PACKING OF SPHERES IN l_p†

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1. Introduction. The Banach space $l_p (p \geqq 1)$ is the space of all infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of real or complex numbers such that $\sum_{i=1}^{\infty} |x_i|^p$ is convergent, with the norm defined by

$$\|\mathbf{x}\| = \left\{ \sum_{i=1}^{\infty} |x_i|^p \right\}^{1/p}.$$

The unit sphere S of l_p is the set of all points $\mathbf{x} \in l_p$ with $\|\mathbf{x}\| \leqq 1$ and the sphere of radius $a \geqq 0$ centred at $\mathbf{y} \in l_p$ is denoted by $S_a(\mathbf{y})$, so that

$$S_a(\mathbf{y}) = \{\mathbf{x} \in l_p : \|\mathbf{x} - \mathbf{y}\| \leqq a\}.$$

A (finite or infinite) set of spheres $S_a(\mathbf{y}_1)$, $S_a(\mathbf{y}_2)$, ... is said to form a *packing* in S if each sphere $S_a(\mathbf{y}_i)$ is contained in S and no two of the spheres overlap (they are, however, allowed to touch). By taking $\mathbf{x} = (1 + a/\|\mathbf{y}\|)\mathbf{y}$ if $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x} = (a/\|\mathbf{u}\|)\mathbf{u}$ for a non-zero $\mathbf{u}$ if $\mathbf{y} = \mathbf{0}$, it is seen that, if $S_a(\mathbf{y}) \subseteq S$, then $\mathbf{x} \in S$ and $\|\mathbf{y}\| \leqq 1 - a$. Conversely, if $\|\mathbf{y}\| \leqq 1 - a$ and $\|\mathbf{x} - \mathbf{y}\| \leqq a$, then $\|\mathbf{x}\| \leqq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| \leqq 1$, so that $S_a(\mathbf{y}) \subseteq S$. Also, two spheres $S_a(\mathbf{y})$ and $S_a(\mathbf{z})$ do not overlap if and only if $\|\mathbf{y} - \mathbf{z}\| \geqq 2a$, so that an equivalent condition for the set of spheres $S_a(\mathbf{y}_1)$, $S_a(\mathbf{y}_2)$, ... to form a packing is that

$$\|\mathbf{y}_i\| \leqq 1 - a \quad \text{for all } i \quad \text{and} \quad \|\mathbf{y}_i - \mathbf{y}_j\| \geqq 2a \quad \text{if } i \neq j. \tag{1.1}$$

Let $\lambda_p = (1 + 2^{1-1/p})^{-1}$ and $\mu_p = (1 + 2^{1/p})^{-1}$. In [1] the following results were proved:

THEOREM 1.1. *If $p > 2$, an infinity of spheres $S_a(\mathbf{y})$ of fixed radius a can be packed in S if and only if $a \leqq \lambda_p$. If $\lambda_p < a \leqq \mu_p$, any finite number of spheres $S_a(\mathbf{y})$ can be packed in S but an infinite number cannot. If $\mu_p < a \leqq 1$, the maximum number of spheres of fixed radius a which can be packed in S does not exceed*

$$M_p(a) = \left[\left\{ 1 - \frac{1}{2} \left(\frac{1-a}{a} \right)^p \right\}^{-1} \right].$$

(The square brackets denote the integral part.)

THEOREM 1.2. *If $1 \leqq p \leqq 2$, an infinity of spheres $S_a(\mathbf{y})$ of fixed radius a can be packed in S if and only if $a \leqq \lambda_p$. If $\lambda_p < a \leqq 1$, the maximum number of spheres $S_a(\mathbf{y})$ which can be packed in S does not exceed $L_p(a)$, where $L_1(a) = 1$ and*

$$L_p(a) = \left[\left\{ 1 - \frac{1}{2} \left(\frac{1-a}{a} \right)^{p/(p-1)} \right\}^{-1} \right] \qquad (1 < p \leqq 2).$$

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It is the purpose of this paper to prove that in the case of complex l_p space the upper bound $M_p(a)$ obtained in Theorem 1.1 is in a sense the best possible, but that in the case of real l_p space both bounds $M_p(a)$ and $L_p(a)$ can be considerably improved.

