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ON THE POSITIVE ROOTS OF AN EQUATION INVOLVING A BESSEL FUNCTION

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1. Introduction

In this paper we shall discuss the positive roots of the equation

$$H(r)=(-Br^2+A)I_q(r)+rI_q'(r)=0, A\in\mathbf{R},\ B>0,\ q>-1, \tag{1.1a}$$

where I_q is the modified Bessel function of the first kind. By means of a recurrence relation for $I_q(r)$ [**2**, (5.7.9)], equation (1.1a) can also be written in the form

$$H(r)=(-Br^2+A+q)I_q(r)+rI_{q+1}(r)=0. \tag{1.1b}$$

We shall show that, depending on the parameters A, B and q, the equation $H(r)=0$ has either one positive simple root, or two positive simple roots, or one positive double root, or no positive roots at all.

Throughout this paper and without further reference we shall freely use the recurrence relation mentioned above and the asymptotic formulas for $I_q(r)$ for small (positive) and large values of r [**2**,(5.16.4), (5.16.5)]. Furthermore, continuity of H as a function of the parameters A, B and q as well as of its argument will tacitly be exploited.

Interest in equation (1.1) arises in connection with the functional properties of the delta function initial condition solution of the generalized Feller equation [**3**, (4.1)] which can be interpreted as a density function [**4**, (3.8)].

2. The reduced equation

For $B=0$, equation (1.1) reduces to

$$h(r)=AI_q(r)+rI_q'(r)=0,\ A\in\mathbf{R},\ q>-1,$$

so that

$$h(r)-H(r)=Br^2I_q(r).$$

The equation $h(r)=0$ can be transformed into a classical one which contains the Bessel function J_q. The corresponding theorem on the roots of that equation is well known [**1**, **5**, 15.25]. Nevertheless, for later references, it is useful to formulate the result for $h(r)=0$ explicitly.

Theorem 2.1 *For $A\in\mathbf{R}$, $q>-1$, the equation $h(r)=0$ has exactly one positive simple root if $A+q<0$ and no positive roots if $A+q\geqq 0$.*

3. The equation $H(r)=0$

Before going into details we note that, if $\rho>0$ is a zero of the function $H(r)$ defined in (1.1), then

$$H'(\rho)=P(\rho)\rho^{-1}I_q(\rho), \tag{3.1}$$

$$H''(\rho)=Q(\rho)\rho^{-2}I_q(\rho), \tag{3.2}$$

$$H'''(\rho)=R(\rho)\rho^{-1}I_q(\rho), \tag{3.3}$$

where

$$P(r)=-[B^2r^4-(1-2B(1-A))r^2+(A^2-q^2)], \tag{3.4}$$

$$Q(r)=-[3B^2r^4-(1-2B(1-A))r^2-(A^2-q^2)], \tag{3.5}$$

$$R(r)=-2[Br^2+1+A+2Bq^2-2B]. \tag{3.6}$$

In some neighbourhood of ρ the power series expansion

$$H(r)=(r-\rho)H'(\rho)+\frac{1}{2!}(r-\rho)^2H''(\rho)+\frac{1}{3!}(r-\rho)^3H'''(\rho)+\cdots \tag{3.7}$$

is valid.

The structure of $H'(\rho)$ at any positive zero ρ of $H(r)$ shows that $H(r)$ can have at most three distinct positive zeros. This follows from the fact that the polynomial $P(r)$ can change sign at, at most, two distinct positive points. An essential consequence of this observation is that there exists a smallest positive zero of $H(r)$ if there are positive zeros at all.

Theorem 3.1 *If $B>0$, $A+q>0$, $q>-1$, the equation $H(r)=0$ has exactly one positive simple root.*

Proof. 1. For small positive values of r the formula

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$$H(r) \sim \frac{1}{\Gamma(1+q)}\left(\frac{r}{2}\right)^q \left[(A+q) - \left(B - \frac{1}{2(1+q)}\right) r^2\right], r \downarrow 0, \tag{3.8}$$

holds. Therefore, if $A+q>0$, $H(r)>0$ for $r>0$ sufficiently small. Furthermore, $H(r)\downarrow -\infty$ as $r \uparrow \infty$. Consequently, under the assumptions, $H(r)$ has at least one positive zero. This fact will tacitly be used in subsequent arguments.

