mathfrak{p}^{(2)}$ is non-Cohen-Macaulay by [15, Proposition (2.4)].

PROPOSITION (6.16). *Suppose that M has type (II) and satisfies $2\alpha' > \alpha$, $2\beta \geq \beta'$ and $2\gamma \geq \gamma'$. Then the following assertions hold.*

(1) *If $\mathrm{ch}\, k \neq 2$, then $[\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)} \neq \mathfrak{p}^{(4)}$ or $\mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)} \neq \mathfrak{p}^{(5)}$.*

(2) *Suppose that $\mathrm{ch}\, k = 2$. Then $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$ if and only if $2\beta = \beta'$ and $2\gamma = \gamma'$. When this is the case, $R_s(\mathfrak{p})$ is not Cohen-Macaulay.*

Proof. (1) Assume $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ and we will prove that $\mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)} \neq \mathfrak{p}^{(5)}$. By the equality $\ell_A(A/(X) + \mathfrak{p}^{(4)}) = \ell_A(A/(X) + [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)})$ we get that

($\sharp$) $10(\beta\gamma + \beta\gamma' + \beta'\gamma') = 15\beta\gamma + 7\beta\gamma' + 12\beta'\gamma' - 3\beta'\gamma$
$$\text{if } 3\beta < 2\beta' \text{ and } 3\gamma \geq 2\gamma',$$
$$= 6\beta\gamma + 13\beta\gamma' + 8\beta'\gamma' + 3\beta'\gamma$$
$$\text{if (i) } 3\beta \geq 2\beta' \text{ or if (ii) } 3\beta < 2\beta' \text{ and } 3\gamma < 2\gamma'.$$

By these equations we find $2\beta = \beta'$ and $2\gamma = \gamma'$, whence $\ell_A(A/(X) + \mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)}) = 108\beta\gamma$ and $\ell_A(A/(X) + \mathfrak{p}^{(5)}) = 105\beta\gamma$. Thus $\mathfrak{p}^{(2)}\mathfrak{p}^{(3)} + \mathfrak{p}\mathfrak{p}^{(4)} \neq \mathfrak{p}^{(5)}$ as required.

(2) Suppose that $R_s(\mathfrak{p}) = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3]$. Then $\mathfrak{p}^{(4)} = [\mathfrak{p}^{(2)}]^2 + \mathfrak{p}\mathfrak{p}^{(3)}$,

which yields the same equations as (#) above so that we have $2\beta = \beta'$ and $2\gamma = \gamma'$. Conversely assume $2\beta = \beta'$ and $2\gamma = \gamma'$ and let e_3 be the element obtained by (2) of (5.6). Then since

$$\ell_A(A/(e_3, c, X)) = \ell_A(A/(Z^{\tau\tau}, Y^{3\beta}, X))$$
$$= 21\beta\gamma$$
$$= 3 \cdot \ell_A(A/(X) + \mathfrak{p}),$$

we get $R_s(\mathfrak{p}) = A[\mathfrak{p}t, d_2t^2, e_3t^3]$ by [5, Corollary (3.8)]. As $A/(c) + \mathfrak{p}^{(2)}$ is non-Cohen-Macaulay by [15, Proposition (2.4)], the A-algebra $R_s(\mathfrak{p})$ cannot be Cohen-Macaulay (cf. [5, Theorem (1.1)] and [20, Corollary (3.4)]).

EXAMPLE (6.17). (1) Let $\mathfrak{p} = \mathfrak{p}(10, 11, 13)$ (resp. $\mathfrak{p}(11, 16, 13)$). Then $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^2 & Y^2 & Z^3 \\ Y & Z & X^3 \end{bmatrix} \quad \left(\text{resp. } \begin{bmatrix} X^2 & Y^2 & Z^3 \\ Y & Z^2 & X^3 \end{bmatrix}\right).$$

Hence by (6.1) $R_s(\mathfrak{p})$ is generated (resp. not generated) in degree three.

(2) Suppose that $\operatorname{ch} k = 2$ and let $\mathfrak{p} = \mathfrak{p}(17, 10, 19)$ (resp. $\mathfrak{p}(7, 11, 12)$). Then $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^3 & Y^4 & Z^2 \\ Y^3 & Z & X \end{bmatrix} \quad \left(\text{resp. } \begin{bmatrix} X^3 & Y^2 & Z^2 \\ Y & Z & X^2 \end{bmatrix}\right)$$

and so $R_s(\mathfrak{p})$ is generated in degree three by (6.16) (resp. (6.17)) but not Cohen-Macaulay.