2. Conditions for the existence of finite packings. In this section the following theorem is proved.

THEOREM 2.1. *If $p \geqq 1$, $n(\geqq 2)$ spheres of radius $a \leqq 1$ can be packed in S if and only if*

$$\left(\frac{1-a}{a}\right)^p \geqq 2^{p-1}\frac{n-1}{K_p(n)}, \tag{2.1}$$

where

$$K_p(n) = \max \frac{\sum_{1 \leqq i<j \leqq n} |z_i - z_j|^p}{\sum_{i=1}^{n} |z_i|^p}, \tag{2.2}$$

the maximum being taken over all n-tuples of complex numbers $(z_1, z_2, \ldots, z_n)$ except $(0, 0, \ldots, 0)$. The theorem is still valid when $n = 1$ if the right-hand side of (2.1) is interpreted as zero.

Proof. Suppose that n spheres of radius a, centred at $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots)$, can be packed in S. Then, since $\|\mathbf{x}_i - \mathbf{x}_j\| \geqq 2a$ for $1 \leqq i < j \leqq n$ by (1.1),

$$\begin{aligned}(2a)^p \tfrac{1}{2} n(n-1) &\leqq \sum_{1 \leqq i<j \leqq n} \sum_{k=1}^{\infty} |x_{ik} - x_{jk}|^p \\ &\leqq \sum_{k=1}^{\infty} K_p(n) \sum_{j=1}^{n} |x_{jk}|^p \\ &= K_p(n) \sum_{j=1}^{n} \|\mathbf{x}_j\|^p \\ &\leqq nK_p(n)(1-a)^p,\end{aligned}$$

from which (2.1) is obtained.

To prove the converse, we first of all note that $K_p(n)$ is certainly an attained upper bound. Suppose that it is attained at the point $(z_1, z_2, \ldots, z_n)$ and consider an $n \times n!$ matrix

$$P = [P_1 P_2 \ldots P_{n!}] = [p_{ij}],$$

where each P_i is a column vector $\{z_{i(1)}, z_{i(2)}, \ldots, z_{i(n)}\}$ obtained from $\{z_1, z_2, \ldots, z_n\}$ by considering, in turn, the $n!$ permutations $(i(1), i(2), \ldots, i(n))$ of $(1, 2, \ldots, n)$. Points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n \in l_p$ are chosen as follows:

$$\mathbf{y}_i = (y_{i1}, y_{i2}, \ldots), \quad \text{where} \quad y_{ij} = \begin{cases} bp_{ij} & (1 \leqq j \leqq n!) \\ 0 & (j > n!), \end{cases}$$

the constant b being chosen in accordance with

$$\frac{(1-a)^p}{(n-1)!} \geqq |b|^p \sum_{j=1}^{n} |z_j|^p \geqq \frac{(2a)^p}{2(n-2)!K_p(n)}. \tag{2.3}$$

This is possible, since we have, by (2.1),

$$\frac{(1-a)^p}{(n-1)!} \geqq \frac{(2a)^p}{2(n-2)!\,K_p(n)}.$$

If all the permutations of $(z_1, z_2, \ldots, z_n)$ are considered, each z_j occupies the ith position $(n-1)!$ times, and consequently

$$\|\mathbf{y}_i\|^p = |b|^p(n-1)! \sum_{j=1}^{n} |z_j|^p \leqq (1-a)^p \qquad (1 \leqq i \leqq n), \tag{2.4}$$

by (2.3). Also, if $i \neq j$, z_i and z_j simultaneously occupy the kth and lth positions respectively in $(n-2)!$ of the permutations, and so, for $1 \leqq k < l \leqq n$,

$$\|\mathbf{y}_k - \mathbf{y}_l\|^p = 2|b|^p(n-2)! \sum_{1 \leqq i < j \leqq n} |z_i - z_j|^p = 2|b|^p(n-2)!\,K_p(n) \sum_{j=1}^{n} |z_j|^p \geqq (2a)^p, \tag{2.5}$$

by (2.3). (2.4) and (2.5) show that if the spheres of radius a are centred at $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$, then the theorem is proved.

