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ON STRONGLY PI-REGULAR RINGS OF STABLE RANGE ONE

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Dedicated to Professor Victor P. Camillo
on the occasion of his fiftieth birthday

An associative ring R is said to have stable range one if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right (equivalently, left) invertible. Call a ring R *strongly π-regular* if for every element $a \in R$ there exist a number n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1}x$. It is an open question whether all strongly π-regular rings have stable range one. The purpose of this note is to prove the following Theorem: If R is a strongly π-regular ring with the property that all powers of every nilpotent von Neumann regular element are von Neumann regular in R, then R has stable range one.

Let R be an associative ring with identity. R is said to *have stable range one* if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right (equivalently, left) invertible. Call a ring R *strongly π-regular* if for every element $a \in R$ there exist a number n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1}x$. This is in fact a two-sided condition [2].

It is an open question whether all strongly π-regular rings have stable range one. Many special classes of strongly π-regular rings have been proved to have stable range one (see [**1**, **4**, **6**, **7**]). Goodearl and Menal [**3**] proved that strongly π-regular regular rings are unit-regular, hence have stable range one (Theorem 5.8, p.278). Here a ring R is called *unit-regular* if for each element x in R there is a unit u in R such that $xux = x$, that is, x is a *unit-regular* element. We say also that an element a of R is *(von Neumann) regular* if $a = axa$ for some element x in R.

The main purpose of this note is to prove the following Theorem:

THEOREM 1. *Let R be a strongly π-regular ring. If all powers of every nilpotent regular element of R are regular in R, then R has stable range one.*

To prove Theorem 1, we need the following characterizationby Goodearl and Menal [**3**, Theorem 6.1] of stable range one for strongly π-regular rings .

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Example 4 was communicated to the author by Professor K.R. Goodearl.

PROPOSITION 2. (Goodearl and Menal) *For a strongly π-regular ring R, R has stable range one if and only if every nilpotent regular element of the ring eRe is unit-regular in eRe, where $e \in R$ is any idempotent of R.*

The proof we are going to give is motivated by a proof of Goodearl and Menal [**3**, Theorem 5.8].

PROOF OF THEOREM 1: Let $e^2 = e \in R$ be any idempotent of R, and $x \in eRe$ be a nilpotent regular element of eRe, with $xyx = x, y \in eRe$. It suffices to prove, by Proposition 2, that x is unit-regular in eRe.

Set $K_i = r.ann_{eRe}(x^i)$, the right annihilator of x^i in eRe for all $i = 0, 1, 2, \cdots$.

CLAIM 1. There exists an integer $n \geqslant 1$ such that $xeRe + K_n = eRe$ and $x^n eRe \cap K_1 = 0$.

This is clear, since we assume that $x^n = 0$ for some $n \geqslant 1$.

CLAIM 2. $xeRe + K_i$ are direct summands of eRe_{eRe} for all $i \geqslant 1$.

It is easy to check that an element of eRe is regular in eRe if and only if it is regular in R. So x^i is regular in eRe for all $i \geqslant 2$ by our assumption on R. We may assume $x^i y_i x^i = x^i$ for some $y_i \in eRe$ for $i \geqslant 2$. Then $K_i = (e - y_i x^i)eRe$. It is easy to check that

$$xeRe + (e - y_i x^i)eRe = y_i x^i x eRe + (e - y_i x^i)eRe.$$

We check below that the element $y_i x^i x$ is actually von Neumann regular in eRe:

$$y_i x^i x \cdot y_{i+1} x^i \cdot y_i x^i x = y_i x^i x y_{i+1} x^i x = y_i x^i x.$$

Put $e_i = y_i x^i x y_{i+1} x^i$ and $f_i = e - y_i x^i$, then $e_i f_i = f_i e_i = 0$. We see that e_i and f_i are orthogonal idempotents, hence $e_i + f_i$ is an idempotent. But $y_i x^i x eRe = e_i eRe$, so $xeRe + K_i = e_i eRe + f_i eRe = (e_i + f_i)eRe$ is a direct summand of eRe_{eRe}.

Recall that we assume $xyx = x$, so $xeRe + K_1$ is a direct summand of eRe_{eRe} for the same reason.

CLAIM 3. $x^i eRe \cap K_1$ are direct summands of eRe_{eRe} for all $i \geqslant 1$.

First, we show $x^i eRe \cap K_1 = x^i K_{i+1}$. Since $x^i K_{i+1} \subset x^i eRe$ and $x^i K_{i+1} \subset K_1$, $x^i K_{i+1} \subset x^i eRe \cap K_1$; on the other hand, pick any $x^i r \in x^i eRe \cap K_1$, then $x x^i r = x^{i+1} r = 0$ so that $r \in K_{i+1}$; thus $x^i r \in x^i K_{i+1}$ and so $x^i eRe \cap K_1 \subset x^i K_{i+1}$.

Second, recall that we assume $x^i y_i x^i = x^i$, so that $K_{i+1} = (e - y_{i+1} x^{i+1})eRe$, and we see that $x^i eRe \cap K_1 = x^i K_{i+1} = x^i (e - y_{i+1} x^{i+1})eRe$. We check below that

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$x^i(e - y_{i+1}x^{i+1})$ is von Neumann regular in eRe:

$$\begin{aligned}
x^i(e - y_{i+1}x^{i+1}) \cdot y_i \cdot x^i(e - y_{i+1}x^{i+1}) &= (x^i - x^i y_{i+1}x^{i+1})y_i x^i(e - y_{i+1}x^{i+1}) \\
&= (e - x^i y_{i+1}x)x^i y_i x^i(e - y_{i+1}x^{i+1}) \\
&= (e - x^i y_{i+1}x)x^i(e - y_{i+1}x^{i+1}) \\
&= (x^i - x^i y_{i+1}x^{i+1})(e - y_{i+1}x^{i+1}) \\
&= x^i(e - y_{i+1}x^{i+1})(e - y_{i+1}x^{i+1}) \\
&= x^i(e - y_{i+1}x^{i+1}).
\end{aligned}$$

Therefore $x^i eRe \cap K_1 = x^i K_{i+1}$ is a direct summand of eRe_{eRe}.