2(a). Let $A-q>0$ so that $A>0$, $A^2-q^2>0$. Within their individual restrictions let A, B and q be arbitrarily given. The sequence of the coefficients of the polynomial $P(r)$ defined in (3.4) has an even number of variations in sign. Therefore, there are three alternatives: $P(r)$ has two distinct positive zeros; $P(r)$ has no positive zeros; $P(r)$ has a positive double zero.

We set $1-2B(1-A)=a$, $4B^2(A^2-q^2)=b$, and $A^2-q^2=c$. Then the roots of the equation $P(r)=B^2r^4-ar^2+c=0$ are

$$\pm\{(2B^2)^{-1}[a \pm \sqrt{a^2-b}]\}^{1/2} \tag{3.9}$$

with $a^2-b=4[B^2(1-2A+q^2)-B(1-A)+\frac{1}{4}]$. Unless the inequalities $A>1$, $1-2A+q^2 \geqq 0$ hold simultaneously, there exists exactly one positive value for B, namely

$$B_* = [2(1-A+\sqrt{c})]^{-1}, \tag{3.10}$$

such that $a>0$ and $a^2-b=0$. For this value of B the equation $P(r)=0$ has the positive double root

$$\sigma = [B_*^{-1}\sqrt{c}]^{1/2}. \tag{3.11}$$

(i) *$P(r)$ has two distinct positive zeros* $\sigma_1<\sigma_2$. Let ρ be the smallest positive zero of $H(r)$. Then ρ cannot be in the interval $(0,\sigma_2)$. To see this, observe that $\rho \uparrow \infty$ as $B \downarrow 0$ (see Section 2) and that $\sigma_1 \downarrow \sqrt{c}$. Therefore, if $0<\rho<\sigma_1$ for some $B>0$, it would have to move across σ_1 into the interval (σ_1,σ_2) as $B\downarrow 0$. But, by (3.1) and (3.4) $H'(\rho)>0$ if $\rho \in (\sigma_1,\sigma_2)$ which is impossible if ρ is the smallest positive zero of $H(r)$. Thus, any positive zero of $H(r)$ is greater than or equal to σ_2. But ρ cannot be equal to σ_2. For, if $\rho=\sigma_2$, the expansion (3.7) holds with $H'(\sigma_2)=0$ and, by (3.2) and (3.5), $H''(\sigma_2)<0$ which implies $H(r)<0$ in some neighbourhood of $\rho=\sigma_2$ which is impossible. Therefore, $\rho>\sigma_2$. But then $H'(\rho)<0$ so that ρ is a simple zero of $H(r)$. Clearly, $H(r)$ cannot have a zero greater than ρ. Consequently, under the current conditions, $H(r)$ has exactly one positive simple zero.

(ii) *$P(r)$ has no positive zeros.* In this case $H'(\rho)<0$ whenever ρ is a positive zero of $H(r)$. Therefore, $H(r)$ has exactly one positive simple zero.

(iii) *$P(r)$ has a positive double zero σ.* By (3.10) and (3.11)

$$H(\sigma) = [-\sqrt{c}+A+q]I_q(\sigma) + \sigma I_{q+1}(\sigma).$$

The factor of $I_q(\sigma)$ is nonnegative for $q \geqq 0$. Therefore, $H(\sigma)>0$ and any positive zero ρ of $H(r)$ must be greater than σ. If $\rho>\sigma$, then $H'(\rho)<0$ so that $H(r)$ has exactly one positive simple zero.

Now let $-1<q<0$ and set $q=-p$. By the result of the previous paragraph

$$H_p(\sigma)=[-\sqrt{c}+A]I_p(\sigma)+\sigma I'_p(\sigma)=\alpha(p)>0,$$

$c=A^2-p^2$. (We observe that $\sigma=\sigma(p)$ is an even function of p by (3.10) and (3.11).) The last inequality holds for $p=0$, i.e., $\alpha(0)>0$. Continuity implies that

$$H_{-p}(\sigma)=[-\sqrt{c}+A]I_{-p}(\sigma)+\sigma I'_{-p}(\sigma)>0$$

if $p>0$ is sufficiently small. Suppose there exists $p>0$ such that $H_{-p}(\sigma)=0$. Then

$$I_{-p}H_p-I_pH_{-p}=\sigma(I_{-p}(\sigma)I'_p(\sigma)-I_p(\sigma)I'_{-p}(\sigma))=\alpha(p)I_{-p}(\sigma). \tag{3.12}$$