We immediately deduce

COROLLARY 2.2. *The maximum number of spheres of radius a that can be packed in S is n if and only if*

$$\frac{2^{p-1}n}{K_p(n+1)} > \left(\frac{1-a}{a}\right)^p \geqq \frac{2^{p-1}(n-1)}{K_p(n)}.$$

Proof. The proof follows immediately from Theorem 2.1.

The inequality

$$K_p(n) \leqq 2^{p-2}n \qquad (p \geqq 2), \tag{2.6}$$

was first derived by Rankin in [2], while in the case $1 \leqq p < 2$ he obtained the estimate $K_p(n) \leqq n^{p-1}(n-1)^{2-p}$ in his paper [3]. These estimates were used in [1] to obtain $M_p(a)$ and $L_p(a)$ defined in Theorems 1.1 and 1.2. We shall show in the next section that, in the case when $K_p(n)$ is defined as a maximum over n-tuples of real numbers, it can be evaluated explicitly and as a consequence $M_p(a)$ and $L_p(a)$ can be improved when the space l_p is real.

However, by taking $n = 2m$, $z_i = -1$ for $1 \leqq i \leqq m$ and $z_i = +1$ for $m < i \leqq 2m$, it is easily seen that

$$K_p(2m) = 2^{p-1}m \qquad (p \geqq 2). \tag{2.7}$$

As a result we may deduce

COROLLARY 2.3. *If $p \geqq 2$ and $\mu_p < a \leqq 1$ and if $M_p(a)$ is even, then exactly $M_p(a)$ spheres of radius a can be packed in S.*

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Proof. We first of all observe that $M_p(a) = 2m$ if and only if

$$\frac{1}{2m} \geqq \left\{1-\frac{1}{2}\left(\frac{1-a}{a}\right)^p\right\} > \frac{1}{2m+1},$$

which is equivalent to

$$\frac{2m-1}{m} \leqq \left(\frac{1-a}{a}\right)^p < \frac{4m}{2m+1}. \tag{2.8}$$

Also, from (2.7),

$$\frac{2^{p-1}(2m-1)}{K_p(2m)} = \frac{2m-1}{m},$$

and, from (2.6),

$$\frac{4m}{2m+1} \leqq \frac{2^p m}{K_p(2m+1)}.$$

Thus, from (2.8), $M_p(a) = 2m$ implies that

$$\frac{2^{p-1}(2m-1)}{K_p(2m)} \leqq \left(\frac{1-a}{a}\right)^p < \frac{2^p m}{K_p(2m+1)},$$

and an application of Corollary 2.2 yields the result.

3. Real l_p space. In this section the l_p spaces under consideration will be real.

Suppose that $p \geqq 1$ and that n is a positive integer > 1. Let

$$f(x_1, x_2, \ldots, x_n) = \frac{\sum_{1 \leqq i < j \leqq n} |x_i - x_j|^p}{\sum_{j=1}^{n} |x_j|^p},$$

and $N_p(n) = \max f(x_1, x_2, \ldots, x_n)$, the maximum being taken over all n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$ except $(0, 0, \ldots, 0)$. Corresponding to Theorem 2.1 and Corollary 2.2 we have

THEOREM 3.1. *If $p \geqq 1$, n spheres of radius a can be packed in S if and only if*

$$\left(\frac{1-a}{a}\right)^p \geqq \frac{2^{p-1}(n-1)}{N_p(n)}.$$

COROLLARY 3.2. *The maximum number of spheres of radius a that can be packed in S is n if and only if*

$$\frac{2^{p-1}n}{N_p(n+} > \left(\frac{1-a}{a}\right)^p \geqq \frac{2^{p-1}(n-1)}{N_p(n)}.$$

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In this section we evaluate $N_p(n)$ and have the following

THEOREM 3.3. (i) *If* $p \geqq 2$, *then*

$$N_p(n) = \begin{cases} 2^{p-2}n, & \text{if } n \text{ is even,} \\ \frac{1}{2}\{(n-1)^{1/(p-1)}+(n+1)^{1/(p-1)}\}^{p-1}, & \text{if } n \text{ is odd.} \end{cases}$$

(ii) *If* $1 \leqq p \leqq 2$, $N_p(n) = n-2+2^{p-1}$.

