PROJECTIVE REPRESENTATIONS OF EXTRA-SPECIAL p-GROUPS

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0. Introduction. Let G be a finite group (with neutral element e) which operates trivially on the multiplicative group R^* of a commutative ring R (with identity 1). Let $H^2(G, R^*)$ denote the second cohomology group of G with respect to the trivial G-module R^*. With every $\bar{f} \in H^2(G, R^*)$ represented by the central factor system $f: G \times G \to R^*$ we associate the so called twisted group algebra (R, G, f) (see [**3**, V, 23.7] for the definition). (R, G, f) is determined by $\bar{f}$ up to R-algebra isomorphism. In this note we shall describe its representations in the case R is an algebraically closed field C of characteristic zero and G is an extra-special p-group P.

1. Some lemmata. We first present a technique for handling a twisted group algebra (K, G, f) of a nilpotent group G over a field K of characteristic zero. Let $Z(G)$ be the center of G and define $N_f = \{z \in Z(G) \mid f(g, z) = f(z, g)$ for all $g \in G\}$. Then by [**6**, (2,4)], the central subgroup N_f of G is trivial if and only if G has a faithful absolutely irreducible f-representation. Let us assume that $N_f \neq \{e\}$. Then by [**10**, Theorem 2.1] there is a factor system $t: G/N_f \times G/N_f \to (K, N_f, f)^*$ such that (K, G, f) and $((K, N_f, f), G/N_f, t)$ are isomorphic K-algebras. Decompose (K, N_f, f) into a direct sum of fields: $(K, N_f, f) \cong \bigoplus_i K_i$. Then there are factor systems $t_i: G/N_f \times G/N_f \to K_i^*$ such that $(K, G, f) \cong \bigoplus_i (K_i, G/N_f, t_i)$ as K-algebras. Applying this process to every component $(K_i, G/N_f, t_i)$, we obtain a decomposition of (K, G, f) into twisted group algebras of type (L, H, s), where L is a certain radical extension of K, H a factor group of G, $s: H \times H \to L^*$ a factor system such that the central subgroup $N_s = \{z \in Z(H) \mid s(z, h) = s(h, z)$ for all $h \in H\}$ of H is trivial or, what amounts to the same thing, H has a faithful absolutely irreducible s-representation. In some cases these conditions are equivalent to the statement that (L, H, s) is central simple, for instance if H is abelian (see [**11**, 5.3]) or nilpotent metacyclic of odd order (see [**6**, (5.3)]). More precisely, for G abelian, we are in a position to deduce the following lemma, part of which is already contained in [**11**, §6].

(1.1) LEMMA. *Let K be a field of characteristic zero, G a finite abelian group and $f: G \times G \to K^*$ a factor system. The central subgroup N_f defined above coincides with the kernel of the symplectic pairing $\omega_f: G \times G \to K^*$ associated with f as defined in* [**4**, §1]. *There is a bijective correspondence between the simple components of (K, G, f) and the simple components of (K, N_f, f). Each simple component of (K, G, f) is isomorphic to $K_i \otimes_K (K, G/N_f, \omega)$, where K_i is some simple component of (K, N_f, f) and $(K, G/N_f, \omega)$ is the K-algebra generated freely by $\{e_x \mid x \in G/N_f\}$ with relations $e_x e_y = \omega(x, y) e_y e_x$ for some nondegenerate pairing $\omega: G/N_f \times G/N_f \to K^*$. The* Hom$(G, K^*)$*-action on the isomorphism* *classes of simple representations of* (K, G, f) *(as defined in* [**11** 5.1]) *is equivalent to the* $\mathrm{Hom}(N_f, K^*)$*-action on the isomorphism classes of simple representations of* (K, N_f, f)*. If* K *contains a primitive* $\exp G$*-th root of unity, then* $\mathrm{Hom}(G, K^*)$ *operates transitively. If* K *is algebraically closed, then each* $\mathrm{Hom}(G, K^*)$*-orbit is isomorphic to* N_f.

The following lemma, which will also be used in the sequel, follows directly from [**6**, (7.6)].

(1.2) LEMMA. *The extra-special* p*-group* P *has an irreducible faithful projective representation over an algebraically closed field of characteristic zero if and only if* P *is the nonabelian group of order* p^3 *and exponent* p *if* $p \neq 2$*, and the dihedral group of order* 8 *if* $p = 2$.

The following lemmata are also useful in determining the projective representations of extra-special p-groups.

(1.3) LEMMA. *Let* C *be an algebraically closed field of characteristic zero. If, for a factor system* $f: G \times G \to C^*$*, the finite group* G *has a faithful* C*-irreducible* f*-representation, then* $\exp Z(G)$ *divides the order of* $\bar{f}$ *in* $H^2(G, C^*)$.

Proof. Let m be the order of $\bar{f}$ in $H^2(G, C^*)$ and W_m the group of mth roots of unity in C. For all $z \in Z(G)$, $g \in G$ we have $1 = \omega_f(z, g)^m = \omega_f(z^m, g)$, where ω_f is the pairing from $G \times Z(G)$ to W_m associated with f. The assumption implies (see [**11**, Proposition 5.2]) $z^m = e$.