Bessel's differential equation for I_q [**2**, (5.7.7)] implies that

$$r[I_{-p}(r)I'_p(r)-I_p(r)I'_{-p}(r)]=\beta(p)=\text{const.}$$

Using the series expansions for I_q and I'_q and letting $r\downarrow 0$, we conclude that

$$\beta(p)=\frac{2p}{\Gamma(1+p)\Gamma(1-p)}.$$

This and (3.12) lead to

$$\frac{2p}{\Gamma(1+p)\Gamma(1-p)}=\{[-\sqrt{c}+A+p]I_p(\sigma)+\sigma I_{p+1}(\sigma)\}I_{-p}(\sigma). \tag{3.13}$$

Using again the series expansion of I_p and that for I_{p+1} we obtain from (3.13), after some manipulations,

$$[\sqrt{c}-A+p]=\{[-\sqrt{c}+A+p](1-p^2)^{-1}+(1+p)^{-1}\}\frac{\sigma^2}{2}+(\text{positive terms}) \tag{3.14}$$

with $\sigma^2/2=(1-A+\sqrt{c})\sqrt{c}$. (It may be helpful to provide a few hints on these manipulations. The right-hand side of (3.13) can be written in the form $[-\sqrt{c}+A+q]$ [power series of positive terms] $+2(\sigma/2)^2$ [power series of positive terms]. Multiplication of the equation by $\Gamma(1+p)\Gamma(1-p)$, subsequent transposition of term $-\sqrt{c}+A+q$ from the right to the left, and grouping of the remaining terms on the right by powers of σ^2 leads to (3.14).) One can show that, in the present circumstances, the left-hand side of (3.14) is less than the first term at the right (the positive terms neglected). In fact, the inequality can finally be brought into the form

$$0<(A-p)(1-p^2)+2c^2(A-c).$$

Therefore, the assumed equality (3.12) is contradictory. We have inequality, the left-hand side being less than the right-hand side in (3.12). This implies that $H_{-p}(\sigma)>0$. Consequently, under the present conditions, $H(r)$ again has exactly one positive simple zero which is greater than σ.

2(b). $A-q\leqq 0$ so that $A>0$, $q>0$, and $A^2-q^2\leqq 0$. Within their individual restrictions let A, B and q be arbitrarily given. There are two alternatives: $P(r)$ has one positive zero; $P(r)$ has no positive zeros.

(i) *$P(r)$ has one positive zero σ_1.* If ρ is the smallest positive zero of $H(r)$ we have $\rho\geqq\sigma_1$ for, otherwise, $H'(\rho)$ would be positive which is impossible. The zero ρ cannot be equal to σ_1. For, if $\rho=\sigma_1$, $H'(\sigma_1)=0$, $H''(\sigma_1)<0$, so that, by (3.7), $H(r)<0$ in some neighbourhood of $\rho=\sigma_1$ which is impossible. Therefore, $\rho>\sigma_1$. But then $H'(\rho)<0$ and, consequently, $H(r)$ has exactly one positive simple zero.

(ii) *$P(r)$ has no positive zeros.* (In this case $A^2-q^2=0$.) Then $H'(\rho)<0$ at any positive zero of $H(r)$. Consequently, $H(r)$ has exactly one positive simple zero. This completes the proof.

Theorem 3.2 *If $B>0$, $A+q=0$, $q>-1$, the equation $H(r)=0$ has exactly one positive simple root if $B<[2(1+q)]^{-1}$, and $H(r)=0$ has no positive roots otherwise.*

Proof. Within the assumptions let A, B and q be arbitrarily given. Then $A=-q<1$, and $A^2-q^2=0$. As (3.4) shows, there are two alternatives: $P(r)$ has one positive zero; $P(r)$ has no positive zeros.

(i) *$P(r)$ has one positive zero.* (The roots of $P(r)=0$ are given in (3.9).) This requires that $a=1-2B(1-A)>0$, i.e., $0<B<[2(1-A)]^{-1}=[2(1+q)]^{-1}$. In these circumstances, as (3.8) shows, $H(r)>0$ for $r>0$ sufficiently small, i.e., $H(r)$ has at least one positive zero. The arguments used in part 2(b)(i) of the proof of Theorem 3.1 can now be applied verbatim.

(ii) *$P(r)$ has no positive zeros.* In this case $a=1-2B(1-A)\leqq 0$, i.e., $B\geqq[2(1+q)]^{-1}$. Let $B=\beta+[2(1+q)]^{-1}$, $\beta\geqq 0$. We write $H(r)=-Br^2I_q(r)+rI_{q+1}(r)$ in the form

$$H(r)=-\beta r^2I_q(r)-[2(1+q)]^{-1}r^2I_q(r)+rI_{q+1}(r)$$
$$=-\beta r^2I_q(r)-\frac{2}{1+q}\sum_{k=1}^{\infty}\frac{(r/2)^{2k+q+2}}{(k-1)!\Gamma(k+q+2)}.$$

This expression shows that $H(r)<0$ for every $r>0$. Consequently, $H(r)$ has no positive zeros. This completes the proof.