Proof. Since

$$\sum_{i<j} |x_i - x_j| \leqq \sum_{i<j} \{|x_i|+|x_j|\} = (n-1)\sum_{j=1}^{n} |x_j|,$$

it is seen that $N_1(n) \leqq n-1$; taking $x_i = 0$ for $1 \leqq i < n$, and $x_n = 1$ shows in fact that $N_1(n) = n-1$. We may therefore suppose that $p > 1$. Also, since the theorem is trivial when $n = 2$, we may suppose that $n \geqq 3$.

Clearly, because of the homogeneity, we may restrict our attention to the compact subset Q of R^n defined by $\frac{1}{2} \leqq \sum_{i=1}^{n} |x_i|^p \leqq 1$ and, since f is continuous on Q, its maximum is attained at some point of Q which we may take to be an interior point of Q. Since $p > 1$, f possesses partial derivatives at every point of Q and consequently, by the above remarks, $N_p(n)$ will be attained at some stationary point of f. If $(a_1, a_2, \ldots, a_n)$ is any stationary point of f, by differentiating

$$f(x_1, x_2, \ldots, x_n) \sum_{i=1}^{n} |x_i|^p = \sum_{i<j} |x_i - x_j|^p$$

with respect to x_l, we see that

$$a_l |a_l|^{p-2} f(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} (a_l - a_j)|a_l - a_j|^{p-2} \qquad (1 \leqq l \leqq n).$$

From this it follows that, if $f(a_1, a_2, \ldots, a_n) \neq 0$, then

$$\sum_{l=1}^{n} a_l |a_l|^{p-2} = 0. \tag{3.1}$$

Since $f(a_1, a_2, \ldots, a_n) = 0$ corresponds to the minimum value of f, we suppose that this is not the case. Thus from (3.1) there exists an integer k $(1 \leqq k \leqq n-1)$ such that

$$a_1 \leqq a_2 \leqq \ldots \leqq a_k \leqq 0 < a_{k+1} \leqq \ldots \leqq a_n, \tag{3.2}$$

where the a_i's have been renumbered if necessary. Then

$$\sum_{i=1}^{k} |a_i|^{p-1} = \sum_{j=k+1}^{n} |a_j|^{p-1}, \tag{3.3}$$

and, for $l \leqq k$,

$$|a_l|^{p-1} f(a_1, a_2, \ldots, a_n) = \sum_{j=l+1}^{n} |a_l - a_j|^{p-1} - \sum_{j=1}^{l} |a_l - a_j|^{p-1}.$$

Hence

$$f(a_1,a_2,\ldots,a_n)\sum_{l=1}^{k}|a_l|^{p-1}=\sum_{l=1}^{k}\sum_{j=l+1}^{n}|a_l-a_j|^{p-1}-\sum_{l=1}^{k}\sum_{j=1}^{l}|a_l-a_j|^{p-1}$$
$$=\sum_{l=1}^{k}\sum_{j=k+1}^{n}|a_l-a_j|^{p-1}. \tag{3.4}$$

We now suppose that $p \geqq 2$ and apply Minkowski's inequality to get

$$\left\{\sum_{l=1}^{k}\sum_{j=k+1}^{n}|a_l-a_j|^{p-1}\right\}^{1/(p-1)} \leqq \left\{\sum_{l=1}^{k}\sum_{j=k+1}^{n}|a_l|^{p-1}\right\}^{1/(p-1)}+\left\{\sum_{l=1}^{k}\sum_{j=k+1}^{n}|a_j|^{p-1}\right\}^{1/(p-1)}$$
$$=\{(n-k)^{1/(p-1)}+k^{1/(p-1)}\}\left\{\sum_{l=1}^{k}|a_l|^{p-1}\right\}^{1/(p-1)},$$

by (3.3). It follows therefore from (3.4) that

$$f(a_1,a_2,\ldots,a_n) \leqq \{(n-k)^{1/(p-1)}+k^{1/(p-1)}\}^{p-1}.$$

Since $N_p(n)=\max_k f(a_1,a_2,\ldots,a_n)$, $(a_1,a_2,\ldots,a_n)$ being a stationary point of f satisfying (3.2), we see that

$$N_p(n) \leqq \begin{cases} 2^{p-2}n, & \text{if } n \text{ is even,}\\ \tfrac{1}{2}\{(n-1)^{1/(p-1)}+(n+1)^{1/(p-1)}\}^{p-1}, & \text{if } n \text{ is odd.}\end{cases}$$