Inasmuch as $xyx = x$, $xeRe \cap K_1 = xK_2$ is a direct summand of eRe_{eRe}.

CLAIM 4. $(xeRe + K_m)/xeRe \cong K_1/(x^m eRe \cap K_1)$ for all m.

The right ideals of eRe involved here are all direct summands of eRe_{eRe} by Claim 2 and Claim 3. We have the ascending and descending chains of direct summands

$$\begin{gathered}
xeRe \subset^{\oplus} xeRe + K_1 \subset^{\oplus} xeRe + K_2 \subset^{\oplus} \cdots \subset^{\oplus} xeRe + K_m \\
K_1^{\oplus} \supset xeRe \cap K_1^{\oplus} \supset x^2 eRe \cap K_1^{\oplus} \supset \cdots^{\oplus} \supset x^m eRe \cap K_1
\end{gathered}$$

which give us the decompositions

$$(xeRe + K_m)/xeRe \cong \bigoplus_{i=0}^{m-1} (xeRe + K_{i+1})/(xeRe + K_i)$$

$$K_1/(x^m eRe \cap K_1) \cong \bigoplus_{i=0}^{m-1} (x^i eRe \cap K_1)/(x^{i+1} eRe \cap K_1)$$

so if we can show that

$$(xeRe + K_{i+1})/(xeRe + K_i) \cong (x^i eRe \cap K_1)/(x^{i+1} eRe \cap K_1)$$

for all i, we are done.

First we note that

$$\begin{aligned}
(xeRe + K_{i+1})/(xeRe + K_i) &= (xeRe + K_i + K_{i+1})/(xeRe + K_i) \\
&\cong K_{i+1}/[(xeRe + K_i) \cap K_{i+1}] = K_{i+1}/[(xeRe \cap K_{i+1}) + K_i].
\end{aligned}$$

As $x^i K_{i+1} \subseteq x^i eRe \cap K_1$ and $x^i[(xeRe \cap K_{i+1}) + K_i] \subseteq x^{i+1} eRe \cap K_1$, left multiplication by x^i gives a module homomorphism

$$f : K_{i+1}/[(xeRe \cap K_{i+1}) + K_i] \to (x^i eRe \cap K_1)/(x^{i+1} eRe \cap K_1).$$

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f is epic: Pick any $r \in x^i eRe \cap K_1$, $r = x^i a$ for some $a \in eRe$. But, since $x^{i+1}a = xr = 0$, $a \in K_{i+1}$. So $f(a) = r$.

f is monic: If $z \in K_{i+1}$ and $x^i z \in x^{i+1}eRe \cap K_1$, then we have $x^i z = x^{i+1}b$ for some $b \in eRe$ and $x^{i+2}b = x(x^i z) = 0$, whence $xb \in K_{i+1} \cap xeRe$. Since $x^i(z - xb) = 0$, $z - xb \in K_i$, thus $z \in (xeRe \cap K_{i+1}) + K_i$, that is, f is monic.

We have proved that f is an isomorphism.

CLAIM 5. x is unit-regular in eRe.

It follows from Claim 1 and Claim 4 that

$$(xeRe + K_n)/xeRe = eRe/xeRe \cong K_1/(x^n eRe \cap K_1) = K_1/0 = K_1.$$

It is assumed that $xyx = x$, hence

$$eRe = yxeRe \oplus K_1 = xeRe \oplus (e - xy)eRe.$$

So $K_1 \cong (e - xy)eRe$. Denote this isomorphism by α. Also, the restriction of the left multiplication by x gives an isomorphism β from $yxeRe$ to $xeRe$. Define $u \in end(eRe_{eRe}) = eRe$ to be the direct sum of α and β^{-1}. Then it is easy to check that u is a unit in eRe and $xux = x$. ∎

Yu [6] proved, among other things, that strongly π-regular rings whose idempotents are all central have stable range one. This result now can be easily deduced as a corollary of Theorem 1.

COROLLARY 3. (Yu, [6]) *Strongly π-regular rings whose idempotents are all central have stable range one.*

PROOF: It is known that a ring R has stable range one if and only if $R/J(R)$ has stable range one, where $J(R)$ denotes the Jacobson radical of R [5, Theorem 2.2)]. Let R be a strongly π-regular ring whose idempotents are all central. We need only to show that $R/J(R)$ has stable range one.

It is trivial to see that for a strongly π-regular ring S with $J(S) = 0$, all idempotents of S are central if and only if S contains no nonzero nilpotent element. Applying this equivalence to the factor ring $R/J(R)$, we see that $R/J(R)$ contains no nonzero nilpotent element, hence all powers of every nilpotent regular element are regular. So the conclusion follows from Theorem 1. ∎

We conclude this note by giving an example, which shows that the converse of Theorem is false.

EXAMPLE. Let R be the 2×2 matrix ring over $F[x]/(x^2)$, where F is any field and $F[x]$ is the polynomial ring over F.

Clearly, R is a finite dimensional algebra, hence strongly π-regular. Of course, R has stable range one. But not all the powers of every nilpotent regular element of R are regular in R. Taking $a = \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix}$ and $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it is easy to see that a is nilpotent and $aua = a$. But $a^2 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in J(R)$ is not regular. So the condition that all powers of every nilpotent regular element are regular is sufficient but not necessary for strongly π-regular rings to have stable range one.

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