Theorem 3.3 *If $B>0$, $A+q<0$, $q>-1$, there exists exactly one positive $\tilde{B}$ such that the equation $H(r)=0$ has two positive simple roots if $B<\tilde{B}$; it has exactly one positive double root if $B=\tilde{B}$; and it has no positive roots if $B>\tilde{B}$.*

Proof 1. The asymptotic formula (3.8), with $A+q<0$, shows that, in the present circumstances, $H(r)<0$ for $r>0$ sufficiently small. Furthermore, $H(r)\downarrow-\infty$ as $r\uparrow\infty$. We also observe that, by Theorem 2.1, the reduced function $h(r)$ has exactly one positive simple zero.

2(a). Let $A-q<0$ so that $A<0$, $A^2-q^2>0$. Let A, B and q be arbitrarily given within their individual restrictions. There are three alternatives: The polynomial $P(r)$ given in (3.4) has two distinct positive zeros; $P(r)$ has a positive double zero; $P(r)$ has no positive zeros.

(i) *$P(r)$ has two distinct positive zeros* $\sigma_1<\sigma_2$. (They are contained in the set defined in (3.9).) Then $B<[2(1-A)]^{-1}$. If $B>0$ is small, $H(r)$ has a positive simple zero ρ_1 as a consequence of Theorem 2.1. Since $H(r)<0$ for $0<r<\rho_1$, $H(r)>0$ for $r>\rho_1$ and r sufficiently close to ρ_1, $H(r)$ must have at least one other zero $\rho_2>\rho_1$. By the properties of $P(r)$ we have $\sigma_1<\rho_1<\sigma_2$, and $\sigma_2\leqq\rho_2$. But ρ_2 cannot be equal to σ_2. The arguments used in part 2(a)(i) of the proof of Theorem 3.1 can be applied verbatim to verify this statement. Therefore, $\rho_2>\sigma_2$ and ρ_2 is simple. Cearly, $H(r)$ cannot have a zero greater than ρ_2 because, at any such zero, H' would be negative. Consequently, under the current conditions, $H(r)$ has two positive simple zeros if $B>0$ is sufficiently small.

(ii) *$P(r)$ has a positive double zero* σ. This requires that $B=B_*<[2(1-A)]^{-1}$. (B_* is given by (3.10), and σ by (3.11).) Suppose $\rho>0$ is the smallest positive zero of $H(r)$. If $0<\rho<\sigma$, then $H'(\rho)<0$ which is impossible. Thus, $\rho\geqq\sigma$. If $\rho=\sigma$, then $H(\sigma)=H'(\sigma)=H''(\sigma)=0$ (by (3.1) and (3.2)). Therefore, by (3.7) in some neighbourhood of $\rho=\sigma$,

$$H(r)=\frac{1}{3!}(r-\sigma)^3H'''(\sigma)+\cdots$$

with

$$H'''(\sigma)=-2B_*\sqrt{c}\,\sigma^{-1}I_q(\sigma)<0$$

according to (3.3) and (3.6). Therefore, in some left-hand neighbourhood of σ, $H(r)>0$ which is impossible. Thus, σ is not a zero of $H(r)$. Finally, if $\rho>\sigma$, then $H'(\rho)<0$ which again is impossible. Consequently, $H(r)$ has no positive zeros.

(iii) *$P(r)$ has no positive zeros.* In this case, which requires $B>B_*$, $H'(\rho)<0$ at any positive zero of $H(r)$. Consequently, $H(r)$ has no positive zeros.