§7. A note on Huckaba's theorem on the analytic spread modulo one element

Let A be an unmixed Noetherian local ring and $\mathfrak{p}$ a prime ideal in A of $\dim A/\mathfrak{p} = 1$. Then S. Huckaba [10] discussed the behavior of the analytic spread $\lambda(\mathfrak{p})$ of $\mathfrak{p}$ modulo one element and gave, in terms of $\lambda(\mathfrak{p})$, a sufficient condition for $R_s(\mathfrak{p})$ to be Noetherian. In this section we shall try to improve his sufficient condition in the case where A is a regular local ring of $\dim A = 3$, clarifying the relationship between Huckaba's condition and Huneke's criterion [13, Theorem 1.4] of $R_s(\mathfrak{p})$ being Noetherian. Our improvement below also provides a partial answer to the question posed by Huckaba [10, Remark 2.6, (3)]:

THEOREM (7.1). *Let A be a 3-dimensional regular local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{p}$ a prime ideal in A of $\dim A/\mathfrak{p} = 1$. Let $0 \neq g \in \mathfrak{p}$*

*and let ℓ be the largest integer with $g \in \mathfrak{p}^{(\ell)}$. Assume that the field $A/\mathfrak{m}$
is infinite. Then the following two conditions are equivalent.*

    (1) $\lambda(\mathfrak{p}/gA) = 1$.

    (2) $\ell_A(A/(f, g, x)) = \ell \cdot \ell_A(A/(X) + \mathfrak{p})$ *for some $f \in \mathfrak{p}$ and $x \in \mathfrak{m} \setminus \mathfrak{p}$.*

When this is the case, $R_s(\mathfrak{p}) = A[\mathfrak{p}^{(n)}t^n \mid 1 \le n \le \ell]$.

    Proof. (1) $\Rightarrow$ (2) We put $B = A_\mathfrak{p}$ and $\mathfrak{n} = \mathfrak{p}A_\mathfrak{p}$. Choose $f \in \mathfrak{p}$ so that
$\mathfrak{p}^{r+1} + gA = f\mathfrak{p}^r + gA$ for some $r \ge 0$. Then $\mathfrak{p} = \sqrt{(f, g)}$ and $\mathfrak{n}^{r+1} + gB = f\mathfrak{n}^r + gB$. Hence

$$\mathfrak{n}^{r+1} = f\mathfrak{n}^r + gB \cap \mathfrak{n}^{r+1}$$
$$= f\mathfrak{n}^r + g\mathfrak{n}^{r+1-\ell}$$

so that f, g forms a super regular sequence in B. Since $\ell_B(B/(f, g)B) = \ell$,
we get by the additive formula [18, p. 126] of multiplicity the required
equality that

$$\ell_A(A/(f, g, x)) = e_{xA}(A/(f, g))$$
$$= \ell_B(B/(f, g)B) \cdot e_{xA}(A/\mathfrak{p})$$
$$= \ell \cdot \ell_A(A/(x) + \mathfrak{p})$$

for any $x \in \mathfrak{m} \setminus \mathfrak{p}$.

    (2) $\Rightarrow$ (1) We put $C = A/gA$ and $P = \mathfrak{p}/gA$. Then since $A/gA + \mathfrak{p}^{(n)}$
is a Cohen-Macaulay ring for $n \ge 1$ and since $\mathfrak{p}^{(n)} = f\mathfrak{p}^{(n-1)} + g\mathfrak{p}^{(n-\ell)}$ for
$n \ge \ell$ (cf. [5, (3.7) (1)], we have that $P^{(n)} = \mathfrak{p}^{(n)}C$ for $n \ge 1$ and that $P^{(n)}
= PP^{(n-1)}$ for all $n \ge \ell$. Therefore $R_s(P)$ is module-finite over $R(P) =
\sum_{n \ge 0} P^n t^n$ and so we get by [4, Theorem (2.10)] that

$$\lambda(P\hat{C} + Q/Q) < \dim \hat{C}/Q = 2$$

for any $Q \in \mathrm{Ass}\,\hat{C}$, where $\hat{C}$ denotes the $\mathfrak{m}/gA$-adic completion of C.
Thus we have $\lambda(P) = 1$, because

$$\lambda(P) = \lambda(P\hat{C}) = \lambda(P\hat{C} + Q/Q)$$

for some $Q \in \mathrm{Ass}\,\hat{C}$ (cf. [14, Lemma 4.2]). The last assertion directly fol-
lows from [5, Corollary (3.8)].

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S. Goto
Department of Mathematics
School of Science and Technology
Meiji University
Higashimita, Tama-ku
Kawasaki-shi 214
Japan

K. Nishida
Department of Mathematics
Faculty of Science
Chiba University
Yayoi-cho, Chiba-shi 260
Japan

Y. Shimoda
College of Liberal Arts and Sciences
Kitasato University
Kitasato, Sagamihara-shi 228
Japan