(1.4) LEMMA. *Let* C *be an algebraically closed field of characteristic zero. Then for any finite group* G *with* $G' \leq Z(G)$ *the transgression map* $\tau: \mathrm{Hom}(G', C^*) \to H^2(G/G', C^*)$ *is injective. For* $G = P$*, an extra-special* p*-group, the inflation map* $\mathrm{inf}: H^2(P/P', C^*) \to H^2(P, C^*)$ *is surjective with kernel isomorphic to* P' *if* P *has no faithful* C*-irreducible projective representation. If* P *has a faithful* C*-irreducible projective representation,* $H^2(P, C^*)$ *is cyclic of order* p.

Proof. Consider the following exact sequence (see [**4**, §1]).

$$1 \longrightarrow \mathrm{Hom}(G/G', C^*) \xrightarrow{\inf} \mathrm{Hom}(G, C^*) \xrightarrow{\mathrm{res}} \mathrm{Hom}(G', C^*) \xrightarrow{\tau} H^2(G/G', C^*)$$

$$\xrightarrow{\inf} H^2(G, C^*) \xrightarrow{\omega} B(G, G'; C^*);$$

here $B(G, G'; C^*)$ denotes the group of pairings from $G \times G'$ to C^* and ω is the "pairing map" $\bar{f} \mapsto \omega_f$ of [**4**, §1]. Now res has trivial image and the first assertion is proved. If $G = P$ has a faithful C-irreducible projective representation, then by (1.3) the group $H^2(P, C^*)$ contains an element of order greater or equal to p. On the other hand, $H^2(P, C^*)$ has not more than p elements, as follows for instance from (1.2) by an application of the Gaschütz-Neubüser-Yen estimation (see [**3**, V, §23]).

(1.5) LEMMA [**9**]. *If* C *is an algebraically closed field of characteristic zero, then, for an arbitrary finite group* G *and any* $\bar{f} \in H^2(G, C^*)$*, the number of* $\mathrm{Hom}(G, C^*)$*-orbits of*

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projective representations belonging to f is equal to the number of those f-regular conjugate classes of G whose elements belong to G′.

2. The representations and the action of the group of linear characters in the algebraically closed case. Let C be an algebraically closed field of characteristic zero, let P be an extra-special p-group, let $|P|=p^{2m+1}$, $P/P'\cong(\mathbb{Z}/p\mathbb{Z})^{2m}$. The following theorem describes the projective representations of P. In its proof we shall use, without explicit reference, the remarks made in Section 1.

(2.1) THEOREM. (a) *If P has no faithful C-irreducible projective representation, then for every* $\bar{f}\in H^2(P, C^*)$ *we have* $(C, P, f)\cong\bigoplus_{i=1}^{p}(C, P/P', t_i)$. *If* p^{2l_i} *is the order of the kernel* N_i *of the symplectic form* ω_{t_i}, *then* (C, P, f) *has* $\sum_{i=1}^{p} p^{2l_i}$ *irreducible characters. There are p orbits under the* Hom(P, C^*)*-action. Each orbit is isomorphic to some* N_i *and is represented by a character of degree* p^{m-l_i}.

(b) *If P has a faithful C-irreducible projective (say f-) representation, then* (C, P, f^j) $(1\le j\le p-1)$ *has p irreducible characters, all of degree p.* Hom(P, C^*) *operates transitively for* $1\le j\le p-1$.

Proof. (a) The pairing $\omega_f: P\times P'\to C^*$ is trivial. Hence $(C, P, f)\cong((C, P', f), P/P', t)\cong\bigoplus_{i=1}^{p}(C, P/P', t_i)$. First note that there are p f-regular conjugate classes contained in P', so by (1.5) there are p Hom(P, C^*)-orbits. Each orbit is isomorphic to a Hom$(P/P', C^*)$-orbit of an irreducible character of some $(C, P/P', t_i)$. All assertions of (2.1a) can now be deduced from (1.1).

To prove part (b), note that only the neutral element of P is contained in P' and is f-regular. So by (1.5) Hom(P, C^*) operates transitively. P is the nonabelian group of order p^3 and exponent p if $p\neq 2$, and the dihedral group of order 8 if $p=2$. We show that each irreducible representation of (C, P, f) has degree p. If $p=2$, this is clear because the dihedral group of order 8 contains a cyclic subgroup of index 2. Let $p\neq 2$ and let χ be a C-irreducible character of (C, P, f). If ψ is a C-irreducible constituent of $\chi_{|P'}$, then the stabilizer group $I(\psi)\neq P$ because the projective representation belonging to χ is faithful and clearly $I(\psi)\neq P'$. By Clifford's theory (which is valid for projective representations too), a C-irreducible constituent ζ of $\chi_{|I(\psi)}$ induces χ. We have $\zeta_{|P'}=n.\psi$. Since $I(\psi)/P'$ is cyclic (of order p) we have $n=1$. Since ψ is one dimensional, so is ζ. Hence χ has degree $[P:I(\psi)]=p$. There are p C-irreducible characters of (C, P, f) because $|P|=p^3$.

The order of $\bar{f}$ in $H^2(P, C^*)$ is equal to p because of (1.3). Hence $\langle\bar{f}\rangle=H^2(P, C^*)$ by (1.4). P has a faithful C-irreducible f^j-representation for $1\le j\le p-1$. Otherwise a generator z of P' satisfies $\omega_{f^j}(z, g)=\omega_f(z, g)^j=\omega_f(z^j, g)=1$ for all $g\in P$. Hence $z^j=e$, which is a contradiction. So the assertion about the irreducible representations of (C, P, f^j), $1\le j\le p-1$, follows as before.

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