That these bounds are attained can be seen by taking $x_1=x_2=\ldots=x_m=-1$, $x_{m+1}=\ldots=x_{2m}=+1$, when $n=2m$, and $x_1=x_2=\ldots=x_{m+1}=1$, $x_{m+2}=\ldots=x_{2m+1}=-\{(m+1)/m\}^{1/(p-1)}$, when $n=2m+1$. This proves part (i) of the theorem.

To prove (ii) we require the following

LEMMA 3.4. *Let n be an integer $\geqq 3$ and let the integer k be such that $1 \leqq k \leqq n-1$. Let*

$$g_k(x_1,x_2,\ldots,x_n)=\sum_{i=1}^{k}\sum_{j=k+1}^{n}(x_i+x_j)^q,$$

where $0<q<1$. If A_k is the bounded closed subset of $(n-2)$-dimensional space defined by

$$\sum_{i=2}^{k}x_i^q \leqq 1, \qquad \sum_{j=k+1}^{n-1}x_j^q \leqq 1, \qquad x_l \geqq 0 \qquad (2 \leqq l \leqq n-1),$$

where, if a summation is empty, it is taken to be zero, and if

$$x_1^q=1-\sum_{i=2}^{k}x_i^q, \qquad x_n^q=1-\sum_{j=k+1}^{n-1}x_j^q,$$

then

(i) $N_q(n,k)=\max_{A_k} g_k(x_1,x_2,\ldots,x_n)$ *is attained on the boundary of A_k (i.e. at least one of the x_i's is zero), and*

(ii) $N_q(n,k)=n-2+2^q$.

Proof. g_k is certainly continuous on A_k and so attains its maximum value there. Since g_k is differentiable on the interior of A_k, if the maximum were attained at some interior point $(b_1, b_2, \ldots, b_n)$, Lagrange's equations would be satisfied:

$$\sum_{i=1}^{k} (b_i+b_j)^{q-1} + \lambda b_j^{q-1} = 0 \quad \text{if} \quad j \geqq k+1,$$

$$\sum_{j=k+1}^{n} (b_i+b_j)^{q-1} + \mu b_i^{q-1} = 0 \quad \text{if} \quad i \leqq k.$$

These equations imply that $b_1 = b_2 = \ldots = b_k$, $b_{k+1} = \ldots = b_n$. For if, for example, $b_i < b_l$ when $i < l \leqq k$, then

$$\left(\frac{b_i}{b_i+b_j}\right)^{1-q} < \left(\frac{b_l}{b_l+b_j}\right)^{1-q} \qquad (0 < q < 1),$$

and the above equations would not be satisfied. For these values of $b_1, b_2, \ldots, b_n$ we have

$$\begin{aligned} g_k(b_1, b_2, \ldots, b_n) &= k(n-k)\{k^{-1/q} + (n-k)^{-1/q}\}^q \\ &= \{(n-k)^{1/q} + k^{1/q}\}^q. \end{aligned}$$

However, this is in fact the minimum value of g_k as an application of Minkowski's inequality shows $(0 < q < 1)$. This contradiction establishes part (i).

(ii) is proved by induction.

When $n = 3$, $k = 1$ or 2 and $N_q(3, k) = \max\{(1+x)^q + (1+y)^q\}$, where $x^q + y^q = 1$, $x \geqq 0$ and $y \geqq 0$. By (i), at the maximum $x = 0$ and consequently $y = 1$. Thus $N_q(3, k) = 1 + 2^q$.