We can now finish up the present part of the proof. Let

$$\sigma_2^2(B)=(2B^2)^{-1}[a+\sqrt{a^2-b}]$$

and consider the function

$$H(\sigma_2(B))=[-B\sigma_2^2(B)+A+q]I_q(\sigma_2(B))+\sigma_2(B)I_{q+1}(\sigma_2(B)).$$

If $B>0$ is sufficiently small, $\sigma_2(B)$ is the larger of the two positive zeros of $P(r)$ (see

(3.9)). By part 2(a)(i), $H(\sigma_2(B))>0$ is $B>0$ is sufficiently small. By part 2(a)(ii), $H(\sigma_2(B))<0$ if $B=B_*$. Therefore, there exists at least one $\tilde{B}\in(0,B_*)$ such that $H(\sigma_2(\tilde{B}))=0$ and $H'(\sigma_2(\tilde{B}))=P(\sigma_2(\tilde{B}))\sigma_2^{-1}(\tilde{B})I_q(\sigma_2(\tilde{B}))=0$. In other words, there exists at least one $\tilde{B}\in(0,B_*)$ such that $H(r;\tilde{B})$ has a double zero at $r=\sigma_2(\tilde{B})$. Clearly, $H(r;\tilde{B})<0$ for $r\neq\sigma_2(\tilde{B})$. But then, if $B>\tilde{B}$, $H(r;B)<0$ for every r including the point $\sigma_2(\tilde{B})$. Consequently, there exists exactly one $\tilde{B}\in(0,B_*)$ such that $H(\sigma_2(\tilde{B}))=H'(\sigma_2(\tilde{B}))=0$. By (3.2) and (3.5) $H''(\sigma_2(\tilde{B}))<0$. $H(r)$ has exactly one positive double zero for $B=\tilde{B}$.

2(b). Let $A-q\geqq 0$ so that $-1<q<0$, $A<1$, and $A^2-q^2\leqq 0$. Within their individual restrictions let A, B and q be arbitrarily given. There are two alternatives: $P(r)$ has one positive zero; $P(r)$ has no positive zeros.

(i) *$P(r)$ has one positive zero σ_1.* If $B>0$ is small, $H(r)$ has a simple positive zero ρ_1 as a consequence of Theorem 2.1. Since $H(r)<0$ for $0<r<\rho_1$, $H(r)>0$ for $r>\rho_1$ and r sufficiently close to ρ_1, $H(r)$ must have at least one other positive zero $\rho_2>\rho_1$. By the properties of $P(r)$ we have $0<\rho_1<\sigma_1$ and $\rho_2\geqq\sigma_1$. But ρ_2 cannot be equal to σ_1. For, if $\rho_2=\sigma_1$, then

$$H(r)=\frac{1}{2!}(r-\sigma_1)^2H''(\sigma_1)+\cdots<0$$

in some neighbourhood of σ_1 which is impossible. Therefore, $\rho_2>\sigma_1$, ρ_2 is simple, and $H(r)$ cannot have a zero greater than ρ_2. Consequently, under the current assumptions, $H(r)$ has two positive simple zeros if $B>0$ is sufficiently small.

(ii) *$P(r)$ has no positive zeros.* In this case $A^2-q^2=0$, $-1<A<0$, and $B\geqq[2(1-A)]^{-1}$. Then $H'(\rho)<0$ at any positive zero ρ of $H(r)$ which is impossible. Therefore, $H(r)$ has no positive zeros.

Now let $P(r)$ have the positive zero σ_1,

$$\sigma_1^2(B)=(2B^2)^{-1}[a+\sqrt{a^2-b}],$$

and consider the function

$$H(\sigma_1(B))=[-B\sigma_1^2(B)+A+q]I_q(\sigma_1(B))+\sigma_1(B)I_{q+1}(\sigma_1(B)).$$

By part 2(b)(i) of this proof, $H(\sigma_1(B))>0$ if $B>0$ is sufficiently small. Relative to $\sigma_1(B)$ we observe that

$$\sigma_1(B)\downarrow 0\begin{cases}\text{as } B\uparrow\infty \text{ if } A^2-q^2<0,\\ \text{as } B\uparrow[2(1-A)]^{-1} \text{ if } A^2-q^2=0.\end{cases}$$

In contrast to this we note that any zero of $H(r)$ is greater than the positive fixed zero

r_1 of the reduced function $h(r)=[A+q]I_q(r)+rI_{q+1}(r)$. Therefore, as B increases, there exists a $B>0$ such that $H(\sigma_1(B))<0$. Consequently, there exists at least one $\tilde{B}>0$ such that $H(\sigma_1(\tilde{B}))=0$, $H'(\sigma_1(\tilde{B}))=P(\sigma_1(\tilde{B}))\sigma_1^{-1}(\tilde{B})I_q(\sigma_1(\tilde{B}))=0$. In other words, there exists at least one $\tilde{B}>0$ such that $H(r;\tilde{B})$ has a double zero at $r=\sigma_1(\tilde{B})$. The uniqueness of $\tilde{B}$ can be established as before. This completes the entire proof.

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