Assume now that $N_q(n-1, k') = n-3+2^q$ for some integer $n \geqq 4$ and all k' such that $1 \leqq k' \leqq n-2$. Then by (i), either

$$N_q(n, k) = \max\left\{\sum_{i=1}^{k-1} \sum_{j=k+1}^{n} (x_i+x_j)^q + \sum_{j=k+1}^{n} x_j^q\right\},$$

where $\sum_{i=1}^{k-1} x_i^q = \sum_{j=k+1}^{n} x_j^q = 1$, taking $x_k = 0$, or

$$N_q(n, k) = \max\left\{\sum_{i=1}^{k} \sum_{j=k+2}^{n} (x_i+x_j)^q + \sum_{i=1}^{k} x_i^q\right\},$$

where $\sum_{i=1}^{k} x_i^q = \sum_{j=k+2}^{n} x_j^q = 1$, taking $x_{k+1} = 0$. (If $k = 1$ or $n-1$ it is clear which of the two alternatives must hold.) In other words,

$$\begin{aligned} N_q(n, k) &= 1 + \max\{N_q(n-1, k-1), N_q(n-1, k)\} \\ &= 1 + n - 3 + 2^q = n - 2 + 2^q, \end{aligned}$$

by the induction hypothesis. This completes the proof of the lemma.

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Returning to Theorem 3.3, (ii), we have, by (3.2), (3.3), and (3.4),

$$N_p(n) \leqq \max_{1 \leqq k \leqq n-1} \max \left\{ \sum_{i=1}^{k} \sum_{j=k+1}^{n} (x_i + x_j)^{p-1} \right\},$$

where (by the homogeneity) $\sum_{i=1}^{k} x_i^{p-1} = \sum_{j=k+1}^{n} x_j^{p-1} = 1$ and $x_l \geqq 0 (1 \leqq l \leqq n)$. Now apply Lemma 3.4 with $p-1 = q$ to obtain $N_p(n) \leqq \max_{1 \leqq k \leqq n-1} N_{p-1}(n, k) = n-2+2^{p-1} (n \geqq 3)$. Taking $x_1 = -x_2 = 1, x_3 = \ldots = x_n = 0$ shows that this upper bound is in fact attained. This completes the proof of Theorem 3.3.

Applying Theorem 3.1 and Corollary 3.2 yields

COROLLARY 3.5. *If $p > 2$, the maximum number of spheres of radius a that can be packed in S is $2m$ if and only if*

$$\frac{2m-1}{m} < \left(\frac{1-a}{a}\right)^p < 2^p \left\{1 + \left(\frac{m+1}{m}\right)^{1/(p-1)}\right\}^{1-p},$$

and the maximum number of spheres of radius a that can be packed in S is $2m-1$ if and only if

$$2^p \left\{1 + \left(\frac{m}{m-1}\right)^{1/(p-1)}\right\}^{1-p} \leqq \left(\frac{1-a}{a}\right)^p < \frac{2m-1}{m}.$$

(The left-hand side of the last inequality is to be taken to be zero when $m = 1$.)

Proof. For $2^{p-1} 2m/N_p(2m+1) = 2^p \left\{1 + \left(\frac{m+1}{m}\right)^{1/(p-1)}\right\}^{1-p}$, and $2^{p-1}(2m-1)/N_p(2m) = (2m-1)/m$.

COROLLARY 3.6. *If $1 < p \leqq 2$ and $a > \lambda_p$, the maximum number of spheres of radius a that can be packed in S is $[1 + \nu_p/(S_p(a) - 1)]$, where $\nu_p = 2^{p-1} - 1$ and $S_p(a) = 2^{p-1} \Big/ \left(\frac{1-a}{a}\right)^p$.*

Proof. By Corollary 3.2, when $1 < p \leqq 2$ the maximum number of spheres of radius a that can be packed in S is m if and only if

$$\frac{2^{p-1} m}{m-1+2^{p-1}} > \left(\frac{1-a}{a}\right)^p \geqq \frac{2^{p-1}(m-1)}{m-2+2^{p-1}}.$$

This condition is easily seen to be equivalent to

$$m+1 > 1 + \frac{\nu_p}{S_p(a) - 1} \geqq m,$$

which completes the proof.

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